

# ADDITION AND REDUCTION THEOREMS FOR MEDIAL PROPERTIES<sup>(1)</sup>

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In [6], I gave a systematic presentation of certain kinds of medial properties such as  $(P, Q)_n$ ,  $(P, Q, \sim)_n$ , etc., and their basic properties, having special regard for their dualities and relations to local properties. The present paper is supplementary to [6], in that it goes further into the relations between open sets and their complements and provides certain addition and reduction theorems not given in [6], as well as their applications.

Most of the proofs depend upon "diagram chasing", and two diagram types recur frequently. To avoid repetition, these types and their relevant properties are given in two lemmas in an Appendix.

Throughout, point sets are assumed to be imbedded in a locally compact Hausdorff space  $X$ .

Point set boundaries are denoted by the symbol " $F$ "; thus  $F(A)$  denotes the boundary of the point set  $A$ . As in [6], a pair  $P, Q$  of open sets is called "canonical" if  $P \supset \bar{Q}$  and  $\bar{Q}$  is compact. Homology and cohomology groups based on compact supports are denoted by lower case " $h$ ". As in [6], Čech homology and cohomology with coefficients in a field are used throughout.

If  $A$  is a subset of a space  $X$ , then a property of  $A$  is called *intrinsic* if it is a topological invariant of  $A$ , and *extrinsic* if it is a positional invariant of  $A$  in  $X$  (see [5, p. 290]). If  $A$  is closed, the distinction is of no consequence, since the open subsets of  $A$  coincide with its intersections with open subsets of  $X$ . But for  $A$  not closed, the distinction is important—medial properties of  $A$  in terms of its own open (rel.  $A$ ) subsets are intrinsic, but in terms of the open subsets of  $X$  they are only extrinsic. (For example, the open set  $M$  of Example I.1 of [6]—a domain in  $E^2$  bounded by a closed curve containing a sine curve of form  $y = \sin 1/x$ —has property  $(P, Q)_0$  intrinsically but not extrinsically.)

The expression " $A$  has property  $(P, Q)_r$ ," will frequently be abbreviated to " $A$  has  $(P, Q)_r$ ," and in similar expressions involving other medial properties.

**1. Relations between open sets and their complements.** In [6], the following question was considered: If  $M$  is closed and both  $X$  and  $M$  have certain medial properties, what can be concluded concerning the medial properties of  $X - M$ ? (Compare Theorems II.2a, II.2 and II.3a of [6].)

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**THEOREM 1.1.** *If  $X$  has  $(P, Q)^{r+1}$  and its closed subset  $M$  has  $(P, Q)^r$ , then  $X - M$  has  $(P, Q)^{r+1}$  extrinsically.*

**Proof.** Let  $U = X - M$  and apply Lemma A1 of the Appendix to the diagram

$$(1) \quad \begin{array}{ccccc} h^r(P, P \cap M) & \longrightarrow & h^{r+1}(P \cap U) & & \\ \uparrow & & \uparrow & & \\ h^r(R, R \cap M) & \longrightarrow & h^{r+1}(R \cap U) & \longrightarrow & h^{r+1}(R) \\ & & \uparrow & & \uparrow \\ & & h^{r+1}(Q \cap U) & \longrightarrow & h^{r+1}(Q) \end{array}$$

in which  $P, Q$  is a canonical pair,  $R$  an open set such that  $P \supset \bar{R} \supset R \supset \bar{Q}$ , the horizontal lines are portions of exact sequences for compact cohomology [1] and the vertical arrows are homomorphisms induced by inclusion.

**REMARK.** Theorem 1.1 is stronger than Theorem II.3a of [6], in which it was assumed that  $p^r(X) \leq \omega$ .

**COROLLARY 1.1.** *If  $U$  is an open set such that  $\bar{U}$  has  $(P, Q)^{r+1}$  and  $F(U)$  has  $(P, Q)^r$ , then  $U$  has  $(P, Q)^{r+1}$  extrinsically.*

**THEOREM 1.2.** *If  $X$  has  $(P, Q, \sim)^{r+1}$  and its closed subset  $M$  has  $(P, Q)^r$ , then the set  $U = X - M$  has  $(P, Q, \sim)^{r+1}$  extrinsically.*

**Proof.** Border the diagram (1), at the top, by the exact sequence

$$h^r(M) \rightarrow h^{r+1}(U) \rightarrow h^{r+1}(X)$$

and apply Lemma A2 (alternative hypothesis) of the Appendix.

