ON THE INTEGRAL REPRESENTATION OF POSITIVE LINEAR FUNCTIONALS

BY
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1. **Introduction.** Let A be a $*$-algebra; i.e., A is an algebra over the field of complex numbers with an involution—that is, a mapping $x \to x^*$ of A onto A such that $(x+y)^* = x^* + y^*$, $(\alpha x)^* = \bar{\alpha} x^*$, $(xy)^* = y^* x^*$, $(x^*)^* = x$ for all x and y in A and complex numbers α. An element $x \in A$ is said to be selfadjoint if $x^* = x$. If $x \in A$, then $x = x_1 + ix_2$, where $x_1 = (x+x^*)/2$ and $x_2 = (x-x^*)/2i$. x_1 and x_2 are selfadjoint elements of A and are called the real and imaginary parts of x, respectively. We write $x_1 = \operatorname{Re} x$ and $x_2 = \operatorname{Im} x$. If B is a subset of A we denote by B^* the set $\{x^* \mid x \in B\}$. A linear functional f on A is said to be *positive* if $f(x^*x) \geqq 0$ for all x in A. A positive linear functional f on a $*$-algebra A is said to be *real* or *hermitian* if $f(x^*) = f(x)^-$ for all x in A. If f is any positive linear functional on A, then $f(x^*y) = f(y^*x)^-$ and $|f(x^*y)| \leqq f(x^*x)^{1/2} f(y^*y)^{1/2}$ (Schwarz's inequality) for all x and y in A. If A has an identity e, we can take $y = e$ and obtain $f(x^*) = f(x)^-$ and $|f(x)|^2 \leqq Mf(x^*x)$, where $M = f(e)$. A positive linear functional which satisfies these extra conditions (i.e., f is real and $|f(x)|^2 \leqq Mf(x^*x)$ for all x in A, where M is a constant independent of x) is called *extendible* for reasons which the following proposition makes clear:

A necessary and sufficient condition that a positive linear functional f on a $*$-algebra A without identity can be extended so as to remain positive when an identity is added to A is that f be extendible in the above sense (cf. [8, p. 96], [9], and [12]).

Let f be a positive linear functional on A. The elements x in A such that $f(x^*x) = 0$ form a left ideal I_f in A. If x is an element in A we denote by x_f the coset of $A/I_f = H'_f$ which contains x and we define by

$$(x_f \mid y_f) = f(y^*x)$$

an inner product on H'_f. Thus H'_f becomes a pre-Hilbert space. Let H_f be the Hilbert space which is the completion of H'_f. If $x \in A$ we denote by U_x the operator in H_f whose domain $D(U_x) = H'_f$ and which maps y_f into $(xy)_f$. Then U_x is a densely defined operator in H_f and $U_{x^*} \subset U_x^*$ (i.e., the adjoint of U_x is an extension of U_{x^*}). Hence U_{x^*} has a closure $[U_{x^*}]$ for every x in A. Furthermore $U_{xy} = U_x U_y$, $U_{\alpha x} = \alpha U_x$, and $U_{x+y} = U_x + U_y$ for all x and y in A and complex numbers α.

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Clearly, the necessary and sufficient condition that U_x, and hence $[U_x]$, be a bounded operator is that there exists a constant M_x such that

$$f(y^*x^*xy) \leqq M_x f(y^*y) \quad \text{for all } y \text{ in } A.$$

If U_x is bounded for every $x \in A$, then f is said to be *unitary* (cf. [6] and [8]). If f is unitary, then $D([U_x])=H_f$ for every $x \in A$ and $x \to T_x=[U_x]$ is a $*$-representation of A by bounded operators on H_f and $f(xyz^*)=(T_x y_f|z_f)$. A $*$-homomorphism of A onto the field C of complex numbers is called a *unitary character of A*. Thus a homomorphism χ of A onto C is a unitary character of A if and only if $\chi(x^*)=\chi(x)^-$ for all $x \in A$. If A is a commutative $*$-algebra, we denote by $\hat{A}$ the set of unitary characters of A together with the weakest topology such that the mappings $\hat{x}: \chi \to \chi(x)$, $x \in A$, are continuous. Clearly $\hat{A}$ is a Hausdorff space. Suppose now that f is a unitary positive linear functional on a commutative $*$-algebra A. Let R be the C^*-algebra generated by $\{T_x\}$, $x \in A$. Using the spectral theorem of the commutative C^*-algebra R, R. Godement has obtained the following integral representation for f, which he has called the Plancherel formula for f:

THEOREM 1 (R. GODEMENT [6, p. 76]). *Let f be a positive linear functional on a commutative $*$-algebra A. If f is unitary, then there exists a positive Radon measure μ_f on a locally compact subset σ_f of $\hat{A}$ such that*

(a) *$\hat{x}(\chi)=\chi(x)$ belongs to $L^2(\mu_f)$ for every $x \in A$;*

(b) *$f(xyz)=\int_{\sigma_f} \chi(xyz)\, d\mu_f(\chi)$ for all x, y, and z in A.*

If, furthermore, f is extendable, then μ_f is a finite measure and

$$f(x) = \int_{\sigma_f} \chi(x)\, d\mu_f(\chi)$$

for all x in A.

(Godement assumes in his definition of a positive linear functional that the functional is real. This condition is not necessary, however, for the proof of (b).)

According to R. Godement the extension of Theorem 1 to arbitrary positive linear functionals is of fundamental importance (cf. loc. cit. p. 78). It follows, however, from the results of R. B. Zarhina [13] on the two-dimensional moment problem that Godement's Plancherel formula is not valid for an arbitrary positive linear functional on an arbitrary commutative $*$-algebra. It is not valid, for example, for every positive linear functional on the $*$-algebra of polynomials in two variables (cf. [7, pp. 232–236]). On the other hand there exist positive linear functionals for which Plancherel's formula ((b) of Theorem 1) holds, but which are not unitary. For example, if A is the commutative $*$-algebra of complex polynomials $p(t)$ with respect to the ordinary operations of addition and multiplication and involution $p^*(t)=p(t)^-$ and

$$f(p) = \int_{-\infty}^{\infty} p(t) e^{-|t|}\, dt$$

for $p \in A$, then f is a positive linear functional on A which by definition has an integral representation of the form (b). But f is not unitary, for otherwise there exists a constant M such that

$$\int_{-\infty}^{\infty} t^2 t^{2n} e^{-|t|}\, dt \leqq M \int_{-\infty}^{\infty} t^{2n} e^{-|t|}\, dt$$

for all $n \geqq 0$. This inequality is obviously false, for the left-hand side is equal to $2\Gamma(2n+3) = 2(2n+2)!$ and the right-hand side is equal to $2M\Gamma(2n+1) = 2M(2n)!$

The main purpose of this paper is to extend Theorem 1 to positive linear functionals which satisfy certain growth conditions, but which are not necessarily unitary.

