ON THE EQUATION $n=p+x^2$ (1)

BY
R. J. MIECH

Let n and x be positive integers, p be a prime, (n/p) be the Legendre symbol, and

$$\mathscr{P}(n) = \prod_p \left[1-\frac{(n/p)}{p-1}\right].$$

Let $\Omega(n)$ denote the number of solutions of the equation $n=p+x^2$. Hardy and Littlewood conjectured that if n is not a square then

$$\Omega(n) \sim \mathscr{P}(n)n^{1/2}/\log n$$

as $n \to \infty$. [3, Conjecture H]. The purpose of this paper is to show that the conjecture holds for nearly every integer n.

To be specific we have the

THEOREM. *Let $n, x, p, \mathscr{P}(n)$ and $\Omega(n)$ be defined as above. Let N be a positive parameter and let A_1 and A_2 be any fixed positive numbers. Then the equation*

$$\Omega(n) = \mathscr{P}(n) \int_1^{\sqrt{(n-3)}} \frac{dt}{\log(n-t^2)} + B_1 \frac{\sqrt{n}}{(\log n)^{A_1}} \tag{1}$$

holds for all but $B_2N(\log N)^{A_2}$ positive integers $n \leqq N$. B_1 and B_2 are numbers whose absolute value is bounded above by some constant that is independent of n and N.

It will be evident later that the Hardy-Littlewood conjecture holds for those integers n which are not exceptions to this theorem.

The proof of (1), given the methods employed by Tschudakoff in [11], follows from Bombieri's recent theorem on the density of the zeroes of the L-functions. This latter result is employed to show that the truncated singular series associated with our problem is asymptotically equal to the product $\mathscr{P}(n)$.

The equation $n=p+x^2$ is, of course, a special case of the general equation

$$n = p_1+\cdots+p_s+x_1^2+\cdots+x_r^2$$

and if we view it as such the theorem of this paper can be considered as an extension of the work of several individuals who proved, under various conditions, that the number of solutions of the general equation satisfies a specific asymptotic equation for all large n if $r+2s>4$, [9], [4]; for almost all even n if $s=2$ [11], for almost all n if $r=2$, $s=1$, [9]; and for all large n if $r=2$ and $s=1$, [5], [7].

Received by the editors October 4, 1966 and, in revised form, June 23, 1967.

(1) This research was supported, in part, by funds received under NSF Grant GP-3933.

1. Several definitions are in order at this point. Suppose that N is an integer and let, for $4 \leqq n \leqq 2N$,

$$\Omega(n, N) = |\{(m, p) : m^2+p = n,\ 1 \leqq m \leqq \sqrt{N},\ 3 \leqq p \leqq N\}|,$$

where $|\{\ \ \}|$ denotes the number of elements in the set $\{\ \ \}$. Let $e(x)=\exp(2\pi i x)$, $y=[\sqrt{N}]$, where $[x]$ is the integral part of x,

$$F(\alpha) = \sum_{m=1}^{y} e(m^2\alpha),$$

and

$$P(\alpha) = \sum_{3\leqq p\leqq N} e(p\alpha).$$

Note that

$$F(\alpha)P(\alpha) = \sum_{m=1}^{y} \sum_{3\leqq p\leqq N} e[(m^2+p)\alpha] = \sum_{n=4}^{2N} \Omega(n, N)e(n\alpha). \tag{2}$$

Suppose next that a/q is a rational number with $a/q \geqq 0$ and $(a, q)=1$. Let

$$W(a, q) = \sum_{m=1}^{q} e(m^2 a/q),$$

$$F_{aq}(\alpha) = \frac{W(a, q)}{q} \sum_{m=1}^{y} e[m^2(\alpha-a/q)]$$

and

$$P_{aq}(\alpha) = \frac{\mu(q)}{\phi(q)} \sum_{u=3}^{N} \frac{e[u(\alpha-a/q)]}{\log u}$$