**REMARK.** Theorems 1.1 and 1.2 supplement Theorem II.2a of [6]. It will be noted that for  $X$  to have  $(P, Q)^{r+1}$  and  $U$  to have  $(P, Q)^{r+1}$  does not imply that  $X - U$  has  $(P, Q)^r$ ; this is shown by the example given in the remark following [6, Theorem II.2], with  $r = 1$ .

**2. Addition theorems.** Consider now the question: If the space  $X$  is the union of sets  $A$  and  $B$  having certain medial properties, what can be said about the medial properties of  $X$ ? This type of question was not considered at all in [6]. It will be understood throughout that  $P, Q$  and  $R$  are open sets such that  $P \supset \bar{R} \supset R \supset \bar{Q}$ , and  $P, R$  as well as  $R, Q$  are canonical pairs.

**THEOREM 2.3.** *If  $M$  is a closed subset of a locally compact space  $X$  and both  $M$  and  $X - M$  have property  $(P, Q)^r$  extrinsically, then  $X$  has  $(P, Q)^r$ .*

**Proof.** Apply Lemma A1 to the diagram below:

$$\begin{array}{ccccc}
 h^r(P-M) & \longrightarrow & h^r(P) & & \\
 \uparrow & & \uparrow & & \\
 h^r(R-M) & \longrightarrow & h^r(R) & \longrightarrow & h^r(R \cap M) \\
 & & \uparrow & & \uparrow \\
 & & h^r(Q) & \longrightarrow & h^r(Q \cap M)
 \end{array}$$

**REMARK.** The corresponding theorem for homology fails, as the following example shows: Let  $X$  consist of the following subspace of the coordinate plane; the sides of the unit square  $S$  in the first quadrant which has two sides on the  $x$ - and  $y$ -axes, and the portions of all lines  $x=1/n, n=2, 3, 4, \dots$ , that lie within  $S$ . Let  $M$  be the subset of  $X$  consisting of all points on the sides of  $S$ . Then  $M$  and  $X-M$  have  $(P, Q)_1$  but  $X$  does not.

**COROLLARY 2.1.** *If  $U$  is an open subset of a locally compact space  $X$  such that  $F(U)$  and  $U$  have property  $(P, Q)^r$  extrinsically, then  $\bar{U}$  has  $(P, Q)^r$  extrinsically.*

**Proof.** Apply Theorem 2.3 with  $M=F(U), X=\bar{U}$ .  
 For the property  $(P, Q, \sim)^r$  we have

**THEOREM 2.4.** *If  $M$  is a closed subset of a locally compact space  $X$  such that  $p^{r-1}(M)$  is finite and both  $M$  and  $X-M$  have  $(P, Q, \sim)^r$  extrinsically,  $r > 0$ , then  $X$  has  $(P, Q, \sim)^r$ .*

**Proof.** Apply Lemma A2 to the diagram below:

$$\begin{array}{ccccccc}
 h^{r-1}(M) & \longrightarrow & h^r(X-M) & \longrightarrow & h^r(X) & \longrightarrow & h^r(M) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & h^r(P-M) & \longrightarrow & h^r(P) & & \\
 & & \uparrow & & \uparrow & & \uparrow \\
 h^r(R-M) & \longrightarrow & h^r(R) & \longrightarrow & h^r(R \cap M) & & \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & h^r(Q) & \longrightarrow & h^r(Q \cap M) & & 
 \end{array}$$

**COROLLARY 2.2.** *If  $U$  is an open subset of a locally compact space  $X$  such that  $p^{r-1}(F(U))$  is finite and both  $F(U)$  and  $U$  have property  $(P, Q, \sim)^r$  extrinsically,  $r > 0$ , then  $\bar{U}$  has property  $(P, Q, \sim)^r$  extrinsically.*

