COBORDISM OPERATIONS AND HOPF ALGEBRAS

BY
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1. **Introduction.** This paper is concerned with the stable operations of the unoriented, complex and symplectic cobordism theories $H^*(\ ;\boldsymbol{MO})$, $H^*(\ ;\boldsymbol{MU})$ and $H^*(\ ;\boldsymbol{MSp})$ on the category of finite CW-pairs (see [5], [6], [15]). The starting point is Theorem (3.1): the algebra (under composition) $\mathscr{A}^*(G)$ of stable operations of $H^*(\ ;MG)$ is additively isomorphic to the algebra $\mathscr{C}^*(G)$ of stable characteristic classes of G-bundles with values in G-cobordism; this is true not only for $\boldsymbol{O}$, $\boldsymbol{U}$ and $\boldsymbol{Sp}$ but also for the other stable groups which give rise to cobordism theories. For $G=\boldsymbol{O}$, $\boldsymbol{U}$ or $\boldsymbol{Sp}$, Conner and Floyd introduced in [5], [7] characteristic classes $E_i(\xi)\in H^{di}(X;MG)$ for G-bundles $\xi\to X$ (see §4); here $d=1$, 2 or 4 for $G=\boldsymbol{O}$, $\boldsymbol{U}$ or $\boldsymbol{Sp}$, resp. The E_i generate a polynomial subalgebra $C^*(G)$ of $\mathscr{C}^*(G)$, to which there corresponds a submodule $A^*(G)=\sum A^i(G)$ $(i\geqq 0)$ of $\mathscr{A}^*(G)$.

It is shown in §5 that $A^*(G)$ is a graded Hopf algebra, over the integers for $G=\boldsymbol{U}$ or $\boldsymbol{Sp}$ and over Z_2 in the unoriented case. In particular $A^*(G)$ is closed under the composition of operations; this product is studied in §6 with the help of the dual Hopf algebra $A_*(G)$. In §7 minimal generating sets are found for $A^*(G)\otimes Z_p$ ($G=\boldsymbol{U}$ or $\boldsymbol{Sp}$, p a prime) and $A^*(\boldsymbol{O})$, and in the following section these Hopf algebras are shown to contain quotients of the Steenrod algebras $\mathscr{S}^*(p)$. For example, $A^*(\boldsymbol{U})\otimes Z_p$ contains an isomorphic copy of $\mathscr{S}^*(p)/(\beta_p)$ with β_p the Bockstein and $A^*(\boldsymbol{O})$ contains a copy of the mod 2 Steenrod algebra $\mathscr{S}^*(2)$.

In the final section some comments are made concerning the action

$$A^i(G)\otimes\Omega^G_n\to\Omega^G_{n-i}$$

of $A^*(G)$ on the G-bordism ring Ω^G_*, obtained via the identification of Ω^G_* with the coefficient ring $H^*(pt;MG)$. Although this action provided the motivation for the present study, it remains quite obscure even in the unoriented case.

2. **Notation and preliminary results.** Let G be $\boldsymbol{O}$, $\boldsymbol{U}$ or $\boldsymbol{Sp}$. When dealing with cohomology, Z_2 will be understood as coefficients if $G=\boldsymbol{O}$ and the integers in the other cases. We assume familiarity with the Stiefel-Whitney classes $w_i(\xi)$ of real bundles and the Chern classes $c_i(\xi)$ of complex bundles. For ξ a right quaternionic bundle, which may also be regarded as a complex bundle, the symplectic Pontrjagin class $p_i(\xi)$ is by definition $(-1)^i c_{2i}(\xi)$; these satisfy the Whitney sum formula (see [4, §9]). Thus in all cases, for $\xi\to X$ a G-bundle over a finite CW-complex there are characteristic classes $e_i(\xi)\in H^{di}(X)$, with $d=1$, 2 or 4 for $G=\boldsymbol{O}$, $\boldsymbol{U}$ or $\boldsymbol{Sp}$, resp.

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For any stable group G, $BG(r)$ can be approximated by finite subcomplexes. If $G=\boldsymbol{O}$, $\boldsymbol{U}$ or $\boldsymbol{Sp}$ we may make this explicit as follows. Let K denote the field $\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ corresponding to the choice of G, and denote by $G_{r,s}(K)$ the Grassmann manifold of r-planes in the (right) vector space K^{r+s}. Thus $G_{r,\infty}(K)=BG(r)$ is the classifying space of $G(r)$, and $G_{r,s}(K)$ approximates $BG(r)$ as $s\to\infty$. In addition to the standard r-plane bundle $\xi_{r,s}$ over $G_{r,s}(K)$, there is an s-plane bundle $\eta_{r,s}$, with $\xi\oplus\eta$ trivial. Let $e_i=e_i(\xi)\in H^{di}(G_{r,s}(K))$ and $\bar{e}_i=e_i(\eta)$ be the characteristic classes of ξ and η. The Whitney sum formula implies that

$$\sum_{i+j=k} e_i\bar{e}_j = 0 \quad (0 < k \leqq r+s).$$

In fact, these generate all relations on the e_i and $\bar{e}_j$ (see [2], [3]):

(2.1) $H^*(G_{r,s}(K))$ is generated by $e_1,\dots,e_r$ and $\bar{e}_1,\dots,\bar{e}_s$ subject to the above relations.

Moreover, $H^*(G_{r,s}(K))$ is generated by $e_1,\dots,e_r$ since the $\bar{e}_j$ can be expressed in terms of the e_i. $H^*(G_{r,s}(K))$ is a polynomial algebra on $e_1,\dots,e_r$ up to dimension ds.

Several points of cobordism theory will now be reviewed; greater detail may be found in [5], [6], [7]. We maintain the convention throughout that $d=1$, 2 or 4 according as $G=\boldsymbol{O}$, $\boldsymbol{U}$ or $\boldsymbol{Sp}$. For (X, A) a finite CW-pair,

$$H^n(X, A; MG) = [S^{dk-n}(X/A), MG(k)]$$

for k large. Let $\xi\to X$ be a $G(n)$-bundle over a finite CW-complex. There is a classifying map $X\to BG(n)$, which gives rise to a map $M(\xi)\to MG(n)$ of Thom spaces, and so to an element u_ξ of $\bar{H}^{dn}(M(\xi); MG)$, the G-cobordism Thom class of ξ. It follows from Dold's theorem [8] that there is a Thom isomorphism

$$\phi\colon H^k(X; MG)\to \bar{H}^{k+dn}(M(\xi); MG)$$

given by $\phi(x)=u_\xi\cdot x$.

There is a natural transformation $\mu\colon H^*(\quad; MG)\to H^*(\quad)$ to cohomology, mod 2 in the unoriented case and integral otherwise. Various forms of the following result may be found in [5], [6, §18].

(2.2) Let (X, A) be a finite CW-pair, with $H^*(X, A)$ free abelian if $G=\boldsymbol{U}$ or $\boldsymbol{Sp}$. Suppose that $K^*\subset H^*(X, A; MG)$ is mapped isomorphically onto $H^*(X, A)$ by μ. Then $\Omega_G^*\otimes K^*\to H^*(X, A; MG)$ is an isomorphism of Ω_G^*-modules.

Here $\Omega_G^*=H^*(pt; MG)$ is the coefficient ring, and the conclusion states that $H^*(X, A; MG)$ is a free Ω_G^*-module.

Let $M^*=\sum_{i\leqq 0} M^i$ and $N^*=\sum_{j\geqq 0} N^j$ be graded abelian groups. The *completed tensor product* of M^* and N^* is the graded group $M^*\mathbin{\hat{\otimes}} N^*=\sum_{k=-\infty}^{\infty}(M^*\mathbin{\hat{\otimes}} N^*)^k$, with

$$(M^*\mathbin{\hat{\otimes}} N^*)^k = \prod_{i+j=k} M^i\otimes N^j.$$

Thus if $M^i=0$ for $i<i_0$ or $N^j=0$ for $j>j_0$, then $M^* \hat{\otimes} N^*$ is simply the graded tensor product $M^* \otimes N^*$. Elements of $(M^* \hat{\otimes} N^*)^k$ will be written as infinite sums $\sum m_i \otimes n_i$, with the understanding that the degrees of the m_i tend to $+\infty$ (and those of the n_i to $-\infty$). The following easily proved result will be used in §4.