We say that a positive linear functional f on a commutative $*$-algebra A is *quasi-unitary* if there exists a subset A_0 of A such that

$$\sum_{n=1}^{\infty} f((xx^*)^n)^{-1/2n} = \infty \quad \text{for all } x \in A_0, \tag{1}$$

and if for every $x \in A$ there exists an element y in the $*$-algebra A_e obtained from A by adjoining an identity element e (if A does not have an identity element) which is a polynomial with complex coefficients in finitely many elements of $A_0 \cup A_0^*$ such that

$$f(xx^*zz^*) \leqq f(yy^*zz^*) \quad \text{for all } z \in A. \tag{2}$$

(Note that condition (2) is automatically satisfied if (1) holds for all $x \in A$.)

In §2 we show that if f is a quasi-unitary positive linear functional on a commutative $*$-algebra A, then $x \to T_x = [U_x]$ is a $*$-representation of A by permuting (in general unbounded) normal operators, and if f is unitary, it is a fortiori quasi-unitary. (For a precise definition of a $*$-representation of A by unbounded normal operators cf. Theorem 2.)

The main result of this paper is Theorem 4 of §3 which states that Godement's theorem remains true mutatis mutandis if unitary is replaced by quasi-unitary and, in addition, the positive linear functional f satisfies the following separability condition (d):

There exists a countable subset D of A_e such that for every $x \in A$ there exists a $y \in A_e$ which is a polynomial with complex coefficients in finitely many elements of D such that

$$f(xx^*zz^*) \leqq f(yy^*zz^*) \quad \text{for all } z \in A.$$

This condition is satisfied if f is unitary, if we take for $D = \{e\}$.

Thus Theorem 4 includes Godement's theorem as a special case, but it also yields the integral representation of the nonunitary positive linear functional of the example given above. Other examples should not be difficult to construct.

2. **$*$-representation.** Let A be a commutative $*$-algebra and f a positive linear functional on A. We denote by A_e the $*$-algebra obtained from A by adjoining an identity element e to A, if A does not have an identity element. If A does have an identity element, we set $A_e = A$. Let $T_x = [U_x]$, where U_x is the operator in H_f defined in the introduction. Then $f(xyz^*) = (T_x y_f | z_f)$ for all x, y, and z in A.

LEMMA 1.

$$\sum_{n=1}^{\infty} f((xx^*)^n)^{-1/2n} = \infty \Leftrightarrow \sum_{n=1}^{\infty} f((xx^*)^{2n})^{-1/4n} = \infty.$$

Proof. That $\sum_{n=1}^{\infty} f((xx^*)^{2n})^{-1/4n} = \infty$ implies $\sum_{n=1}^{\infty} f((xx^*)^n)^{-1/2n} = \infty$ is obvious. To prove the reverse implication we may assume without loss of generality that $f(xx^*) = 1$, for if $f(xx^*) = 0$ then $f((xx^*)^n) = 0$ for all $n \geqq 1$ by Schwarz's inequality. We assume, therefore, that $f(xx^*) = 1$. Then $f((xx^*)^{n+1})^{1/2n}$ is a nondecreasing function of $n \geqq 1$. Indeed,

$$f((xx^*)^2) = f(x(x^*xx^*)) \leqq f(xx^*)^{1/2} f((xx^*)^3)^{1/2} = f((xx^*)^3)^{1/2}.$$

Hence

$$f((xx^*)^2)^{1/2} \leqq f((xx^*)^3)^{1/4}.$$

Assume now that $f((xx^*)^{n+1})^{1/2n} \leqq f((xx^*)^{n+2})^{1/(2n+2)}$, then

$$f((xx^*)^{n+2}) = f(x^{n+1}(xx^{*n+2})) \leqq f((xx^*)^{n+1})^{1/2} f((xx^*)^{n+3})^{1/2}$$
$$\leqq f((xx^*)^{n+2})^{n/(2n+2)} f((xx^*)^{n+3})^{1/2}.$$

Hence

$$f((xx^*)^{n+2})^{(n+2)/(2n+2)} \leqq f((xx^*)^{n+3})^{1/2}$$

and therefore $f((xx^*)^{n+2})^{1/(2n+2)} \leqq f((xx^*)^{n+3})^{1/(2n+4)}$. Hence, by finite induction, $f((xx^*)^{n+1})^{1/2n}$ is a nondecreasing function of n. It follows that

$$\sum_{n=1}^{\infty} f((xx^*)^{n+1})^{-1/2n} = \infty \Rightarrow \sum_{n=1}^{\infty} f((xx^*)^{2n})^{-1/(4n-2)} = \infty,$$

and hence

$$\sum_{n=1}^{\infty} f((xx^*)^{n+1})^{-1/2n} = \infty \Rightarrow \sum_{n=1}^{\infty} f((xx^*)^{2n})^{-1/4n} = \infty.$$

Now, if $M_n > 0$ for $n \geqq 1$ and if p is an arbitrary but fixed real number, then $\sum_{n=1}^{\infty} (M_n)^{-1/(n+p)}$ converges if and only if $\sum_{n=1}^{\infty} (M_n)^{-1/n}$ converges (cf. [3, p. 106]). Hence

$$\sum_{n=1}^{\infty} f((xx^*)^{n+1})^{-1/2n} = \infty \Leftrightarrow \sum_{n=1}^{\infty} f((xx^*)^n)^{-1/2n} = \infty$$

and therefore

$$\sum_{n=1}^{\infty} f((xx^*)^n)^{-1/2n} = \infty \Rightarrow \sum_{n=1}^{\infty} f((xx^*)^{2n})^{-1/4n} = \infty.$$

THEOREM 2. *Suppose there exists a subset* A_0 *of* A *such that*

1. $\sum_{n=1}^{\infty} f((xx^*)^n)^{-1/2n} = \infty$ *for all* $x \in A_0$;

2. *for every* $x \in A$ *there exists an element* $y \in A_e$ *which is a polynomial with complex coefficients in finitely many elements of* $A_0 \cup A_0^*$ *such that*

$$f(xx^*zz^*) \leqq f(yy^*zz^*) \quad \textit{for all } z \in A.$$

Then $x \to T_x$ *is a* $*$*-representation of* A *by permuting (in general unbounded) normal operators. That is,* $\{T_x\}$, $x \in A$, *are permuting normal operators (i.e., their resolutions of the identity permute),* $T_{x+y} = [T_x + T_y]$, $T_{xy} = [T_x T_y]$, $T_{\alpha x} = [\alpha T_x]$ *and* $T_{x^*} = T_x^*$ *for all* $x, y \in A$ *and complex numbers* α.