Another interesting corollary is

**COROLLARY 2.3.** *If  $X$  is a connected, locally compact space, and  $X$  has a 0- $lc$  closed and connected subspace  $M$  such that  $X - M$  has property  $(P, Q, \sim)^1$ , then  $X$  is 0- $lc$ .*

**Proof.**  $M$  has  $(P, Q, \sim)_0$  and hence  $(P, Q, \sim)^1$ . Hence  $X$  has  $(P, Q, \sim)^1$  by Theorem 2.4. Therefore  $X$  has  $(P, Q, \sim)_0$ .

**REMARK.** If, in the theorem just proved, it had been assumed that  $X - M$  has  $(P, Q)^r$ , then the hypothesis that  $p^{r-1}(M)$  is finite would not have been needed. (See Lemma A2, alternative hypothesis.) Consequently the following theorem holds:

**THEOREM 2.5.** *If  $M$  is a closed subset of a locally compact space  $X$  such that  $M$  has property  $(P, Q, \sim)^r$  and  $X - M$  has  $(P, Q)^r$  extrinsically,  $r \geq 0$ , then  $X$  has  $(P, Q, \sim)^r$ .*

**COROLLARY 2.4.** *If  $U$  is an open subset of a locally compact space  $X$  such that  $F(U)$  has property  $(P, Q, \sim)^r$  and  $U$  has property  $(P, Q)^r$  extrinsically,  $r \geq 0$ , then  $\bar{U}$  has  $(P, Q, \sim)^r$ .*

In the cases so far considered in this section, the two sets whose union is  $X$  have been disjoint. The following theorems do not make this assumption.

**THEOREM 2.6.** *If  $X$  is a locally compact space which is the union of closed subsets  $X_1$  and  $X_2$  having property  $(P, Q)_{r+1}$ , such that  $X_1 \cap X_2$  has  $(P, Q)_r$ , then  $X$  has  $(P, Q)_{r+1}$ .*

**Proof.** Apply Lemma A1 to the following diagram, in which the horizontal lines are portions of Mayer-Vietoris sequences, and the vertical mappings are induced by inclusion:

$$\begin{array}{ccccc}
 H_{r+1}(\bar{P} \cap X_1) + H_{r+1}(\bar{P} \cap X_2) & \longrightarrow & H_{r+1}(\bar{P}) & & \\
 \uparrow & & \uparrow & & \\
 H_{r+1}(\bar{R} \cap X_1) + H_{r+1}(\bar{R} \cap X_2) & \longrightarrow & H_{r+1}(\bar{R}) & \longrightarrow & H_r(\bar{R} \cap X_1 \cap X_2) \\
 & & \uparrow & & \uparrow \\
 & & H_{r+1}(\bar{Q}) & \longrightarrow & H_r(\bar{Q} \cap X_1 \cap X_2)
 \end{array}$$

The following corollary is a consequence of Theorem 2.6 and [6, Theorem III.2].

**COROLLARY 2.5.** *Let  $X$  be a locally compact space which is the union of closed subsets  $X_1, X_2$  having property  ${}_k(P, Q)_n, 0 \leq k \leq n$ , and such that  $X_1 \cap X_2$  is  $lc_k^{n-1}$  and has property  $(P, Q)_{k-1}$ . Then  $X$  has property  ${}_k(P, Q)_n$ .*

**REMARK.** It is well known that if a compact space  $X = X_1 \cup X_2$  where  $X_1$  and  $X_2$  are closed and  $lc^n$  and  $X_1 \cap X_2$  is  $lc^{n-1}$ , then  $X$  is  $lc^n$ . The above corollary generalizes this.

**THEOREM 2.7.** *If a locally compact space  $X$  is the union of open sets  $X_1$  and  $X_2$  having  $(P, Q)^r$  extrinsically, and  $X_1 \cap X_2$  has property  $(P, Q)^{r+1}$  extrinsically, then  $X$  has property  $(P, Q)^r$ .*

**Proof.** Use Mayer-Vietoris sequences for cohomology with compact supports, and proceed as in the proof of Theorem 2.6.