(2.3) Let $M^*=\sum_{i\leqq 0} M^i$ and $N^*=\sum_{j\geqq 0} N^j$ be graded abelian groups, with the N^* forming an inverse system. If all the M^i are free abelian of finite rank, then

$$\operatorname{lim\,inv} (M^* \hat{\otimes} N^*_\alpha) \approx M^* \hat{\otimes} (\operatorname{lim\,inv} N^*_\alpha).$$

3. **Cobordism operations and characteristic classes.** Throughout this section G will denote a fixed stable group such as $\boldsymbol{O}$, $\boldsymbol{U}$, $\boldsymbol{Sp}$ or one of the other groups giving rise to a cobordism theory. Thus there is a Thom spectrum MG and the notion of G-bundle.

By a *G-cobordism operation* θ of degree i is meant a family of linear mappings

$$\theta\colon H^k(X, A; MG) \to H^{k+i}(X, A; MG),$$

defined for all CW-pairs, which commutes with induced maps and suspensions. The totality of these operations is denoted $\mathscr{A}^i(G)$. Thus $\mathscr{A}^*(G)=\sum_{i=-\infty}^{\infty} \mathscr{A}^i(G)$ is a graded algebra with addition "pointwise" and multiplication given by the composition of operations.

Since $H^*(X, A; MG)$ is a left graded module over the coefficient ring

$$\Omega_G^* = H^*(pt; MG),$$

Ω_G^* may be regarded as a subalgebra of $\mathscr{A}^*(G)$.

A *G-cobordism characteristic class* γ of degree i is a functor which assigns to each G-bundle $\xi \to X$ over a finite CW-complex a cobordism class $\gamma(\xi) \in H^i(X; MG)$, and which is natural with respect to G-bundle maps and is stable: $\gamma(\xi \oplus 1)=\gamma(\xi)$. Let $\mathscr{C}^i(G)$ denote these characteristic classes, and put $\mathscr{C}^*(G)=\sum_{i=-\infty}^{\infty} \mathscr{C}^i(G)$. Thus $\mathscr{C}^*(G)$ is a graded algebra with pointwise operations. For $n \in \Omega_G^i$,

$$\xi \to n\cdot 1 \in H^i(X; MG)$$

is an element of $\mathscr{C}^i(G)$, so Ω_G^* is trivially a subalgebra of $\mathscr{C}^*(G)$.

The rest of this section is devoted to showing that $\mathscr{A}^*(G)$ and $\mathscr{C}^*(G)$ are isomorphic additively. Let $\Psi\colon \mathscr{A}^*(G) \to \mathscr{C}^*(G)$ be defined in the following familiar manner. For $\theta \in \mathscr{A}^*(G)$ and $\xi \to X$ a G-bundle, put

$$\Psi(\theta)(\xi) = \phi^{-1}\,\theta\phi(1),$$

where ϕ is the G-cobordism Thom isomorphism of ξ.

(3.1) THEOREM. *$\Psi\colon \mathscr{A}^*(G) \to \mathscr{C}^*(G)$ is an isomorphism of graded abelian groups.*

Proof. It suffices to define an inverse Φ for Ψ. Let $\gamma \in \mathscr{C}^*(G)$ be given. If u_ξ is the G-cobordism Thom class of a G-bundle ξ, put $\Phi(\gamma)\cdot u_\xi=\phi(\gamma(\xi))$, to assure that Φ be an inverse for Ψ. If now $\alpha \in H^n(X, A; MG)$ is any cobordism class, represent α by a map $f\colon S^{dk-n}(X/A) \to MG(k)$, and assume $MG(k)$ replaced by a finite approximation. Thus the suspension $\sigma^{dk-n}(\alpha)$ equals $f^*(u_k)$, with u_k the G-cobordism Thom class of the (finite) universal G-bundle over $BG(k)$. Then put

$$\Phi(\gamma)\cdot\alpha = (\sigma^{dk-n})^{-1}f^*(\Phi(\gamma)\cdot u_k).$$

It is easily checked that Φ is well defined, linear and degree preserving, and it was constructed to be an inverse for Ψ. This completes the proof.

More generally, if $\mathscr{H}^*$ is a generalized cohomology theory with a functorial Thom isomorphism for G-bundles, then there is an analogous correspondence between the stable operations $H^*(\ \ ; MG)\to\mathscr{H}^*(\ \ )$ and the stable $\mathscr{H}^*$-characteristic classes of G-bundles. Notice that a G-cobordism operation is determined by its values on the Thom classes of G-bundles, in fact by its values on the Thom classes of the finite universal G-bundles.

4. **Construction of characteristic classes.** Let $G=\boldsymbol{O}$, $\boldsymbol{U}$ or $\boldsymbol{Sp}$ and $d=1$, 2 or 4 accordingly. For $\xi\to X$ a $G(1)$-bundle over a finite complex, define $E_1(\xi)$ in $H^d(X; MG)$ as follows. Choose a classifying map $X\to BG(1)$ and compose it with the inclusion $BG(1)\subset MG(1)$ to obtain a map $X\to MG(1)$, which gives rise to an element $E_1(\xi)$ of $H^d(X; MG)$. It is easily seen that $\mu(E_1(\xi))=e_1(\xi)\in H^d(X)$, with μ the natural transformation from G-cobordism to cohomology. Thus the hypotheses of [7, (7.5)] and its analogues for the real and complex case are satisfied, so we obtain

(4.1) THEOREM. *For $G=\boldsymbol{O}$, $\boldsymbol{U}$ or $\boldsymbol{Sp}$, there are unique G-cobordism classes $E_i\in\mathscr{C}^{di}(G)$ with $E_0=1$ satisfying*

(1) $E_k(\xi\oplus\eta)=\sum_{i+j=k}E_i(\xi)E_j(\eta)$;

(2) *for ξ a $G(1)$-bundle, $E_1(\xi)$ is defined as above and $E_i(\xi)=0$, $i>1$.*

Moreover, $\mu(E_i(\xi))=e_i(\xi)\in H^{4i}(X)$ for $\xi\to X$ a G-bundle over a finite CW-complex.

Let $C^*(G)\subset\mathscr{C}^*(G)$ be the subalgebra generated by the E_i. Applying μ, we see that $C^*(G)$ is a polynomial algebra on the E_i. Recall the definition of the completed tensor product (§2).

(4.2) THEOREM. *$\mathscr{C}^*(G)\approx\Omega_G^*\mathbin{\hat\otimes}C^*(G)$ additively.*

Proof. Since Ω_G^* and $C^*=C^*(G)$ are contained in $\mathscr{C}^*=\mathscr{C}^*(G)$, there is a map $\rho\colon\Omega_G^*\mathbin{\hat\otimes}C^*\to\mathscr{C}^*$ defined on the completed tensor product, given by

$$\rho\Big(\sum n_i\otimes c_i\Big) = \sum n_ic_i;$$

these are infinite sums with the convention (§2) that the degrees of the c_i tend to $+\infty$. In fact, if ξ is a G-bundle over a finite complex then $c_i(\xi)=0$ for all but a finite number of the c_i, so $(\sum n_ic_i)(\xi)=\sum n_i\cdot c_i(\xi)$ is essentially a finite sum.

There is a map $\sigma\colon\mathscr{C}^*\to\lim\operatorname{inv}H^*(G_{r,s}; MG)$ $(r, s\to\infty)$, which associates to $\gamma\in\mathscr{C}^*$ the element $(\gamma(\xi_{r,s}))$ of the inverse limit; here $\xi_{r,s}\to G_{r,s}(K)$ is the universal r-plane bundle with $K=\boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{H}$ depending on G. Since the $\xi_{r,s}$ are approximations to universal bundles, it is clear that *σ is an isomorphism.*

In $H^*(G_{r,s}; MG)$, let $K_{r,s}$ be the subalgebra generated by the cobordism characteristic classes $E_1,\dots,E_r$ of the bundle $\xi_{r,s}$. According to (2.1), $H^*(G_{r,s})$ is generated by the cohomology characteristic classes $e_1, \ldots, e_r$ of $\xi_{r,s}$, and all the relations on the generators are consequences of relations expressed by the Whitney sum formula applied to the Whitney sum $\xi \oplus \eta$. According to (4.1), these same relations hold among the E_i. Thus the assignment $e_i \to E_i$ extends to an algebra homomorphism ν: $H^*(G_{r,s}) \to H^*(G_{r,s}; MG)$ with $\mu \circ \nu = $ identity and

$$\nu\{H^*(G_{r,s})\} = K_{r,s}.$$

So by (2.2), the natural map $\Omega_G^* \otimes K_{r,s} \to H^*(G_{r,s}; MG)$ is an isomorphism.