Proof. We first observe that if condition 1 holds for a given x, then it also holds for $x_1 = \operatorname{Re} x$ and $x_2 = \operatorname{Im} x$ since

$$f((xx^*)^n) = \sum_{k=0}^{n} \binom{n}{k} f(x_1^{2k} x_2^{2(n-k)})$$

and hence

$$f((xx^*)^n) \geqq f(x_1^{2n}) \quad \text{and} \quad f((xx^*)^n) \geqq f(x_2^{2n}) \quad \text{for all } n \geqq 1.$$

Since $U_{x^*} \subset U_x^*$, it follows that $T_{x^*} \subset T_x^*$ for every $x \in A$. Hence, if x is selfadjoint, T_x is a closed symmetric operator. To prove the theorem, it is sufficient to show that (i) T_x is selfadjoint for every selfadjoint element x in A and (ii) T_x and T_y permute if x and y are any two selfadjoint elements of A. Indeed, suppose that (i) and (ii) hold. Let x be any element in A. Then $U_{xx^*} \subset U_x U_{x^*} \subset T_x T_{x^*} \subset T_x T_x^*$. Hence $T_{xx^*} = U_{xx^*}^* \supset T_x T_x^*$, since $T_x T_x^*$ is selfadjoint. But this implies that $T_{xx^*} = T_x T_x^*$, since T_{xx^*} is symmetric. Similarly $U_{xx^*} = U_{x^*x} \subset T_{x^*} T_x \subset T_x^* T_x$. Hence $T_{xx^*} = U_{xx^*}^* \supset T_x^* T_x$ and therefore $T_{xx^*} = T_x^* T_x$. Hence $T_x T_x^* = T_x^* T_x$; i.e., T_x is normal for every $x \in A$. Suppose x and y are any two elements in A. Write $x = x_1 + ix_2$, $y = y_1 + iy_2$, where x_1, y_1 and x_2, y_2 are the real and imaginary parts of x and y, respectively. Now $U_x = U_{x_1} + iU_{x_2} \subset T_{x_1} + iT_{x_2}$ and T_{x_1} and T_{x_2} are permuting selfadjoint operators. Hence $T_{x_1} + iT_{x_2}$ is a normal operator and $T_x = [U_x] \subset T_{x_1} + iT_{x_2}$. But T_x is normal as we have seen. Hence $T_x = T_{x_1} + iT_{x_2}$. Similarly $T_y = Y_{y_1} + iT_{y_2}$. But T_{x_1}, T_{x_2}, T_{y_1}, T_{y_2} are permuting selfadjoint operators by (ii). Hence T_x and T_y permute. Moreover, $T_{\alpha x} = [\alpha T_x]$ for $U_{\alpha x} = \alpha U_x$ and hence $T_{x^*} = T_{x_1} + iT_{-x_2} = T_{x_1} - iT_{x_2} = T_x^*$. Now, $T_x + T_y = (T_{x_1} + iT_{x_2}) + (T_{y_1} + iT_{y_2}) = (T_{x_1} + T_{y_1}) + i(T_{x_2} + T_{y_2}) \subset T_{x_1+y_1} + iT_{x_2+y_2} = T_{x+y}$, for $[T_{x_1} + T_{y_1}] = T_{x_1+y_2}$, and $[T_{x_2} + T_{y_2}] = T_{x_2+y_2}$. (Because $U_{x_1+y_1} = U_{x_1} + U_{y_1} \subset T_{x_1} + T_{y_1}$ and hence taking adjoints: $T_{x_1+y_1} \supset T_{x_1} + T_{y_1}$. But $[T_{x_1} + T_{y_1}]$ is selfadjoint by the operational calculus for normal operators. Hence $[T_{x_1} + T_{y_1}] = T_{x_1+y_1}$. Similarly, $[T_{x_2} + T_{y_2}] = T_{x_2+y_2}$.) From $T_x + T_y \subset T_{x+y}$, and the fact that T_x, T_y, T_{x+y} are normal and T_x and T_y permute, follows by the operational calculus for normal operators and the fact that a normal operator is maximal (in the sense that it does not have a proper normal extension) that $[T_x + T_y] = T_{x+y}$. Finally, $U_x U_y = U_{x_1 y_1 - x_2 y_2} + iU_{x_1 y_2 + x_2 y_1}$. Hence $[U_x U_y]$

$=T_{x_1y_1-x_2y_2}+iT_{x_1y_2+x_2y_1}=T_{xy}$. But $[U_xU_y]=(U_xU_y)^{**}\supset(U_y^*U_x^*)^*\supset U_x^{**}U_y^{**}=T_xT_y$. Hence $T_xT_y\subset T_{xy}$. From this follows, since T_x, T_y, T_{xy} are normal and T_x and T_y permute—as above—that $[T_xT_y]=T_{xy}$.

Let x now be the real or imaginary part of an element of A_0 and y any element of A, then

$$\|T_x^n y_f\|^2 = \|(x^n y)_f\|^2 = f(x^{2n}yy^*) \leqq f(x^{4n})^{1/2}f((yy^*)^2)^{1/2}$$

and hence

$$\sum_{n=1}^{\infty} \|T_x^n y_f\|^{-1/n} \geqq \sum_{n=1}^{\infty} f(x^{4n})^{-1/4n}f((yy^*)^2)^{-1/4n}.$$

But $\sum_{n=1}^{\infty} f(x^{2n})^{-1/2n}=\infty$ by condition 1 and the above remark and hence $\sum_{n=1}^{\infty} f(x^{4n})^{-1/4n}=\infty$ by Lemma 1. Therefore

$$\sum_{n=1}^{\infty} \|T_x^n y_f\|^{-1/n} = \infty.$$

That is, every element of H_f' is a quasi-analytic vector for T_x (for the theory of quasi-analytic vectors cf. [11]). Hence T_x is selfadjoint for every x which is the real or imaginary part of an element of A_0 by Theorem 2 of loc. cit. If x and y are the real or imaginary parts of any two elements of A_0, then T_x and T_y permute by Theorem 6 of loc. cit.