Analogously, the following theorem holds for the  $(P, Q, \sim)_r$  property.

**THEOREM 2.8.** *If a locally compact space  $X$  is the union of closed sets  $X_1$  and  $X_2$  which have  $(P, Q, \sim)_{r+1}$ ,  $r \geq 0$ , and  $X_1 \cap X_2$  has  $(P, Q, \sim)_r$  and  $H_{r+1}(X_1 \cap X_2)$  is finitely generated, then  $X$  has  $(P, Q, \sim)_{r+1}$ .*

**Proof.** Border the diagram in the proof of Theorem 2.6, at the top, with the portion of the Mayer-Vietoris sequence of the triad  $X, X_1, X_2$  which extends from  $H_{r+1}(X_1 \cap X_2)$  to  $H_r(X_1 \cap X_2)$ ; apply Lemma A2 of the Appendix.

**REMARK.** To show the necessity for assuming  $H_{r+1}(X_1 \cap X_2)$  finitely generated in Theorem 2.8, consider the following example:

$X$  is a subspace of coordinate 3-space consisting of a denumerable collection of finite, circular, hollow cylinders  $C_1, C_2, C_3, \dots$ , closed at both ends, with bases in the planes  $z=1$  and  $z=-1$  and successively tangent along common line elements lying in the plane  $x=0$  which converge to the interval  $E$  between  $z=1$  and  $z=-1$  on the  $z$ -axis. (See the figure on p. 343 of [4].) Let  $M_n$  denote the set of all points in the intersections of  $C_n$  with the planes  $z = \pm p/2^{n-1}$ ,  $p=0, 1, \dots, 2^{n-1}-1$ . Let  $M = E \cup \bigcup_{n=1}^{\infty} M_n$ . The circles constituting  $M_n$  divide  $C_n$  into components  $K_{n1}, K_{n2}, \dots, K_{nk}$  where  $k=2^n$ . Let  $X_{2n} = \bar{K}_{n2} \cup \bar{K}_{n4} \cup \dots \cup \bar{K}_{nk}$ . Let  $X_{1n}$  = closure of  $C_n - X_{2n}$  and  $X'_i = \bigcup_{n=1}^{\infty} X_{in}$ ,  $i=1, 2$ , and  $X_i = \bar{X}'_i$ . Then  $X_1$  and  $X_2$  have  $(P, Q, \sim)_1$ ; for  $E^3 - X_i$  has  $(P, Q, \sim)_0$  and consequently  $X_i$  has  $(P, Q, \sim)_1$  by virtue of Theorem II.5 ("Fourth fundamental duality theorem") of [6]. Also,  $M = X_1 \cap X_2$  has  $(P, Q, \sim)_0$  since  $M$  is 0-lc. However,  $X$  does not have property  $(P, Q, \sim)_1$ .

**THEOREM 2.9.** *If a locally compact space  $X$  is the union of open sets  $X_1$  and  $X_2$  having property  $(P, Q, \sim)^r$  extrinsically, while  $X_1 \cap X_2$  has property  $(P, Q, \sim)^{r+1}$  extrinsically and  $h^r(X_1 \cap X_2)$  is finitely generated, then  $X$  has  $(P, Q, \sim)^r$ .*

**Proof.** Apply Lemma A2 to the diagram

$$\begin{array}{ccccccc}
 h^r(X_1 \cap X_2) & \longrightarrow & h^r(X_1) + h^r(X_2) & \longrightarrow & h^r(X) & \longrightarrow & h^{r+1}(X_1 \cap X_2) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & h^r(P \cap X_1) + h^r(P \cap X_2) & \longrightarrow & h^r(P) & & \\
 & & \uparrow & & \uparrow & & \uparrow \\
 h^r(R \cap X_1) + h^r(R \cap X_2) & \longrightarrow & h^r(R) & \longrightarrow & h^{r+1}(R \cap X_1 \cap X_2) & & \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & & & h^r(Q) & \longrightarrow & h^{r+1}(Q \cap X_1 \cap X_2)
 \end{array}$$

where the horizontal lines are portions of Mayer-Vietoris sequences for cohomology with compact supports (see [1]).