By virtue of the naturality of the E_i, for $r \leqq r'$ and $s \leqq s'$ the induced map

$$H^*(G_{r',s'}; MG) \to H^*(G_{r,s}; MG)$$

restricts to a map $K_{r',s'} \to K_{r,s}$. Thus the graded modules $K_{r,s}$ form an inverse system, and the homomorphisms $C^* \to K_{r,s}$ sending E_i to $E_i(\xi_{r,s})$ give rise to a homomorphism $C^* \to \lim\operatorname{inv} K_{r,s}$. It follows from (2.1) that this is an isomorphism. We now make use of (2.3) to obtain an isomorphism of $\Omega_G^* \hat{\otimes} C^*$ with

$$\lim\operatorname{inv} H^*(G_{r,s}; MG),$$

which is just the composite $\sigma \circ \rho$. Hence ρ is an isomorphism, as was to be shown.

5. **The Hopf algebra** $A^*(G)$. Under the additive isomorphism Ψ of $\mathscr{A}^*(G)$ with $\mathscr{C}^*(G)$, there corresponds to $C^*(G)$ a graded submodule $A^*(G) = \sum_{i \geqq 0} A^i$ of $\mathscr{A}^*(G)$. The elements of $A^*(G)$ will be called *basic G-cobordism operations*. Theorem (4.2) has an immediate corollary.

(5.1) Theorem. *$\mathscr{A}^*(G) \approx \Omega_G^* \hat{\otimes} A^*(G)$ additively.*

Under this isomorphism $\sum n_i \otimes \theta_i$ corresponds to $\sum n_i\theta_i$, with the usual convention concerning infinite sums. In fact, if $\alpha \in H^*(X, A; MG)$ is a cobordism class of a finite CW-pair, then $(\sum n_i\theta_i)\alpha = \sum n_i(\theta_i\alpha)$ is essentially a finite sum since $\theta_i\alpha = 0$ for almost all i.

A basis for $A^*(G)$ is obtained as follows. For $\omega = (i_1, \ldots, i_r)$ a partition with $i_1 + \cdots + i_r = i$, let $S_\omega = S_\omega(E)$ be the S_ω-symmetric function (or symmetrized monomial; see [9, §1]) of the E_i. Thus $S_\omega \in C^{di}(G)$, and as ω runs through all partitions we obtain a basis for $C^*(G)$. The Whitney sum formula for these characteristic classes is

$$S_\omega(\xi \oplus \eta) = \sum_{\omega_1\omega_2 = \omega} S_{\omega_1}(\xi)S_{\omega_2}(\eta).$$

We shall *identify* $A^*(G)$ and $C^*(G)$ additively, and so $\{S_\omega\}$ is a basis for $A^*(G)$.

In order to determine the action on products of cobordism classes, we define a *coproduct* $\psi^*: A^* \to A^* \otimes A^*$ by

$$\psi^*(S_\omega) = \sum_{\omega_1\omega_1 = \omega} S_{\omega_1} \otimes S_{\omega_2};$$

as a map $C^* \to C^* \otimes C^*$ this is the algebra homomorphism sending E_k to

$$\sum_{i+j=k} E_i \otimes E_j.$$

Let $A^* \otimes A^*$ act on $H^*(X, A; MG) \otimes H^*(Y, B; MG)$ with values in

$$H^*(X \times Y, X \times B \cup A \times Y; MG)$$

by the formula

$$\left(\sum \theta_i' \otimes \theta_i''\right)(\alpha \otimes \beta) = \sum (\theta_i'\alpha)(\theta_i''\beta).$$

(5.2) For $\theta \in A^*$ and α, β G-cobordism classes, $\theta(\alpha \cdot \beta) = \psi^*(\theta) \cdot (\alpha \otimes \beta)$.

Proof. It is sufficient to verify this for $\theta = S_\omega$ and $\alpha = u_\xi$, $\beta = u_\eta$ cobordism Thom classes of G-bundles $\xi \to X$ and $\eta \to Y$ over finite CW-complexes. Let

$$\xi \times \eta \to X \times Y$$

be the Cartesian product of ξ and η; this is a G-bundle with Thom space

$$M(\xi \times \eta) = M(\xi) \wedge M(\eta)$$

and G-cobordism Thom class $u_{\xi \times \eta} = u_\xi \cdot u_\eta$ (see [6, §11]). One may now easily check that $S_\omega(u_\xi \cdot u_\eta) = \psi^*(S_\omega) \cdot (u_\xi \otimes u_\eta)$ to complete the proof.

In particular, for X a finite complex $H^*(X; MG)$ is a graded *algebra over the co-algebras* (A^*, ψ^*), i.e. (5.2) holds for all α and β in $H^*(X; MG)$; see [13]. By an algebra over A^* we shall mean, throughout this section, an algebra over the co-algebra (A^*, ψ^*).

The rest of this section is devoted to showing that $A^*(G)$ is a subalgebra of $\mathscr{A}^*(G)$, and that it is in fact a Hopf algebra.

(5.3) THEOREM. *$A^*(G)$ is a subalgebra of $\mathscr{A}^*(G)$.*

Assuming this proved, let $\phi^*: A^* \otimes A^* \to A^*$ denote the product homomorphism. It will be left for the reader to convince himself that the coproduct

$$\psi^*: A^* \to A^* \otimes A^*$$

is an algebra homomorphism, i.e., to check the formula

$$\psi^*(S_\omega \circ S_{\omega'})(\alpha \otimes \beta) = (\psi^* S_\omega \cdot \psi^* S_{\omega'})(\alpha \otimes \beta)$$

for every pair of cobordism classes α, β. Thus we obtain

(5.4) THEOREM. *$(A^*(G), \phi^*, \psi^*)$ is a graded connected Hopf algebra with commutative coproduct. The ground ring is Z_2 in the unoriented case and the integers if $G = U$ or Sp.*

Before turning to the proof of (5.3), we describe the action of $A^*(G)$ on

$$H^*(P^n(K); MG);$$

this information will also be used in the following section. Let $\xi \to P^n(K)$ be the Hopf $G(1)$-bundle, and put $\alpha = E_1(\xi) \in H^d(P^n(K); MG)$. It follows from (4.1) and (2.2) that as an Ω_G^*-module, $H^*(P^n(K); MG)$ is free on the basis $\{1, \alpha, \ldots, \alpha^n\}$ and that $\alpha^{n+1} = 0$. Thus it will suffice to determine the action of S_ω on α. It is first of all immediate that:

(5.5) If ξ is a $G(1)$-bundle with characteristic class $E_1(\xi)$, then the G-cobordism classes $S_\omega(\xi)$ are given by

$$S_k(\xi) = (E_1(\xi))^k, \qquad S_\omega(\xi) = 0 \quad \text{otherwise}.$$

(5.6) LEMMA. *Let $\alpha \in H^d(P^n(K); MG)$ be the generator. The action of the basic cobordism operations S_ω on α is*

$$S_k\alpha = \alpha^{k+1}, \qquad S_\omega\alpha = 0 \quad \textit{otherwise}.$$

Proof. It is well known that the Thom space of the Hopf $G(1)$-bundle $\xi \to P^n(K)$ may be identified with $P^{n+1}(K)$, and that the inclusion of $P^n(K)$ in this Thom space is the usual inclusion $P^n(K) \subset P^{n+1}(K)$. Thus the Thom class $u \in \bar{H}^d(P^{n+1}(K); MG)$ of ξ is carried onto $\alpha = E_1(\xi)$ in $H^d(P^n(K); MG)$. The conclusion now follows directly from (5.5), in view of the isomorphism of §3.