Next, let x_1 and x_2 be any two selfadjoint elements of A. Let $x=x_1+ix_2$ and choose, using condition 2, an element $y\in A_e$ which is a polynomial in the elements $a_1, a_2, \dots, a_m, a_1^*, a_2^*, \dots, a_m^*$, where $a_1, a_2, \dots, a_m$ are elements of A_0 such that

$$f(xx^*zz^*) \leqq f(yy^*zz^*) \quad \text{for all } z\in A.$$

Replacing a_k by $\operatorname{Re} a_k+i\operatorname{Im} a_k$, $k=1, 2, \dots, m$, we see that

$$y = \sum c_{i_1\cdots i_n} y_1^{i_1}\cdots y_n^{i_n},$$

where $y_1, \dots, y_n$ are the real or imaginary parts of elements of A_0 and the $c_{i_1\cdots i_n}$ are complex numbers. We may assume that y is selfadjoint, for otherwise replace y by $y_1^2+y_2^2+e$, where $y_1=\operatorname{Re} y$ and $y_2=\operatorname{Im} y$. For, if $u=y_1^2+y_2^2$, then

$$f(yy^*zz^*) = f((y_1^2+y_2^2)zz^*) \leqq f((u^2+2u+e)zz^*) = f((u+e)^2zz^*).$$

Finally, we may assume that the coefficients $c_{i_1\cdots i_n}$ are real, for

$$y = \frac{y+y^*}{2} = \tfrac{1}{2}\sum (c_{i_1\cdots i_n}+\bar{c}_{i_1\cdots i_n})y_1^{i_1}\cdots y_n^{i_n} = \sum (\operatorname{Re} c_{i_1\cdots i_n})y_1^{i_1}\cdots y_n^{i_n}.$$

If $w\in A_e$, we denote by U_w the operator $x\to(wy)_f$ in H_f with domain H_f'. Clearly $U_y=\sum c_{i_1\cdots i_n}U_{y_1}^{i_1}\cdots U_{y_n}^{i_n}\subset\sum c_{i_1\cdots i_1}T_{y_1}^{i_1}\cdots T_{y_n}^{i_n}=V$. Let $\{E_i(t)\}$ be the resolution of the identity of T_{y_i}, $i=1, \dots, n$. Let k_i be any nonnegative integer and $E_i^{(k_i)}=E_i(k_i)-E_i(-k_i)$ and $E_{(k_1,\dots,k_n)}=E_n^{(k_1)}\cdots E_n^{(k_n)}$. $E_i(t)$ permutes with T_{x_1}, T_{x_2}, $T_{y_1}, \dots, T_{y_n}$

by Corollary 5 of [11]. Hence $E_k = E_{(k_1,\ldots,k_n)}$ permutes with $T_{x_1}, T_{x_2}, T_y = [U_y]$, $T_{y_1}, \ldots, T_{y_n}$ and hence with V. Now,

$$E_k U_k \subset E_k V \subset \sum c_{i_1 \cdots i_n} (T_{y_1} E_1^{(k_1)})^{i_1} \cdots (T_{y_n} E_n^{(k_n)})^{i_n} \subset VE_k$$

and $T_{y_i} E_i^{(k_i)}$ is a bounded selfadjoint operator which permutes with $T_{y_j} E_j^{(k_j)}$, for $j = 1, \ldots, n$. Hence VE_k is a bounded selfadjoint operator and therefore

$$T_y E_k = (E_k U_y)^* \supset VE_k.$$

Hence $T_y E_k = VE_k$ and therefore $T_y E_k$ is a bounded selfadjoint operator.

Now,

$$f(xx^*zz^*) = f((x_1^2 + x_2^2)zz^*) = \|T_{x_1} z_f\|^2 + \|T_{x_2} z_f\|^2 \leqq f(yy^*zz^*) = \|T_y z_f\|^2.$$

That is, $\|T_{x_1} z_f\|^2 + \|T_{x_2} z_f\|^2 \leqq \|T_y z_f\|^2$ for all $z_f \in H_f'$. It follows that $D(T_y) \subset D(T_{x_1})$, $D(T_y) \subset D(T_{x_2})$ and

$$\|T_{x_1} u\|^2 + \|T_{x_2} u\|^2 \leqq \|T_y u\|^2 \quad \text{for all } u \in D(T_y).$$

Hence $\|T_{x_1} E_k u\|^2 + \|T_{x_2} E_k u\|^2 \leqq \|T_y E_k u\|^2 \leqq \|T_y E_k\|^2 \|u\|^2$ for all $u \in H_f$. Hence $T_{x_1} E_k$ and $T_{x_2} E_k$ are bounded. From this and the fact that E_k permutes with T_{x_1} and T_{x_2} and $E_k \to I$ as $k_1 \to \infty, \ldots, k_n \to \infty$, follows by standard Hilbert space methods that T_{x_1} and T_{x_2} are selfadjoint. It can also be seen as follows:

$$\|T_{x_1}^n E_k u\| = \|(T_{x_1} E_k)^n u\| \leqq \|T_{x_1} E_k\|^n \|u\|$$

and similarly

$$\|T_{x_2}^n E_k u\| \leqq \|T_{x_2} E_k\|^n \|u\| \quad \text{for all } u \in H_f \text{ and all } k.$$

Hence every vector of the set $D = \{E_k u \mid u \in H_f, \text{ all } k\}$ is a quasi-analytic vector for T_{x_1} and T_{x_2}, respectively. Since D is dense in H_f it follows from Theorem 2 of [11] that T_{x_1} and T_{x_2} are selfadjoint.

Finally,

$$(E_k U_{x_1}^m U_{x_2}^n)^* \supset T_{x_2}^n T_{x_1}^m E_k \supset (T_{x_2} E_k)^n (T_{x_1} E_k)^m$$

and therefore $(E_k U_{x_1^m x_2^n})^* = T_{x_2}^n T_{x_1}^m E_k$. Similarly,

$$(E_k U_{x_1^m x_2^n})^* = (E_k U_{x_2^n x_1^m})^* = T_{x_1}^m T_{x_2}^n E_k.$$

Hence $T_{x_1}^m T_{x_2}^n E_k = T_{x_2}^n T_{x_1}^m E_k$ for all n and $m \geqq 1$ and all k. Hence $T_{x_1}^m T_{x_2}^n u = T_{x_2}^n T_{x_1}^m u$ for all $u \in D$ and n and $m \geqq 1$. Hence T_{x_1} and T_{x_2} permute by Theorem 6 of [11]. (That T_{x_1} and T_{x_2} permute follows also by standard Hilbert space techniques from the fact that E_k reduces T_{x_1} and T_{x_2}, respectively, to bounded permuting selfadjoint operators and the fact that $E_k \to I$ as $k_1 \to \infty, \ldots, k_n \to \infty$.)