**3. Reduction theorems.** If a space  $X$  and closed subsets  $X_1, X_2$  of which  $X$  is the union all have certain medial properties, what can be said about medial properties of the intersection of  $X_1$  and  $X_2$ ? Analogous questions concerning global properties have been studied in the past; for instance, if a 1-acyclic continuum is expressed as the union of subcontinua  $X_1$  and  $X_2$ , then  $X_1 \cap X_2$  is connected (unicoherence).

**THEOREM 3.10.** *If a locally compact space  $X$  having property  $(P, Q)_{r+1}$  is expressed as the union of closed sets  $X_1, X_2$  having property  $(P, Q)_r$ , then  $X_1 \cap X_2$  has property  $(P, Q)_r$ .*

**Proof.** Extend the sequences in the diagram of the proof of Theorem 2.6 one step to the right, again applying Lemma A1.

**REMARKS.** Theorem 3.10 is the analogue, for medial properties, of the theorem which states that *if a locally compact,  $(r+1)$ -lc space  $X$  is expressed as the union of closed sets  $X_1, X_2$  which are  $r$ -lc, then  $X_1 \cap X_2$  is  $r$ -lc.* (The proof of the latter may be given by a diagram like that for Theorem 3.10, with localized  $P, R$  and  $Q$ .)

Examples can be given to show that for locally compact spaces  $X$  having property  $(P, Q)_{r+1}$ , the converse of Theorem 3.10 does not generally hold.

The theorems just proved have interesting applications to common boundaries. For example, it is known that if a closed set separates  $n$ -space into two  $ulc^k$  domains of which it is common boundary, then it is  $lc^k$  [5]; this is an immediate corollary of Theorem 3.10, since  $ulc^k$  domains have  $lc^k$  closures [5, p. 301, Theorem 5.8] and hence closures having property  $(P, Q)_k$ . Even more generally, however, it follows from Theorem 3.10 that *if any locally compact space is known to be  $lc^{k+1}$  and is separated into  $ulc^k$  open subsets by a common boundary  $B$  thereof (see [2]), then  $B$  is  $lc^k$ .* Also, it follows from Theorem 3.10 that *if a locally compact space  $X$  having property  $(P, Q)_1$  is the union of closed subsets  $X_1, X_2$  having  $(P, Q)_0$ , then  $X_1 \cap X_2$  is locally connected.*

**THEOREM 3.11.** *If a locally compact space  $X$  which has property  $(P, Q, \sim)_{r+1}$  is expressed as the union of closed subsets  $X_1$  and  $X_2$  which have property  $(P, Q, \sim)_r$ , and such that  $H_{r+1}(X_1)$  and  $H_{r+1}(X_2)$  are finitely generated, then  $X_1 \cap X_2$  has property  $(P, Q, \sim)_r$ .*

The proof may be obtained by an extension of the diagram used in proving Theorem 3.10 and applying Lemma A2.

The theorems for cohomology which correspond to the two preceding theorems are stated below without proof (the proofs are quite analogous to those of the preceding theorems, being based on Mayer-Vietoris sequences for cohomology with compact support, with applications of Lemmas A1 and A2).

**THEOREM 3.12.** *If a locally compact space having property  $(P, Q)^r$  is expressed as the union of open sets  $U_1, U_2$  having property  $(P, Q)^{r+1}$  extrinsically, then  $U_1 \cap U_2$  has property  $(P, Q)^{r+1}$  extrinsically.*

**THEOREM 3.13.** *If a locally compact space having property  $(P, Q, \sim)^r$  is expressed as the union of open sets  $U_1, U_2$  having property  $(P, Q, \sim)^{r+1}$  extrinsically and such that  $h^r(U_1)$  and  $h^r(U_2)$  are both finitely generated, then  $U_1 \cap U_2$  has property  $(P, Q, \sim)^{r+1}$  extrinsically.*

Consider now the situation where medial properties are known to hold for a space  $X$  as well as for the intersection of closed sets whose union is  $X$ ; can one say what properties are inherited by these closed sets?