(5.7) COROLLARY. *Let ξ be a $G(1)$-bundle. Then $S_k \cdot E_1(\xi) = (E_1(\xi))^{k+1}$ and $S_\omega \cdot E_1(\xi) = 0$ otherwise.*

Proof of (5.3). Let θ, $\theta' \in A^*$ and put $\psi^*(\theta') = \sum \theta_i' \otimes \theta_i''$. Let γ and γ_i' in C^* correspond to θ and θ_i' under the isomorphism $\Psi\colon C^* \approx A^*$. If u_ξ is the G-cobordism Thom class of a G-bundle $\xi \to X$ and $\pi\colon D(\xi) \to X$ the associated disk bundle, then $\theta u_\xi = u_\xi \cdot \pi^*(\gamma(\xi))$, and a computation yields

$$(\theta' \circ \theta)u_\xi = u_\xi \cdot \pi^*\Big\{\sum \gamma_i'(\xi) \cdot \theta_i''\gamma(\xi)\Big\}.$$

Thus in order to show that A^* is closed under composition it suffices to prove the following statement.

(5.8) If $\theta \in A^*(G)$ and $\gamma \in C^*(G)$, then there exists $\gamma' \in C^*(G)$ so that for all G-bundles ξ, $\theta \cdot \gamma(\xi) = \gamma'(\xi)$.

Moreover, in view of the splitting principle and (2.2) it is enough to prove (5.8) under the assumption that ξ splits into a Whitney sum of $G(1)$-bundles.

Let $R = \boldsymbol{Z}$ if $G = \boldsymbol{U}$ or $\boldsymbol{Sp}$ and $\boldsymbol{Z}_2$ if $G = \boldsymbol{O}$. Consider a graded polynomial algebra $R[\alpha_1, \ldots, \alpha_n]$ on generators of degree d, and identify the G-cobordism classes $E_1, \ldots, E_n$ with the elementary symmetric functions of the α_i. Make $R[\alpha_1, \ldots, \alpha_n]$ an algebra over (the coalgebra) $A^*(G)$ by requiring that for $i = 1, \ldots, n$

$$S_k \cdot \alpha_i = \alpha_i^{k+1}, \qquad S_\omega \cdot \alpha_i = 0 \quad \text{otherwise};$$

together with the condition $\theta(\beta_1 \cdot \beta_2) = \psi^*\theta \cdot (\beta_1 \otimes \beta_2)$ for $\theta \in A^*$ and β_1, β_2 in

$$R[\alpha_1, \ldots, \alpha_n],$$

this specifies the action of A^*. It is obvious that if β is a symmetric polynomial in the α_i, so is $S_\omega\beta$. Thus the subalgebra $R[E_1, \ldots, E_n]$ is also an algebra over A^*.

Now let $\xi = \xi_1 \oplus \cdots \oplus \xi_n$ be a Whitney sum of $G(1)$-bundles over a finite CW-complex X. We define an algebra homomorphism $f_\xi\colon R[\alpha_1, \ldots, \alpha_n] \to H^*(X; MG)$ by setting $f_\xi(\alpha_i) = E_1(\xi_i)$. In virtue of (5.7), f_ξ is a homomorphism of algebras over A^*; so is the restriction $f_\xi\colon R[E_1, \ldots, E_n] \to H^*(X; MG)$, and $f_\xi(E_i) = E_i(\xi)$. It follows that $f_\xi(\gamma) = \gamma(\xi)$ for each $\gamma \in R[E_1, \ldots, E_n]$. Hence

$$(\theta \cdot \gamma)(\xi) = \theta \cdot \gamma(\xi)$$

for $\theta \in A^*$ and $\gamma \in R[E_1, \ldots, E_n]$.

It remains to be shown that the actions of A^* on $R[E_1, \ldots, E_n]$ combine to make $C^*=R[E_1, E_2, \ldots]$ an algebra over A^*. For $n \leqq m$, the homomorphism

$$R[\alpha_1, \ldots, \alpha_m] \to R[\alpha_1, \ldots, \alpha_n]$$

sending $\alpha_i \to \alpha_i$ $(i \leqq n)$ and $\alpha_i \to 0$ $(i > n)$ restricts to a homomorphism

$$R[E_1, \ldots, E_m] \to R[E_1, \ldots, E_n]$$

of algebras over A^*, sending $E_i \to E_i$ $(i \leqq n)$, $E_i \to 0$ $(i > n)$. Taking the inverse limit, we see that C^* becomes an algebra over the co-algebra A^*.

Now let $\theta \in A^*$ and $\gamma \in C^*$. For ξ a Whitney sum of n $G(1)$-bundles let

$$\gamma_1 \in Z_2[E_1, \ldots, E_n]$$

be the projection of γ. Then also $\theta\gamma_1$ is the projection of $\theta\gamma$, and so

$$(\theta \cdot \gamma)(\xi) = (\theta \cdot \gamma_1)(\xi) = \theta \cdot \gamma_1(\xi) = \theta \cdot \gamma(\xi).$$

Thus (5.8) is proved for ξ a Whitney sum of $G(1)$-bundles, which completes the proof of Theorem (5.3).

6. **Computation of products in $A^*(G)$.** It was shown in the previous section that $(A^*(G), \phi^*, \psi^*)$ is a graded connected Hopf algebra with commutative coproduct. The product ϕ^* is given by the composition of basic G-cobordism operations, and ψ^* is given in terms of the basis $\{S_\omega\}$ by

$$\psi^* S_\omega = \sum_{\omega_1 \omega_2 = \omega} S_{\omega_1} \otimes S_{\omega_2}.$$

The product in $A^*(G)$ is complicated, but with the help of the dual Hopf algebra $A_*(G)$ and the methods of [11] we can compute products in $A^*(G)$.

Let (A_*, ψ_*, ϕ_*) denote the Hopf algebra dual to $A^*=A^*(G)$. Thus $A_*=\sum_{i \geqq 0} A_i$ and $A_i = \mathrm{Hom}\,(A^i, R)$, with $R=\boldsymbol{Z}$ for $G=\boldsymbol{U}$ or $\boldsymbol{Sp}$ and $R=\boldsymbol{Z}_2$ for $G=\boldsymbol{O}$. Let $\{\sigma_\omega\}$ be the basis of A_* dual to the basis $\{S_\omega\}$ of A^*. It is clear that $\psi_*(\sigma_\omega \otimes \sigma_{\omega'})=\sigma_{\omega\omega'}$, and so A_* is a polynomial algebra $R[\sigma_1, \sigma_2, \ldots]$ on generators σ_k of degree dk. The coproduct $\phi_*\colon A_* \to A_* \otimes A_*$ is an algebra homomorphism, so is determined by its values on the algebra generators σ_k. We shall obtain a formula for $\phi_*(\sigma_k)$.

In order to state this formula, we introduce some notation. If $\omega=(i_1, \ldots, i_r)$ is a partition, let r_p denote the number of occurrences of the integer p in ω and put $R=(r_1, r_2, \ldots)$. Thus there is a correspondence $\omega \leftrightarrow R$ between partitions and finitely nonzero sequences of nonnegative integers. We write $S_\omega = S^R$ and $\sigma_\omega=\sigma^R$ if $\omega \leftrightarrow R$. Sequences are added component-wise; denote by Δ_k the sequence

$$(r_1, r_2, \ldots)$$

with $r_k=1$ and $r_p=0$ for $p \neq k$. In terms of the S^R, the coproduct of A^* is given by

$$\psi^* S_R = \sum_{R_1+R_2=R} S^{R_1} \otimes S^{R_2}.$$

We make the definitions

$$\|R\| = \sum pr_p, \qquad |R| = \sum r_p,$$

$$\binom{m}{R} = \frac{m!}{r_1!\, r_2! \cdots (m-|R|)!} \qquad (|R| \leqq m).$$

Thus S^R has degree $d\|R\|$ and $|R|$ is the number of terms of the partition which corresponds to the sequence R.