DEFINITION 1. *A positive linear functional f on a commutative $*$-algebra A will be called quasi-unitary, if there exists a subset A_0 of As such that conditions* 1 *and* 2 *of Theorem* 2 *hold.*

PROPOSITION 1. *Every unitary positive linear functional on a commutative $*$-algebra is quasi-unitary.*

Proof. Let f be a unitary positive linear functional on a commutative $*$-algebra A. Let x be any element in A, then $f((xx^*)^n) \leqq M_x f((xx^*)^{n-1})$ for all $n \geqq 2$ and hence

$$f((xx^*)^n) \leqq M_x^{n-1} f(xx^*) \quad \text{for all } n \geqq 1.$$

Hence $\sum_{n=1}^{\infty} f((xx^*)^n)^{-1/2n} = \infty$. To satisfy conditions 1 and 2 of Theorem 2 we may therefore take $A_0 = A$. However, it is sufficient to take $A_0 = \{x_0\}$, where x_0 is an arbitrary element in A, for we may choose for $x \in A$ the element y in condition 2 to be $M_x^{1/2} e$, which is a polynomial in x_0 and x_0^*.

The positive linear functional which we have considered in the introduction is quasi-unitary, but not unitary (as we have seen). Indeed, let $A_0 = \{t\}$. Then

$$f(t^{2n}) = \int_{-\infty}^{\infty} t^{2n} e^{-|t|}\, dt = 2\int_0^{\infty} t^{2n} e^{-t}\, dt = 2(2n)! < 2(2n)^{2n};$$

that is, $f(t^{2n}) < 2(2n)^{2n}$ for all $n \geqq 1$. Hence $\sum_{n=1}^{\infty} f(t^{2n})^{-1/2n} = \infty$. Condition 2 of Theorem 2 is obviously satisfied, for every element in A is a polynomial in t.

3. **Integral representation of quasi-unitary positive linear functionals.** Let f be a quasi-unitary positive linear functional on a commutative $*$-algebra A and $x \to T_x$ the corresponding $*$-representation (cf. Theorem 2). Let R be the bi-commutant of $\{T_x \mid x \in A\}$, then R is the von Neumann algebra generated by the spectral projections of the normal operators $\{T_x\}$, $x \in A$. Let $T \to \hat{T}$ be the Gelfand representation of the C^*-algebra R onto $C(\mathfrak{M}) - \mathfrak{M}$ is the spectrum of R. Let $\bar{C}(\mathfrak{M})$ be the algebra of continuous functions on $\mathfrak{M}$ which are ∞ only on a nowhere dense set. (If f and g are elements of $\bar{C}(\mathfrak{M})$, then fg and $f+g$ are defined to be the unique elements in $\bar{C}(\mathfrak{M})$ such that $(fg)(x) = f(x)g(x)$ and $(f+g)(x) = f(x)+g(x)$, respectively, except on a set of the first category (cf. [5] and [10]).) Let $E(\sigma)$ be the spectral measure of R. If $\hat{T} \in \bar{C}(\mathfrak{M})$, let T be the normal operator (in general unbounded) $T = \int_{\mathfrak{M}} \hat{T}(M)\, dE(M)$. ($u \in D(T)$ if and only if $\int_{\mathfrak{M}} |\hat{T}(M)|^2\, d\|E(M)u\|^2 < \infty$.) Let $\bar{R}$ be the set of all normal operators $\{T \mid \hat{T} \in \bar{C}(\mathfrak{M})\}$ and define the sum and product of any two operators T and S in $\bar{R}$ to be $[T+S]$ and $[TS]$, respectively. $\bar{R}$ together with these operations and the usual operations of multiplications by scalars and adjunction is a commutative $*$-algebra and the mapping $\hat{T} \to T$ is a $*$-isomorphism of $\bar{C}(\mathfrak{M})$ onto $\bar{R}$ (cf. loc. cit.). Now, $T_x \in \bar{R}$ for all $x \in A_e$ (the proof is the same as that of Theorem 4 in [10]), and hence $x \to \hat{T}_x$ is a $*$-homomorphism of A_e into $\bar{C}(\mathfrak{M})$ and

$$f(xyz^*) = (T_x y_f | z_f) = \int_{\mathfrak{M}} \hat{T}_x(M)\, d(E(M)y_f | z_f)$$

for all x, y, and z in A. We denote by μ_x, if $x \in A$, the Radon measure which for every Borel set $\sigma \subset \mathfrak{M}$ is defined by $\mu_x(\sigma) = \|E(\sigma)x_f\|^2$. If $x \in A_e$, let S_x be the set of all M such that $|\hat{T}_x(M)| = \infty$. S_x is nowhere dense and hence $E(S_x) = 0$ (for $\hat{E}(S_x)$

is the characteristic function of $\varnothing$ (cf. [10, p. 134])) and therefore $\mu_y(S_x)=0$ for all $y \in A$. Therefore $\hat{T}_x$ is finite μ_y-a.e. for every $y \in A$. Now, for every x and y in A and $\hat{T} \in C(\mathfrak{M})$,

$$\int_{\mathfrak{M}} \hat{T}(M)|\hat{T}_y(M)|^2\, d\mu_x(M) = (TT_yT_y^*x_f|x_f)$$
$$= (TT_xT_x^*y_f|y_f) = \int_{\mathfrak{M}} \hat{T}(M)|T_x(M)|^2\, d\mu_y(M)$$

and hence $|\hat{T}_y(M)|^2\, d\mu_x(M)=|\hat{T}_x(M)|^2\, d\mu_y(M)$ for all x and y in A.

Let X be the set of $M \in \mathfrak{M}$ such that $\hat{T}_x(M)\neq 0$ for some $x \in A$. X is an open subset of $\mathfrak{M}$ and hence locally compact. Let ν_x be the restriction of the Radon measure μ_x to X and denote the restriction of a function $\hat{T} \in \bar{C}(\mathfrak{M})$ to X by $\tilde{T}$. (In this note we follow Bourbaki's approach to measure theory [1], [2].) Then $|\tilde{T}_x(M)|^2\, d\nu_y(M)=|\tilde{T}_y(M)|^2\, d\nu_x(M)$ for all x and y in A.