**THEOREM 3.14.** *If a locally compact space  $X$  having property  $(P, Q)_r$  is expressed as the union of closed sets  $X_1, X_2$  whose intersection has property  $(P, Q)_r$ , then both  $X_1$  and  $X_2$  have property  $(P, Q)_r$ .*

**COROLLARY 3.6.** *If a locally compact space  $X$  having properties  $(P, Q)_r$  and  $(P, Q)_{r+1}$  is expressed as the union of closed subsets  $X_1$  and  $X_2$ , then a necessary and sufficient condition that  $X_1$  and  $X_2$  each have property  $(P, Q)_r$  is that  $X_1 \cap X_2$  have property  $(P, Q)_r$ .*

**Proof.** The necessity follows from Theorem 3.10. The sufficiency follows from Theorem 3.14.

**REMARK.** The case  $r=0$  of Theorem 3.14 reduces to the statement that if  $X$  is lc and is expressed as the union of closed sets  $X_1$  and  $X_2$  whose intersection is lc, then both  $X_1$  and  $X_2$  are lc. Other applications include notably the  $n$ -gms, which all have “ $P, Q$ ”-properties; thus if an open set  $U$  in an  $n$ -gm has an  $lc^k$  boundary, then  $\bar{U}$  is  $lc^k$  (as is also the complement of  $U$ ).

**THEOREM 3.15.** *Let a locally compact space  $X$  have property  $(P, Q, \sim)_r$  and  $h_{r+1}(X)$  finitely generated. If  $X$  is expressed as the union of closed sets  $X_1$  and  $X_2$  whose intersection has property  $(P, Q, \sim)_r$ , then  $X_1$  and  $X_2$  have property  $(P, Q, \sim)_r$ .*

For cohomology, the theorems corresponding to the last two above are:

**THEOREM 3.16.** *If a locally compact space having property  $(P, Q)^r$  is expressed as the union of open sets  $U_1$  and  $U_2$  whose intersection has property  $(P, Q)^r$  extrinsically, then  $U_1$  and  $U_2$  have property  $(P, Q)^r$  extrinsically.*

**THEOREM 3.17.** *Let a locally compact space  $X$  have property  $(P, Q, \sim)^r$  and  $h_{r-1}(X)$  finitely generated. If  $X$  is expressed as the union of open sets  $U_1$  and  $U_2$  whose intersection has property  $(P, Q, \sim)^r$  extrinsically, then  $U_1$  and  $U_2$  have property  $(P, Q, \sim)^r$  extrinsically.*

**4. Applications to local connectedness in  $n$ -manifolds.** It was shown in [5] (see p. 301, Theorem 5.8) that if  $D$  is a  $ulc^k$  open subset of an orientable  $n$ -gcm  $X$ , then

$\bar{D}$  is  $lc^k$ . Since  $(P, Q)_r$  is generally weaker than  $r$ -ulc, it is of interest to note that the result cited can be generalized as follows:

**LEMMA 4.1.** *Let  $D$  be an open subset of an orientable  $n$ -gcm  $X$  such that  $D$  is  $ulc^k$  and has property  $(P, Q)_{k+1}$  extrinsically. Then  $\bar{D}$  is  $lc^{k+1}$ .*

**Proof.** It is sufficient to show that  $\bar{D}$  has property  $(P, Q)_{k+1}$  since for  $\bar{D}$  to be  $lc^{k+1}$  is equivalent to  $\bar{D}$  having property  ${}_0(P, Q)_{k+1}$ ; and it is already known, from the result cited, that  $\bar{D}$  is  $lc^k$  and hence has property  ${}_0(P, Q)_k$ .