(6.1) THEOREM.

$$\phi_*(\sigma_k) = \sum \binom{m+1}{R} \sigma_R \otimes \sigma_m$$

over all pairs (R, m) *with* $\|R\|+m=k$ *and* $|R| \leqq m+1$.

For example, we have

$$\phi_*\sigma_1 = \sigma_1 \otimes 1 + 1 \otimes \sigma_1,$$
$$\phi_*\sigma_2 = \sigma_2 \otimes 1 + 2\sigma_1 \otimes \sigma_1 + 1 \otimes \sigma_2,$$

hence

$$\phi_*(\sigma_1)^2 = (\sigma_1)^2 \otimes 1 + 2\sigma_1 \otimes \sigma_1 + 1 \otimes (\sigma_1)^2.$$

Thus σ_1 and $\sigma_2 - (\sigma_1)^2$ are primitive elements of A_* (see §7).

The proof of (6.1) will be preceeded by several lemmas. Let $M^* = \sum_{i \geqq 0} M^i$ be a graded algebra over the Hopf algebra A^*, with each M^i free and finitely generated over R $(=Z$ or $Z_2)$. The action of A^* on M^* induces an action on the graded dual M_* of M^*. Hence there is a homomorphism $M_* \otimes A^* \to M_*$, which we dualize to obtain a homomorphism

$$\lambda\colon M^* \to M^* \otimes A_*.$$

The following results may be proved as in [11].

(6.2) LEMMA. *For* $\alpha \in M^*$ *we have* $(\lambda \otimes 1)\lambda(\alpha) = (1 \otimes \phi_*)\lambda(\alpha)$ *in* $M^* \otimes A_* \otimes A_*$.

(6.3) LEMMA. $\lambda\colon M^* \to M^* \otimes A_*$ *is an algebra homomorphism.*

(6.4) LEMMA. If $\lambda(\alpha) = \sum \alpha_i \otimes \tau_i$ $(\alpha, \alpha_i \in M^*,\ \tau_i \in A_*)$, *then for any* $\theta \in A^*$

$$\theta \cdot \alpha = \sum \langle \theta, \tau_i \rangle \alpha_i.$$

Proof of (6.1). Let $\alpha \in H^d(P^n(K); MG)$ be the canonical generator of

$$H^*(P^n(K); MG),$$

as in §5. Let $M^* \subset H^*(P^n(K); MG)$ be the submodule with basis $1, \alpha, \ldots, \alpha^n$. By (5.6), M^* is an algebra over the Hopf algebra A^*, and with (6.4) this gives

$$\lambda\alpha = \alpha \otimes 1 + \alpha^2 \otimes \sigma_1 + \cdots + \alpha^n \otimes \sigma_{n-1}.$$

Hence

$$\begin{aligned}(\lambda \otimes 1)\lambda\alpha &= \sum_m (\lambda\alpha)^{m+1} \otimes \sigma_m \\ &= \sum_m \left\{ \sum_{|R| \leqq m+1} \binom{m+1}{R} \alpha^{\|R\|+m+1} \otimes \sigma^R \right\} \otimes \sigma_m \\ &= \sum_k \alpha^{k-1} \otimes \left\{ \sum \binom{m+1}{R} \sigma^R \otimes \sigma_m \right\},\end{aligned}$$

the final sum over all pairs (R, m) with $\|R\| + m = k$ and $|R| \leqq m+1$. On the other hand,

$$(1 \otimes \phi_*)\lambda\alpha = \sum_k \alpha^{k+1} \otimes \phi_*\sigma_k,$$

and so for $k < n$ we obtain the desired formula for $\phi_*(\sigma_k)$. Letting $n \to \infty$, the formula holds for all k and the proof is complete.

Notice that $A^*(\boldsymbol{U})$ and $A^*(\boldsymbol{Sp})$ are therefore isomorphic Hopf algebras, although the degrees are doubled. Similarly, $A^*(\boldsymbol{O})$ and $A^*(\boldsymbol{U}) \otimes Z_2$ are isomorphic Hopf algebras, doubling degrees. In the unoriented case, the binomial coefficients in (6.1) are to be taken mod 2.

We shall have occasion to use the following computations:

$$(6.5)\qquad \begin{aligned} S_n \circ S_n &= (n+1)S_{2n} + 2S_{n,n}, \\ S_m \circ S_n &= (n+1)S_{m+n} + S_{m,n} \qquad (m \neq n), \\ S_2 \circ S_{2n,2n} &\equiv S_{2,2n,2n} + S_{2n,2n+2} \qquad (\text{mod } 2), \\ S_{2n,2n} \circ S_2 &\equiv S_{2,2n,2n} + S_{2n,2n+2} + S_{4n+2} \qquad (\text{mod } 2). \end{aligned}$$

For example, consider $S_2 \circ S_{2n,2n}$. We write $S_2 \circ S_{2n,2n} = \sum_\omega a_\omega S_\omega$ and must determine the coefficients a_ω. Notice that

$$a_\omega = \langle S_2 \otimes S_{2n,2n}, \phi_*\sigma_\omega \rangle.$$

Now $\phi_*\sigma_{2,2n,2n} = (\phi_*\sigma_2)(\phi_*\sigma_{2n})^2$ contains the term

$$(\sigma_2 \otimes 1)(1 \otimes \sigma_{2n})^2 = \sigma_2 \otimes \sigma_{2n,2n}$$

once if $n > 1$ and three times if $n = 1$; thus $a_{2,2n,2n} \equiv 1$ (mod 2). Also $\phi_*\sigma_{2n,2n+2} = (\phi_*\sigma_{2n})(\phi_*\sigma_{2n+2})$ contains the term $(1 \otimes \sigma_{2n})(\sigma_2 \otimes \sigma_{2n}) = \sigma_2 \otimes \sigma_{2n,2n}$ an odd number of times, namely $2n+1$, so $a_{2n,2n+2} \equiv 1$ (mod 2). It may be seen that $a_\omega = 0$ for any other ω, so $S_2 \circ S_{2n,2n}$ is as stated.

7. Generators of $A^*(G) \otimes Z_p$. Let $A^* = A^*(G)$ for $G = \boldsymbol{U}$ or $\boldsymbol{Sp}$, and let p be a prime. Thus $A^* \otimes Z_p$ is a Hopf algebra over the field Z_p, isomorphic to $A^*(\boldsymbol{O})$ if $p = 2$. In this section we shall obtain a minimal set of generators of the algebra $A^* \otimes Z_p$, selected from the basis $\{S^R\} = \{S_\omega\}$.

(7.1) Theorem. *A minimal set of generators of $A^* \otimes Z_p$ is provided by*

$$\{S^{p^k\Delta_1}, S^{p^k\Delta_2}\}_{k \geqq 0}.$$

We remark that if p is odd, another minimal set of generators is $\{S^{p^k\Delta_1}, S^{2p^k\Delta_1}\}_{k\geqq 0}$. If $p=2$, then among the elements of the basis $\{S^R\}$ only the members of the minimal generating set $\{S^{2^k\Delta_1}, S^{2^k\Delta_2}\}_{k\geqq 0}$ are indecomposable; all others are decomposable in $A^* \otimes Z_2$.

For the rest of this section we shall fix a prime p and write A^* for $A^* \otimes Z_p$. Thus A^* is regarded as a Hopf algebra over Z_p.

An element τ of the dual Hopf algebra A_* is called *primitive* if

$$\phi_*(\tau) = \tau \otimes 1 + 1 \otimes \tau.$$

For example, it was shown after the statement of (6.1) that σ_1 and $\sigma_2 - (\sigma_1)^2$ are primitive. We denote by $P(A_*)$ the space of primitive elements of A_*. If A^*_+ denotes the ideal of elements of positive degree in A^*, then $A^*_+ \cdot A^*_+$ is the ideal of *decomposable* elements of A^*. $Q(A^*) = A^*_+ / A^*_+ \cdot A^*_+$ is called the space of *indecomposable* elements of A^*. The following results are proved in [13] and [14] respectively.