THEOREM 3. *There exists a positive Radon measure ν on X such that $\tilde{T}_x \in L^2(\nu)$ and $d\nu_x(M)=|\tilde{T}_x(M)|^2\, d\nu(M)$ for all $x \in A$ and $f(xyz^*)=\int_X \tilde{T}_{xyz^*}(M)\, d\nu(M)$ for all x, y, and z in A.*

Proof. If K is a compact set in X, then there exists an $x \in A$ such that $\tilde{T}_x(M)\neq 0$ for all $M \in K$. Indeed, if $M \in K$, there exists an element $y=y_M \in A$ such that $\tilde{T}_y(M)\neq 0$. Hence there exists an open neighborhood U_y of M on which $\tilde{T}_y$ does not vanish. Since K is compact, there exist finitely many such U_y : U_{y_i}, $i=1,2,\ldots,n$, such that $K\subset\bigcup_{i=1}^n U_{y_i}$. Let $x=y_1y_1^*+y_2y_2^*+\cdots+y_ny_n^*$, then $\tilde{T}_x(M)=|\tilde{T}_{y_1}(M)|^2+|\tilde{T}_{y_2}(M)|^2+\cdots+|\tilde{T}_{y_n}(M)|^2$ for all $M \in X$ (equality holds for all M because the sum of the right-hand side is everywhere continuous) and hence $\tilde{T}_x(M)>0$ on K.

Let $C_{00}(X)$ be the vector space of complex-valued continuous functions on X with compact support. If $\varphi \in C_{00}(X)$ and σ_φ is the support of φ, we choose an element $x \in A$ such that $\tilde{T}_x(M)\neq 0$ on σ_φ. Then $\varphi/|\tilde{T}_x|^2 \in C_{00}(X)$ ($\varphi/|\tilde{T}_x|^2$ denotes the function which is equal to $\varphi(M)/|\tilde{T}_x(M)|^2$ for $M \in \sigma_\varphi$ and 0 for $M \notin \sigma_\varphi$) and set $\nu(\varphi)=\int (\varphi/|\tilde{T}_x|^2)\, d\nu_x$. The definition of ν is independent of the particular choice of x, for if y is another element in A such that $\tilde{T}_y(M)\neq 0$ on σ_φ, then

$$\int \frac{\varphi}{|\tilde{T}_x|^2}\, d\nu_x = \int_{\sigma_\varphi} \frac{\varphi(M)}{|\tilde{T}_x(M)|^2}\, d\nu_x(M) = \int_{\sigma_\varphi} \frac{\varphi(M)}{|\tilde{T}_y(M)|^2}\frac{|\tilde{T}_y(M)|^2}{|\tilde{T}_x(M)|^2}\, d\nu_x(M)$$
$$= \int_{\sigma_\varphi} \frac{\varphi(M)}{|\tilde{T}_y(M)|^2}\frac{|\tilde{T}_x(M)|^2}{|\tilde{T}_x(M)|^2}\, d\nu_y(M) = \int_{\sigma_\varphi} \frac{\varphi(M)}{|\tilde{T}_y(M)|^2}\, d\nu_y(M) = \int \frac{\varphi}{|\tilde{T}_y|^2}\, d\nu_y.$$

Now, $\nu(\varphi)\geqq 0$ if $\varphi\geqq 0$ and hence ν is a positive Radon measure on X.

Let $N_x=\{M \in X \mid \tilde{T}_x(M)=0\}$, then $\nu_x(N_x)=0$. Indeed, if C is a compact subset of N_x choose $y \in A$ such that $\tilde{T}_y(M)\neq 0$ on C. Then

$$\int_C |\tilde{T}_y(M)|^2\, d\nu_x(M) = \int_C |\tilde{T}_x(M)|^2\, d\nu_y(M) = 0,$$

and therefore $\nu_x(C)=0$. Hence $\nu_x(N_x)=0$. We assert also that $\nu(S_x)=0$. Indeed, for every integer $n>0$ let $G_n=\{M\in X \mid |\tilde{T}_x(M)|>n\}$, then $\bar{G}_n$ (closure of G_n in X) is clopen and compact and hence

$$\nu(S_x) \leqq \nu(\bar{G}_n) = \int_{\bar{G}_n} \frac{d\nu_x(M)}{|\tilde{T}_x(M)|^2} \leqq \frac{f(xx^*)}{n^2}.$$

Therefore $\nu(S_x)=0$.

Let $\varphi\in C_{00}^{+}(X)$ (nonnegative real-valued elements of $C_{00}(X)$) and $x\in A$. For every integer $n>0$ let $\sigma_n=\{M\in X \mid 1/n<|\tilde{T}_x(M)|<n\}$. $\bar{\sigma}_n$ (closure of σ_n in X) is clopen and contained in $\{M\in X \mid 1/n\leqq|\tilde{T}_x(M)|\leqq n\}$ and therefore is compact. Hence

$$\int_{\bar{\sigma}_n} \varphi(M)\,d\nu_x(M) = \int_{\bar{\sigma}_n} \varphi(M)\frac{|\tilde{T}_x(M)|^2}{|\tilde{T}_x(M)|^2}\,d\nu_x(M) = \int_{\bar{\sigma}_n} \varphi(M)|\tilde{T}_x(M)|^2\,d\nu(M).$$

Letting $n\to\infty$ we obtain

$$\int_{X-(S_x\cup N_x)} \varphi(M)\,d\nu_x(M) = \int_{X-(S_x\cup N_x)} \varphi(M)|\tilde{T}_x(M)|^2\,d\nu(M)$$

by the monotone convergence theorem. But $\nu_x(N_x)=\nu_x(S_x)=\nu(S_x)=0$ as we have seen. Hence

$$\int_X \varphi(M)\,d\nu_x(M) = \int_X \varphi(M)|\tilde{T}_x(M)|^2\,d\nu(M).$$

Therefore $\int_X \varphi(M)\,d\nu_x(M)=\int_X \varphi(M)|\tilde{T}_x(M)|^2\,d\nu(M)$ for all $\varphi\in C_{00}(X)$; that is, $d\nu_x(M)=|\tilde{T}_x(M)|^2\,d\nu(M)$ (and therefore $\tilde{T}_x\in L^2(\nu)$).