Let  $P, Q$  be a canonical pair of open sets of  $X$ . Since  $D$  has  $(P, Q)_{k+1}$  extrinsically,  $h_{k+1}(Q \cap D \mid \bar{P} \cap D)$  is finitely generated [6, p. 207]. Consider

$$h_{k+1}(Q \cap \bar{D} \mid \bar{P} \cap \bar{D});$$

let  $\{Z_{k+1}^i\}$  be any collection of independent elements of it. By [5, p. 168, Theorem 19.7], there exists an open set  $U$  containing  $\bar{P} \cap \bar{D}$  such that the  $Z_{k+1}^i$  are also independent elements of  $h_{k+1}(U)$ . However, by [5, p. 300, Lemma 5.7], this implies the existence of equally many independent elements in  $h_{k+1}(Q \cap D \cap U \mid U)$ . But the latter, because of the inclusion maps

$$h_{k+1}(Q \cap D \cap U) \rightarrow h_{k+1}(Q \cap D) \rightarrow h_{k+1}(\bar{P} \cap D) \rightarrow h_{k+1}(U)$$

cannot have more independent elements than  $h_{k+1}(Q \cap D \mid \bar{P} \cap D)$ —which, as observed above, is finitely generated. It follows that  $h_{k+1}(Q \cap \bar{D} \mid \bar{P} \cap \bar{D})$  must be finitely generated.

It was shown by R. L. Moore [3] that if the simply connected domain  $D$  in  $S^2$  has property  $(P, Q)_0$ , then  $F(D)$  is 0-lc. In [6, Theorem V.2], I gave an  $n$ -dimensional generalization of this result, viz.; if  $U$  is an open subset of an orientable  $n$ -gcm having properties  $(P, Q, \sim)_0$  and  $(P, Q, \sim)_{n-2}$  as well as  $p_{n-1}(U)$  finite, then the boundary of every component of  $U$  is 0-lc. A generalization of another kind can now be given, in the form of a condition sufficient that a boundary be  $lc^{n-2}$ .

**THEOREM 4.1.** *Let  $U$  be an open subset of an orientable  $n$ -gcm  $X$  which is  $ulc^{n-3}$  (if  $n > 2$ ), has property  $(P, Q)_{n-2}$  extrinsically, and  $p_{n-1}(U)$  finite. Then the boundary of  $U$  is  $lc^{n-2}$ .*

**Proof.** By [4, Corollary VII.3] and [6, Corollary IV.1],  $X - U$  is  $lc^{n-2}$ . By Lemma 4.1,  $\bar{U}$  is  $lc^{n-2}$ . And as a consequence of Theorem 3.10,  $(X - U) \cap \bar{U}$  is  $lc^{n-2}$ .

**REMARKS.** We could not conclude, in addition, that  $F(U)$  is  $(n-1)$ -lc. For if  $M$  is the union of an infinite set of solid balls  $S_n$  is  $S^3$ , successively tangent but otherwise disjoint, and converging to a point  $p$ , the complement  $U$  of  $M$  is 0-ulc, has property  $(P, Q)_1$  and  $p_2(U) = 0$ ; but  $F(U)$  is not 2-lc. However, we could conclude above that  $X - U$  is  $lc^{n-1}$ , since  $U$  has property  $(P, Q)_{n-1}$  (see [6, Lemma IV.1]) and hence  ${}_0(P, Q)_{n-1}$  (see [6, Theorem IV.3]).

If  $U$  were actually  $ulc^{n-2}$ , then  $F(U)$  would be much more restricted; in particular each component of  $F(U)$  would be either a point or an orientable  $(n-1)$ -gcm [5, p. 311, Theorem 8.3].



The necessity for assuming  $p_{n-1}(U)$  finite is shown by the example, in  $S^3$ , of a sequence of points approaching a limit point  $p$ ; the complement  $U$  of this sequence is 0-ulc, and has property  $(P, Q)_1$ , but does not have  $p_2(U)$  finite.

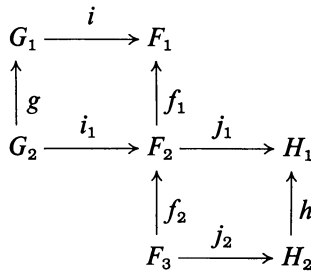
Finally, the  $ulc^{n-3}$  condition could not be weakened through replacement by “ $P, Q$ ” conditions. For example, consider the example in the Remark following Theorem 2.8 above. The space  $X$  of this example, considered as a subset of  $S^3$ , is the boundary of a domain which has property  ${}_0(P, Q)_1$ , but  $X$  is not  $lc^1$ .