(7.2) Lemma. *$A^*_+ \cdot A^*_+$ is the annihilator of $P(A_*)$ in A^*_+, hence $P(A_*)$ is canonically isomorphic to the dual of $Q(A^*)$.*

(7.3) Lemma. *Any set of algebra generators of A^* contains a subset whose image in $Q(A^*)$ is a vector space basis; such a subset is minimal and generates A^*.*

As was just mentioned, σ_1 and $\sigma_2 - (\sigma_1)^2$ are primitive elements of A_*. It follows that for all $k \geqq 0$, the powers $(\sigma_1)^{p^k}$ and $(\sigma_2 - (\sigma_1)^2)^{p^k} = (\sigma_2)^{p^k} - (\sigma_1)^{2p^k}$ are also primitive. In fact, we shall prove

(7.4) Lemma. *$P(A_*)$ has as a basis the set $\{(\sigma_1)^{p^k}, (\sigma_2)^{p^k} - (\sigma_1)^{2p^k}\}_{k\geqq 1}$.*

Theorem (7.1) *follows directly from these lemmas.* In fact, the image of the set $\{S^{p^k\Delta_1}, S^{p^k\Delta_2}\}_{k\geqq 0}$ in $Q(A^*)$ is a vector space basis by virtue of (7.2) and (7.4), so this set is a minimal set of generators of A^* according to (7.3). Notice that if $p=2$, $P(A_*)$ also has as basis the set $\{(\sigma_1)^{2^k}, (\sigma_2)^{2^k}\}$. Thus the remarks following the statement of the theorem follow just as easily.

Thus it remains to prove (7.4). We first show that among the elements S_k of A^*, only S_1 and S_2 are indecomposable.

(7.5) Lemma. *S_k is decomposable if $k > 2$.*

Proof. To begin with, notice that the computations (6.5) imply that for $m \neq n$,

$$[S_m, S_n] = S_m S_n - S_n S_m = (n-m) S_{m+n}.$$

Suppose first that p is an odd prime. If $k = 2j > 2$, then $[S_{j-1}, S_{j+1}] = 2S_k$ so S_k is decomposable since $(2, p) = 1$. If $k = 2j+1 > 2$, then $[S_j, S_{j+1}] = S_k$ so S_k is again decomposable. Now take $k=2$ and consider three cases: k odd, $k \equiv 0$ (mod 4) and $k \equiv 2$ (mod 4). For $k = 2j+1 > 2$, $[S_j, S_{j+1}] = S_k$. For $k = 4j > 2$, it follows from (6.5) that $S_{2j}\, S_{2j} = S_k$ (mod 2). Finally for $k = 4j+2 > 0$, (6.5) implies that $[S_{2j,2j}, S_2] \equiv S_k$ (mod 2). Thus the lemma is proved in all cases.

We now borrow a result from [13]. Let $\xi\colon A_* \to A_*$ be defined by $\xi(a)=a^p$; this is a homomorphism of Hopf algebras which multiplies degrees by p. There is a natural map $\chi\colon P(A_*) \to Q(A_*)$, namely the composition

$$P(A_*) \to A_*^+ \to A_*^+/A_*^+\cdot A_*^+.$$

Since $A_*=Z_p[\sigma_1, \sigma_2, \ldots]$, $Q(A_*)$ has a basis $\{\bar{\sigma}_k\}$, where $\bar{\sigma}_k \in Q(A_*)$ denotes the image of $\sigma_k \in A_*^+$. We remark that $P(A_*)$ is closed under the map $\xi\colon A_* \to A_*$ and that $P(\xi A^*)=\xi P(A^*)$. As a consequence of [13, (4.21)] we have:

(7.6) LEMMA. *There is an exact sequence*

$$0 \longrightarrow \xi P(A_*) \longrightarrow P(A_*) \xrightarrow{\chi} Q(A_*).$$

We now study the map $\chi\colon P(A_*) \to Q(A_*)$. Recall that $Q(A_*)$ has a basis $\{\bar{\sigma}_k\}$; since σ_1 and $\sigma_2-(\sigma_1)^2$ lie in $P(A_*)$, $\bar{\sigma}_1$ and $\bar{\sigma}_2$ are in the image of χ.

(7.7) LEMMA. *For $k>2$ $\bar{\sigma}_k$ is not in the image of χ.*

Proof. Suppose $\bar{\sigma}_k \in Q(A_*)$ were in the image of χ. Then there would be an element $\tau \in P(A_*)$ of degree k with $\tau \equiv \sigma_k$ modulo decomposable elements of A_*. Now $S_k \in A^*$ is clearly a primitive element of A^*, and so $\langle S_k, \rho\rangle=0$ for all decomposable elements ρ of A_* by (7.2). Thus $\langle S_k, \tau\rangle=\langle S_k, \sigma_k\rangle=1$. But if $k>2$ (7.5) states that S_k is decomposable, and so by (7.2) again $\langle S_k, \tau\rangle=0$ since $\tau \in P(A_*)$. This is a contradiction, which completes the proof.

Proof of (7.4). We write $P=P(A_*)$ and let $P' \subset P$ be the subspace with σ_1 and $\sigma_2-(\sigma_1)^2$ as basis. It follows from (7.6) and (7.7) that $P=\xi P \oplus P'$. We may apply ξ repeatedly to obtain for $k \geqq 0$

$$P = \xi^{k+1} P \oplus (\xi^k P' \oplus \cdots \oplus \xi P' + P').$$

Since $\bigcap_k \xi^k P=(0)$, it follows that $P=\bigoplus_{k=0}^{\infty} \xi^k P'$. Now $\xi^k\sigma_1=(\sigma_1)^{p^k}$ and

$$\xi^k(\sigma_2-(\sigma_1)^2) = (\sigma_2)^{p^k}-(\sigma_1)^{2p^k},$$

hence the lemma, and with it Theorem (7.1), is proved.

8. **Subalgebras of** $A^*(G) \otimes Z_p$**.** In this section we shall show that the Hopf algebras $A^*(G) \otimes Z_p$, $G=\boldsymbol{U}$ or $\boldsymbol{Sp}$, and $A^*(\boldsymbol{O})$ contain as Hopf subalgebras suitable quotients of the mod p Steenrod algebras $\mathscr{S}^*(p)$. In addition, we determine the primitively generated subalgebras of $A^*(G) \otimes Z_p$ and $A^*(\boldsymbol{O})$.

(8.1) THEOREM. *There are degree-preserving homomorphisms of Hopf algebras $\Gamma\colon \mathscr{S}^*(p) \to A^*(G) \otimes Z_p$ for $G=\boldsymbol{U}$ or $\boldsymbol{Sp}$, and $\Gamma\colon \mathscr{S}^*(2) \to A^*(0)$, such that for $\theta \in \mathscr{S}^*(p)$ and $\theta' \in A^*(G)$ the diagrams*

$$\begin{array}{ccc} H^*(X, A; MG) & \xrightarrow{\theta'} & H^*(X, A; MG) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ H^*(X, A; Z_p) & \xrightarrow{\theta} & H^*(X, A; Z_p) \end{array}$$

are commutative for all finite CW-pairs if and only if $\Gamma(\theta)=\theta' \otimes 1$.

Let (β_p) denote the two-sided ideal generated by the Bockstein in $\mathscr{S}^*(p)$. Let $\xi\colon \mathscr{S}^*(2) \to \mathscr{S}^*(2)$ be the dual of the squaring map in the dual Hopf algebra $\mathscr{S}_*(2)$. Thus ξ and $\xi^2 = \xi \circ \xi$ are Hopf algebra homomorphisms of $\mathscr{S}^*(2)$ into itself, so their kernels are Hopf ideals in $\mathscr{S}^*(2)$ (see Liulevicius [10]). Notice that $\beta_2 = Sq^1$ and $(\beta_2) = \ker(\xi)$ in $\mathscr{S}^*(2)$, by Milnor [12].