Finally, let x and y be arbitrary elements in A and $\Delta_n=\{M\in X \mid 1/n<|\tilde{T}_x(M)|<n, 1/n<|\tilde{T}_y(M)|<n\}$. Then $\bar{\Delta}_n$ is clopen and compact and

$$\int_{\bar{\Delta}_n} \tilde{T}_x(M)\,d\nu_y(M) = \int_{\bar{\Delta}_n} \tilde{T}_x(M)|\tilde{T}_y(M)|^2\,d\nu(M) = \int_{\bar{\Delta}_n} \tilde{T}_{xyy^*}(M)\,d\nu(M).$$

Now $\tilde{T}_x\in L^1(\nu_y)$ and $\tilde{T}_{xyy^*}\in L^1(\nu)$ since $\tilde{T}_x$ and $\tilde{T}_{yy^*}$ belong to $L^2(\nu)$. Hence, letting $n\to\infty$ we obtain

$$\int_X \tilde{T}_x(M)\,d\nu_y(M) = \int_X \tilde{T}_{xyy^*}(M)\,d\nu(M)$$

by Lebesgue's dominated convergence theorem and the fact that $\nu(S_x)=\nu(S_y)=\nu_y(N_y)=\nu_y(S_y)=0$. But $f(xyy^*)=\int_X \tilde{T}_x(M)\,d\nu_y(M)$. Therefore

$$f(xyy^*) = \int_X \tilde{T}_{xyy^*}(M)\,d\nu(M)$$

and hence, using the identity

$$f(xyz^*) = \tfrac{1}{4}\{f(x(y+z)(y+z)^*)-f(x(y-z)(y-z)^*)+if(x(y+iz)(y+iz)^*) - if(x(y-iz)(y-iz)^*)\}$$

we obtain

$$f(xyz^*) = \int_X \tilde{T}_{xyz^*}(M)\, d\nu(M) \quad \text{for all } x, y, \text{ and } z \text{ in } A.$$

Corollary 1. *If N is a ν-measurable subset of X such that $\nu_x(N)=0$ for all $x \in A$, then N is ν-locally negligible.*

Proof. We first observe that if N is a ν-measurable set, then N is ν_x-measurable, because $N-N_x$ is ν-measurable (since N_x is closed) (cf. [1, p. 43]). Let C be a compact subset of X and N a ν-measurable set such that $\nu_x(N)=0$ for all $x \in A$. Choose an element $y \in A$ such that $\tilde{T}_y(M) \neq 0$ on C. Then

$$0 = \nu_y(C \cap N) = \int_{C \cap N} |\tilde{T}_y(M)|^2\, d\nu(M)$$

and therefore $\nu(C \cap N)=0$.

Let $S=\bigcup_{x \in A} S_x$, where S_x is as above the set of $M \in X$ such that $|\tilde{T}_x(M)|=\infty$. We shall give a sufficient condition for $\bar{S}$ to be ν-locally negligible.

Lemma 2. *If f satisfies the additional condition* (d): *there exists a countable subset D of A_e such that for every $x \in A$ there exists a $y \in A_e$ which is a polynomial with complex coefficients in finitely many elements of D such that*

$$f(xx^*zz^*) \leqq f(yy^*zz^*) \quad \textit{for all } z \in A,$$

then $\bar{S}$ is ν-locally negligible.

Proof. We shall show that if f satisfies condition (d), then $S=\bigcup_{x \in D} S_x$. Indeed, let x be an arbitrary element in A and y be an element in A_e which is a polynomial in finitely many elements of D such that $f(xx^*zz^*) \leqq f(yy^*zz^*)$ for all $z \in A$. This inequality is equivalent with the inequality $\|T_x u\| \leqq \|T_y u\|$ for all $u \in H_f'$. This implies that $D(T_y) \subset D(T_x)$ and $\|T_x u\| \leqq \|T_y u\|$ for all $u \in D(T_y)$, since T_y and T_x are the closures of their restrictions, respectively, to H_f'. This implies in turn that $|\hat{T}_x(M)| \leqq |\hat{T}_y(M)|$ for all $M \in \mathfrak{M}$. Indeed, suppose that $|\hat{T}_x(M_0)| > |\hat{T}_y(M_0)|$. Then $|\hat{T}_y(M_0)| < \infty$ and hence there exists a clopen neighborhood σ of M_0 and a positive number ε such that $|\hat{T}_x(M)|^2 > |\hat{T}_y(M)|^2 + \varepsilon$ for all $M \in \sigma$. $E(\sigma) \neq 0$ since $\sigma \neq \varnothing$. We may therefore choose a nonzero vector u in the range of $E(\sigma)$ and since $\hat{T}_y$ is bounded on σ it follows that $u \in D(T_y) \subset D(T_x)$ and hence

$$\|T_x u\|^2 = \int_{\mathfrak{M}} |\hat{T}_x(M)|^2\, d\|E(M)u\|^2 = \int_\sigma |\hat{T}_x(M)|^2\, d\|E(M)u\|^2$$

$$\geqq \int_\sigma (|\hat{T}_y(M)|^2 + \varepsilon)\, d\|E(M)u\|^2 = \|T_y u\|^2 + \varepsilon\|u\|^2.$$

This is a contradiction. From the fact that $|\hat{T}_x| \leqq |\hat{T}_y|$ follows that $S_x \subset S_y$. But clearly $S_y \subset \bigcup_{z \in D} S_z$(2). Hence $S \subset \bigcup_{z \in D} S_z$ and therefore $S=\bigcup_{z \in D} S_z$.

(2) For $S_{x+y} \subset S_x \cup S_y$ and $S_{xy} \subset S_x \cup S_y$ (cf. [10, p. 136]).

Since D is a countable set and every S_z is nowhere dense, it follows that S is a set of the first category in $\mathfrak{M}$. But in $\mathfrak{M}$ every set of the first category is nowhere dense (cf. [10] or [1, p. 65]). Hence S is nowhere dense in $\mathfrak{M}$ and therefore the closure $\bar{S}^{\mathfrak{M}}$ of S in $\mathfrak{M}$ is nowhere dense in $\mathfrak{M}$. Hence $\mu_x(\bar{S}^{\mathfrak{M}})=0$ for all $x \in A$ (for $E(\bar{S}^{\mathfrak{M}})=0$). Hence $\nu_x(\bar{S})=0$ for all $x \in A$ ($\bar{S}$ denotes the closure of S in X) and therefore $\bar{S}$ is ν-locally negligible by Corollary 1.

REMARK. If f is unitary we know a priori that $S=\varnothing$, since the operators T_x are bounded in that case. But f clearly satisfies also condition (d) (cf. Introduction).

We are now ready to prove the main theorem which is an extension of Theorem 1 of R. Godement.