Lemma 4.1 also enables us to strengthen the theorem which states that a set which separates the  $n$ -sphere into two  $ulc^k$  open sets of which it is the common boundary must be  $lc^k$  (see Remarks following Theorem 3.10):

**THEOREM 4.2.** *If  $X$  is an orientable  $n$ -gcm and  $F$  is a closed subset of  $X$  which separates  $X$  into  $ulc^k$  open sets  $U_1$  and  $U_2$ , each of which has property  $(P, Q)_{k+1}$  and which have  $F$  as common boundary, then  $F$  is  $lc^{k+1}$ .*

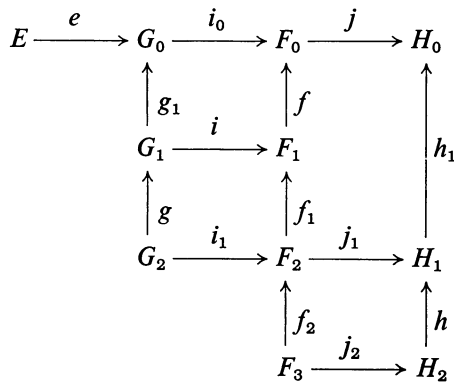
**APPENDIX.** The following two lemmas concern types of diagrams that occur repeatedly in the establishing of reduction and addition theorems.

**LEMMA A1.** *Consider the commutative diagram*



in which the  $G$ 's,  $F$ 's and  $H$ 's are vector spaces, the  $i$ 's,  $f$ 's,  $j$ 's and  $h$  are homomorphisms, and the middle horizontal line of homomorphisms forms an exact sequence. If  $\text{Im } g$  and  $\text{Im } hj_2$  are  $f.g.$  (=finitely generated), then  $\text{Im } f_1 f_2$  is  $f.g.$

**LEMMA A2.** *Consider the commutative diagram:*



of vector spaces in which  $E$ ,  $g(\text{Kern } g_1g)$  and  $h(\text{Kern } h_1h)$  are f.g., and the horizontal lines are exact. Then  $f_1f_2(\text{Kern } ff_1f_2)$  is f.g.

Alternatively, if  $\text{Im } g$  is f.g., then the same conclusion follows without the assumption that  $E$  is f.g.

**Proof.** Let  $K_0 = \text{Kern } ff_1f_2$ . Since  $fff_1f_2K_0 = h_1h(j_2K_0) = 0$ ,  $j_2K_0$  is a subspace of  $\text{Kern } h_1h$  and therefore  $h(j_2K_0) = j_1f_2K_0$  is f.g. We may, then, represent  $f_2K_0$  as a direct sum  $K+L$  where  $L$  is f.g. and  $K \subset \text{Kern } j_1$ .

Since the horizontal lines are exact,  $K$  has antecedent  $K_1$  in  $G_2$ . And since  $i_0(g_1gK_1) = ff_1i_1K_1 = ff_1K = 0$ ,  $g_1gK_1$  must be a subspace of  $\text{Kern } i_0$ . Hence  $g_1gK_1$  has antecedent  $K_2$  in  $E$  and must be f.g. Then  $K_1$  is representable as a direct sum  $K_3+L_1$  where  $L_1$  is f.g. and  $K_3 = \text{Kern } g_1gK_1$ . But then  $gK_3$  must be f.g., since  $g(\text{Kern } g_1g)$  is f.g. by hypothesis. Hence  $gK_1$  is f.g. and  $igK_1 = f_1i_1K_1 = f_1K$  is f.g. It follows that  $f_1f_2K_0 = f_1(K+L)$  is f.g.

The proof under the alternative hypothesis should be clear.

**REMARK.** In the application of Lemma A2, it frequently happens that  $K_1$  of the proof is a homology (cohomology) group based on chains with compact supports. In this case, the antecedent  $K_2$  in  $E$  of  $g_1gK_1$  may be f.g. even though  $E$  itself is not. In such cases, reference is made to "Lemma A2, Remark."

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