(8.2) THEOREM. *The kernel of* Γ *for* $G = \boldsymbol{U}$ *or* $\boldsymbol{Sp}$ *and* p *an odd prime is exactly* (β_p). *For* $p = 2$ *and* $G = \boldsymbol{O}$, $\boldsymbol{U}$ *or* $\boldsymbol{Sp}$, *the kernel of* Γ *is* (0), $\ker(\xi)$ *or* $\ker(\xi^2)$ *respectively.*

It is convenient to introduce the stable operations from G-cobordism to cohomology. Let $R = \boldsymbol{Z}$ or $\boldsymbol{Z}_p$, p a prime, for $G = \boldsymbol{U}$ or $\boldsymbol{Sp}$, and $R = \boldsymbol{Z}_2$ if $G = \boldsymbol{O}$. We denote by $\mathscr{A}^i(G; R)$ the set of linear operations

$$\nu\colon H^k(X, A; MG) \to H^{k+i}(X, A; R)$$

defined for all finite CW-pairs, which commute with induced maps and suspensions. Under addition of operations, $\mathscr{A}^*(G; R) = \sum_{i \geq 0} \mathscr{A}^i(G; R)$ is a graded R-module. Using finite approximations to the Thom spaces $MG(n)$ as in §4, it is easily shown that $\mathscr{A}^*(G; R)$ is naturally isomorphic to $H^*(MG; R)$. Moreover, there is an isomorphism $\Psi\colon \mathscr{A}^*(G; R) \approx R[e_1, e_2, \ldots]$ entirely analogous to the isomorphism Ψ of §3 (see the concluding remarks of §3).

There is a linear degree-preserving map $\mu^L\colon A^*(G) \to \mathscr{A}^*(G; R)$ defined as follows. Let $\mu\colon H^*(\quad; MG) \to H^*(\quad; R)$ be the natural transformation. For $\theta \in A^*(G)$, put $\mu^L(\theta) = \mu \circ \theta$, composition with μ on the left. It is clear that there is a commutative diagram

$$\begin{array}{ccc} A^*(G) \otimes R & \xrightarrow{\mu^L \otimes 1} & \mathscr{A}^*(G; R) \\ \Big\downarrow {\scriptstyle \Psi \otimes 1} & & \Big\downarrow {\scriptstyle \Psi} \\ R[E_1, E_2, \ldots] & \xrightarrow{\mu} & R[e_1, e_2, \ldots] \end{array}$$

hence,

(8.3) LEMMA. $\mu^L \otimes 1\colon A^*(G) \otimes R \to \mathscr{A}^*(G; R)$ *is an additive isomorphism.*

In addition, there are linear maps $\mu^R\colon \mathscr{S}^*(p) \to \mathscr{A}^*(G; Z_p)$, defined by putting $\mu^R(\theta) = \theta \circ \mu$ for $\theta \in \mathscr{S}^*(p)$ ($p = 2$ if $G = \boldsymbol{O}$). In fact, there is a map

$$\mathscr{S}^*(p) \otimes \mathscr{A}^*(G; Z_p) \to \mathscr{A}^*(G; Z_p)$$

sending $\theta \otimes \nu$ to the composition $\theta \circ \nu$, thereby making $\mathscr{A}^*(G, Z_p)$ a graded module over $\mathscr{S}^*(p)$. Since $\mu \in \mathscr{A}^*(G; Z_p)$, the homomorphism μ^R is given by the action of $\mathscr{S}^*(p)$ on μ. Under the natural isomorphism of $\mathscr{A}^*(G; Z_p)$ with $H^*(MG; Z_p)$, μ corresponds to the "unit class" in $H^*(MG; Z_p)$, and the action (by composition) of $\mathscr{S}^*(p)$ on $\mathscr{A}^*(G; Z_p)$ is seen to correspond to the usual module structure of $H^*(MG; Z_p)$ over $\mathscr{S}^*(p)$.

Proof of (8.1). Let $\Gamma\colon \mathcal{S}^*(p) \to A^*(G) \otimes Z_p$ denote the composition

$$(\mu^L \otimes 1)^{-1} \circ \mu^R.$$

Thus Γ is linear and degree-preserving, and for $\theta \in \mathcal{S}^*(p)$ we have $(\mu^L \otimes 1)\Gamma(\theta) = \mu^R(\theta)$. In view of the definitions of μ^L and μ^R, for $\theta' \in A^*(G)$ we have $\Gamma(\theta) = \theta' \otimes 1$ if and only if $\mu \circ \theta' = \theta \circ \mu$.

To show that Γ is an algebra homomorphism, let $\theta_1, \theta_2 \in \mathcal{S}^*(p)$ with $\Gamma(\theta_1) = \theta_1' \otimes 1$ and $\Gamma(\theta_2) = \theta_2' \otimes 1$. Then $\mu \circ \theta_i' = \theta_i \circ \mu$ $(i=1, 2)$, from which it follows that $\mu \circ (\theta_1' \circ \theta_2') = (\theta_1 \circ \theta_2) \circ \mu$, hence $\Gamma(\theta_1 \circ \theta_2) = \Gamma(\theta_1)\Gamma(\theta_2)$. In a more tedious fashion, it may be shown that Γ commutes with the coproducts; recall that the coproduct in $\mathcal{S}^*(p)$ satisfies an analogue of (5.2). Thus Γ is a homomorphism of Hopf algebras.

Proof of (8.2). From the definition of Γ, we see that it has the same kernel as $\mu^R\colon \mathcal{S}^*(p) \to \mathcal{A}^*(G; Z_p)$. Hence ker (Γ) is the annihilator in the Steenrod algebra $\mathcal{S}^*(p)$ of the unit class in $H^*(MG; Z_p)$. Thus (8.2) follows from results of Milnor [12]; for $p=2$ see Liulevicius [10].

Remark. Unless $G = \boldsymbol{Sp}$ and p is odd, the image of Γ in $A^*(G) \otimes Z_p$ has as a Z_p-basis the elements S_ω for ω a partition involving only integers of the form $p^k - 1$.

For example, let $G = \boldsymbol{U}$. Milnor [12] has shown, in slightly different notation, that $\mathcal{S}^*(p)/(\beta_p)$ has a basis $\{t_\omega\}$ over Z_p, $\omega = (i_1, \ldots, i_r)$ running through partitions involving only integers of the form $p^k - 1$, such that

(a) t_ω has degrees $2(i_1 + \cdots + i_r)$;

(b) the coproduct sends t_ω to $\sum_{\omega_1 \omega_2 = \omega} t_{\omega_1} \otimes t_{\omega_2}$;

(c) if $x \in H^2(X; Z_p)$, then $t_{p^k - 1} x = x^{p^k}$ and $t_\omega x = 0$ otherwise.

Thus if $\alpha \in H^2(P^n(C); \boldsymbol{MU})$ is the generator and ω involves only integers of the form $p^k - 1$, $\mu(S_\omega(\alpha)) = t_\omega \mu(\alpha)$ in $H^*(P^n(C); Z_p)$. It follows easily from the splitting principle and universality that $\mu(S_\omega(u_\xi)) = t_\omega \mu(u_\xi)$ whenever u_ξ is the U-cobordism Thom class of a U-bundle $\xi \to X$ over a finite complex, hence

$$\mu \circ S_\omega = t_\omega \circ \mu \in \mathcal{A}^*(U; Z_p).$$

But this says exactly that $\Gamma(t_\omega) = S_\omega$, verifying the statement.

Let $A^* = A^*(G) \otimes Z_p$ for fixed G and p $(p=2$ if $G = \boldsymbol{O})$. The nonzero primitive elements of A^* are evidently the S_k for $k > 0$. Let B^* denote the subalgebra generated by these elements and the identity. In fact, since the S_k are primitive, B^* is a Hopf subalgebra of A^*.

(8.4) Theorem. *A basis for B^* consists of the elements S_ω for which no integer has $\geq p$ occurrences in ω.*

Proof. Let $\bar{B}^*$ be the submodule of A^* with the basis described in the theorem. We first show that $\bar{B}^*$ is a Hopf subalgebra of A^*. It is clear that $\psi^*(\bar{B}^*) \subset \bar{B}^* \otimes \bar{B}^*$.