THEOREM 4. *Let f be a positive linear functional on a commutative $*$-algebra A. If f is quasi-unitary and satisfies condition* (d), *then there exists a positive Radon measure μ_f on a locally compact subset σ_f of $\hat{A}$ such that*

(a) *$\hat{x}(\chi)=\chi(x)$ belongs to $L^2(\mu_f)$ for every $x \in A$;*

(b) *$f(xyz)=\int_{\sigma_f} \chi(xyz)\, d\mu_f(\chi)$ for all x, y, and z in A.*

If, furthermore, f is extendible, then μ_f is a finite measure and

$$f(x) = \int_{\sigma_f} \chi(x)\, d\mu_f(\chi)$$

for all x in A.

Proof. By Theorem 3 and Lemma 2

$$f(xyz) = \int_{X'} T'_{xyz}(M)\, d\nu'(M) \quad \text{for all } x, y, \text{ and } z \text{ in } A,$$

where $X'=X-\bar{S}$, T'_x is the restriction of $\tilde{T}_x$ to X' and ν' is the restriction of the Radon measure ν to X' (X' is an open subset of X and hence locally compact). The mapping $x \to T'_x(M)$ is a unitary character of A for every $M \in X'$, since $x \to \hat{T}_x$ is a $*$-homomorphism of A into $\bar{C}(\mathfrak{M})$. Let φ be the mapping of X' into $\hat{A}$ which maps M into $T'_{(\cdot)}(M)$. φ is continuous because $M \to T'_x(M)$ is a continuous mapping on X' for every fixed $x \in A$.

Let $\sigma_f=\varphi(X')$. σ_f is locally compact, for if $\varphi(M_0)=T'_{(\cdot)}(M_0) \in \sigma_f$, let x_0 be an element of A such that $T'_{x_0}(M_0)\neq 0$, $\varepsilon=|T'_{x_0}(M_0)|/2$ and

$$\hat{N} = \{\chi \in \sigma_f \mid |\chi(x_0)-T'_{x_0}(M_0)| \leqq \varepsilon\}.$$

$\hat{N}$ is clearly a neighborhood of $\varphi(M_0)$ and

$$N = \varphi^{-1}(\hat{N}) = \{M \in X' \mid |T'_{x_0}(M)-T'_{x_0}(M_0)| \leqq \varepsilon\}.$$

N is a compact neighborhood of M_0 for $\{M \in \mathfrak{M} \mid |\hat{T}_{x_0}(M)-\hat{T}_{x_0}(M_0)| \leqq \varepsilon\}$ is a compact neighborhood of M_0 in $\mathfrak{M}$ and $N=\{M \in \mathfrak{M} \mid |\hat{T}_{x_0}(M)-\hat{T}_{x_0}(M_0)| \leqq \varepsilon\}$. Since φ is continuous and $\hat{N}=\varphi(N)$, it follows that $\hat{N}$ is compact.

Next, we show that φ is a proper mapping; that is, if C is a compact set in σ_f, then $\varphi^{-1}(C)$ is a compact set in X'. Let C be a compact set in σ_f and $K=\varphi^{-1}(C)$.

Since C is compact, there exists by what precedes a finite number of compact neighborhoods $\hat{N}_1, \hat{N}_2, \ldots, \hat{N}_n$ of points in C such that $\varphi^{-1}(\hat{N}_i)=N_i$ is compact for $i=1, 2, \ldots, n$, and $C\subset\bigcup_{i=1}^n \hat{N}_i$. Hence $K\subset\bigcup_{i=1}^n N_i$. Since K is closed and $\bigcup_{i=1}^n N_i$ is compact, it follows that K is compact.

Let now μ_f be the image of the Radon measure ν' under φ (i.e. μ_f is the Radon measure on σ_f defined by $\int_{\sigma_f} g\, d\mu_f=\int_{X'} (g \circ \varphi)\, d\nu'$ for all $g \in C_{00}(\sigma_f)$) then

$$f(xyz) = \int_{\sigma_f} \chi(xyz)\, d\mu_f(\chi) \quad \text{for all } x, y, \text{ and } z \text{ in } A.$$

That $\hat{x}(\chi)=\chi(x)$ belongs to $L^2(\mu_f)$ follows from the fact that $T'_x \in L^2(\nu')$ by Theorem 3 for every $x \in A$.

Finally, if f is extendible, let $\tilde{f}$ be the positive linear functional on A_e which extends f. We may assume that $f\neq 0$, for otherwise the assertion of the theorem is trivially true. It is easily seen that $\tilde{f}$ is quasi-unitary. Let $x \rightarrow T_x$ be the $*$-representation of A_e corresponding to $\tilde{f}$ (cf. Theorem 2). Then

$$f(x) = \tilde{f}(x) = (T_x e_{\tilde{f}} | e_{\tilde{f}}) = \int_{\mathfrak{M}} \hat{T}_x(M)\, d\mu(M)$$

for all $x \in A$, where $\mu(\sigma)=\|E(\sigma)e_{\tilde{f}}\|^2$. Clearly μ is a bounded measure and $\hat{T}_x \in L^2(\mu)$ for all $x \in A$. It is easily seen that $\tilde{f}$ satisfies condition (d) and hence $\bar{S}$ is μ-locally negligible. Let X be the set of all M in $\mathfrak{M}$ such that $\hat{T}_x(M)\neq 0$ for some $x \in A$. X is an open subset of $\mathfrak{M}$ and hence $X'=X-\bar{S}$ is an open subset of $\mathfrak{M}$ and therefore locally compact. Let—using the same notation as above—T'_x be the restriction of $\hat{T}_x$ to X' and ν' the restriction of the Radon measure μ to X', then

$$f(x) = \int_{X'} T'_x(M)\, d\nu'(M) \quad \text{for all } x \in A.$$

Let as above φ be the mapping of X' into $\hat{A}$ which maps M into $T'_{(\cdot)}(M)$. The rest of the proof is identical with the preceding argument and we obtain the formula

$$f(x) = \int_{\sigma_f} \chi(x)\, d\mu_f(\chi) \quad \text{for all } x \in A,$$

where μ_f is a bounded measure on σ_f (as the image under φ of the bounded measure ν').

REMARK: If f is unitary, then the functions $\hat{x}(\chi)=\chi(x)$ are bounded on σ_f. In fact, in that case $|\hat{x}(\chi)|\leqq M_x$ for all $\chi \in \sigma_f$. If f is not unitary but quasi-unitary and satisfies condition (d), then the functions $\hat{x}$ are not in general bounded on σ_f.

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