The annihilator of $\bar{B}^*$ in the dual Hopf algebra $A_* = Z_p[\sigma_1, \sigma_2, \ldots]$ is the ideal I_* generated by $\sigma_1^p, \sigma_2^p, \ldots$. It follows from (6.1) that

$$\phi_*(\sigma_k^p) = \sum \binom{m+1}{R} (\sigma^R)^p \otimes \sigma_m^p,$$

summed over certain pairs (R, m); hence $\phi_*(\sigma_k^p) \in I_* \otimes A_* + A_* \otimes I_*$. Thus

$$\phi_*(I_*) \subset I_* \otimes A_* + A_* \otimes I_*,$$

which means that $\bar{B}^*$ is a subalgebra of A^*. Since $S_k \in \bar{B}^*$ for all k, $B^* \subset \bar{B}^*$. Hence it remains to show that $\bar{B}^* \subset B^*$.

We shall show that each basis element $S_{i_1, i_2, \ldots, i_r} \in \bar{B}^*$ belongs to B^* by induction on the number of terms r. If $r = 1$, then S_{i_1} is a generator of B^*. If $r > 1$, suppose that $i_1 \leqq i_2 \leqq \cdots \leqq i_r$ and notice that the number m of occurrences of i_r is less than p. A judicious application of (6.1) now shows that

$$S_{i_r} \circ S_{i_1, \ldots, i_{r-1}} = m S_{i_1, \ldots, i_r} + \sum a_\omega S_\omega, \; a_\omega \in Z_p,$$

with the sum over partitions of the form

$$\omega = (i_1, \ldots, i_{n-1}, i_{n+1}, \ldots, i_{r-1}, i_n + i_r),$$

for $1 \leqq n \leqq r-1$, having $(r-1)$ terms. Since m is prime to p and the basis elements $S_{i_1, \ldots, i_{r-1}}$ and the S_ω of the sum belong to $\bar{B}^*$, so to B^* by hypothesis of the induction, the proof is complete.

9. **Operations on the bordism rings.** The coefficient ring $\Omega_G^* = H^*(pt; MG)$ of the G-cobordism theory is a graded module over the Hopf algebra $A^*(G)$. Recall that $\Omega_G^{-n} = \pi_{dk+n}(MG(k))$ for k large; in particular $\Omega_G^n = 0$ for $n > 0$. Since the bordism group Ω_n^G of n-dimensional G-manifolds is also naturally isomorphic to this stable homotopy group, we may identify Ω_n^G with Ω_G^{-n} for $n \geqq 0$. There results an action

$$A^i(G) \otimes \Omega_n^G \to \Omega_{n-i}^G$$

on the G-bordism ring. Thus $\theta(\theta'[M]) = (\theta \circ \theta')[M]$ and if $\psi^*(\theta) = \sum \theta_i' \otimes \theta_i''$ then $\theta([M] \cdot [N]) = \sum (\theta_i'[M])(\theta_i''[N])$.

Remark. This action may also be given the following description; the proof of the equivalence of the two definitions is omitted. Let M^n be an n-dimensional G-manifold and let $\theta \in A^i(G)$ correspond to the G-cobordism characteristic class γ. Thus if $\xi \to M^n$ denotes the stable normal G-bundle of M, $\gamma(\xi) \in H^i(M^n; MG)$. By Poincaré duality [6, §13; 16] there results a G-bordism class in $H_{n-i}(M^n; MG)$; such a class has a representative (V^{n-i}, f) consisting of a G-manifold V^{n-i} and a continuous map $f: V^{n-i} \to M^n$. Put $\theta[M^n] = [V^{n-i}]$.

The remainder of the discussion will be restricted to the complex case. If $[M^{2n}] \in \Omega_{2n}^U$ and $\theta \in A^{2n}(U)$, the result of the action is $\theta[M^{2n}] \in \Omega_0^U = Z$. Thus we obtain characteristic numbers of the bordism class, which may be easily identified with the dual Chern numbers. Hence there is a homomorphism $\Delta: \Omega_{2n}^{U} \to A_{2n}(\boldsymbol{U})$ so that $\langle \theta, \Delta[M^{2n}]\rangle = \theta[M^{2n}]$ for all $\theta \in A^{2n}(U)$. In fact, $\Delta: \Omega_{*}^{U} \to A_{*}(\boldsymbol{U})$ is an algebra homomorphism into the dual of $A^{*}(U)$, by virtue of the coproduct formula for $\theta([M]\cdot[N])$.

Now fix a prime p and put $A^{*} = A^{*}(\boldsymbol{U}) \otimes Z_p$. Let S^{*} denote the image of Γ in A^{*}. There is a ring homomorphism $\Delta \otimes 1: \Omega_{*}^{U} \otimes Z_p \to A_{*}(\boldsymbol{U}) \otimes Z_p = A_{*}$, whose image we shall study. The following result is due to Atiyah and Hirzebruch [1, §5], and is essentially the determination of the relations on mod p Chern numbers.

(9.1) Theorem. *The image of* $\Delta \otimes 1$ *in* A_{*} *has for annihilator in* A_{*} *the right ideal* $S_{+}^{*}A^{*}$ *generated by the set* S_{+}^{*} *of elements of* S^{*} *with positive degree.*

Proof. Let $\theta \in \mathcal{S}^{2n}(p)$, $n > 0$, so that $\Gamma(\theta) = \theta' \otimes 1$ with $\theta' \in A^{2n}(U)$. By (8.1), if $[M^{2n}] \in \Omega_{2n}^{U} \approx \Omega_{U}^{-2n}$ then $\mu\theta'[M^{2n}] = \theta\mu[M^{2n}] = 0$ since $\mu[M^{2n}] \in H^{-2n}(pt; Zp)$. Thus $\theta'[M^{2n}]$ is divisible by p, so $\langle \Gamma(\theta), \Delta[M^{2n}] \otimes 1\rangle = \Gamma(\theta)([M^{2n}] \otimes 1) = 0$. Hence S_{+}^{*} annihilates the image of $\Delta \otimes 1$, and so does $S_{+}^{*}A^{*}$ since the annihilator is easily seen to be a right ideal in A^{*}. By Milnor [12], the image of $\Delta \otimes 1$ in $A_{*} \approx H_{*}(\boldsymbol{MU}; Z_p)$ is a polynomial algebra over Z_p with one generator in each dimension $2k$ not of the form $2p^{i} - 2$; moreover, $A^{*} \approx H^{*}(\boldsymbol{MU}; Z_p)$ is a free left module over

$$S^{*} \approx \mathcal{S}^{*}(p)/(\beta_p),$$

with a basis consisting of the S_{ω} for which ω involves no integers of the form $p^{i} - 1$. A dimension count shows that $S_{+}^{*}A^{*}$ is exactly the annihilator of the image of $\Delta \otimes 1$.

Remark. In the unoriented case, put $I^{*} = \sum I^{n}$ with I^{n} consisting of all $\theta \in A^{n}(\boldsymbol{O})$ for which $\theta[M^{n}] = 0$ $\forall [M^{n}] \in \Omega_{n}^{O}$. It is easily seen that I^{*} is a right ideal in $A^{*}(\boldsymbol{O})$, and the argument just given shows that I^{*} is the right ideal generated by the elements of the image of $\Gamma: \mathcal{S}^{*}(2) \to A^{*}(\boldsymbol{O})$ of positive degree. In the symplectic case, there is an analogue of (9.1) for p odd but not for $p = 2$.

Conjecture. The right ideal $S_{+}^{*}A^{*}$ in $A^{*} = A^{*}(\boldsymbol{U}) \otimes Z_p$ (see (9.1)) contains no nontrivial left ideals. The analogous assertion in the unoriented case implies that the action of $A^{*}(\boldsymbol{O})$ on Ω_{*}^{O} is effective, i.e., if $\theta \in A^{*}(\boldsymbol{O})$ satisfies $\theta[M] = 0$ $\forall [M] \in \Omega_{*}^{O}$ then $\theta = 0$.

Added in proof. The present paper has considerable overlap with recent work of S. P. Novikov and J. M. Boardman. Novikov studies the Adams spectral sequence for U-cobordism; an announcement has appeared in Dokl. Akad. Nauk SSSR **172** (1967), 33–36. Boardman studies the nonoriented case, with applications to smooth involutions; see Bull. Amer. Math. Soc. **73** (1967), 136–138, and Chapter VI of his Warwick notes "Stable Homotopy Theory."

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