where $\mu(n)$ is the Möbius function and $\phi(n)$ is Euler's function. The sums $F_{aq}(\alpha)$ and $P_{aq}(\alpha)$ are, as we shall see later, approximations to $F(\alpha)$ and $P(\alpha)$ at the point a/q. Finally, let

$$Q(\alpha) = \sum_{q=1}^{[\Delta]} \sum_{a}^{q}{}' F_{aq}(\alpha)\cdot P_{aq}(\alpha)$$

where $\Delta=[\exp(\log N)^{1/4}]+\frac{1}{2}$ and the prime (′) indicates the inner summation is taken over the set of integers $\{a\}$ satisfying the conditions: $0 \leqq a < q$, $(a, q)=1$.

If we substitute the defining sums for $F_{aq}(\alpha)$ and $P_{aq}(\alpha)$ in this last equation and make several rearrangements we have

$$Q(\alpha) = \sum_{n=4}^{2N} \Psi(n, N)e(n\alpha) \tag{3}$$

where

$$\Psi(n, N) = L(n, N)H(n, N),$$

$$L(n, N) = \sum_{1\leqq m\leqq y,\, 3\leqq u\leqq N;\, m^2+u=n} \frac{1}{\log u}$$

and

$$H(n, N) = \sum_{q=1}^{[\Delta]} \sum_{a}^{q}{}' \frac{W(a, q)}{q} \frac{\mu(q)}{\phi(q)} e(-na/q).$$

Equations (2) and (3) give us

$$F(\alpha)P(\alpha) - Q(\alpha) = \sum_{n=4}^{2N} (\Omega(n, N) - \Psi(n, N))e(n\alpha).$$

Consequently, it follows that

$$\sum_{n=4}^{2N} |\Omega(n, N) - \Psi(n, N)|^2 = \int_0^1 |F(\alpha)P(\alpha) - Q(\alpha)|^2 \, d\alpha. \tag{4}$$

For the balance of this paper the symbol B will denote a number whose absolute value is bounded by a constant that does not depend on N or n; its value will usually be different each time it occurs. The symbol $F \ll G$, i.e. $F = BG$, will also be employed from time to time.

2. The main result of this section is

LEMMA A. *If t is any fixed positive number then*

$$\sum_{n=4}^{2N} |\Omega(n, N) - \Psi(n, N)|^2 = BN^2(\log N)^{-t},$$

where B depends on t.

Following Tschudakoff, [11], we shall prove this lemma by finding an appropriate bound for the integral in (4). To this end, let $\tau = N \exp(-9(\log N)^{1/4})$, $a = -\tau^{-1}$, $b = 1 - \tau^{-1}$,

$$\begin{aligned}
\Gamma_{aq} &= \{\alpha : |\alpha - a/q| \leqq 1/\tau q\}, \\
\mathscr{M}_1 &= \{\Gamma_{aq} : 0 \leqq a < q, (a, q) = 1, 1 \leqq q \leqq (\log N)^{2t}\}, \\
\mathscr{M}_2 &= \{\Gamma_{aq} : 0 < a < q, (a, q) = 1, (\log N)^{2t} < q \leqq \tau\},
\end{aligned}$$

and $\mathscr{M}$ be the union of $\mathscr{M}_1$ and $\mathscr{M}_2$. Then, by the periodicity of the integrand and by the well-known properties associated with the Farey dissection of the unit interval,

$$\begin{aligned}
\int_0^1 |F(\alpha)P(\alpha) - Q(\alpha)|^2 \, d\alpha &= \int_a^b |F(\alpha)P(\alpha) - Q(\alpha)|^2 \, d\alpha \\
&\ll \sum_{\mathscr{M}} \int_{\Gamma_{aq}} |F(\alpha)P(\alpha) - Q(\alpha)|^2 \, d\alpha.
\end{aligned}$$

Furthermore since

$$|F(\alpha)P(\alpha) - Q(\alpha)| \leqq |F(\alpha)P(\alpha) - F_{aq}(\alpha)P_{aq}(\alpha)| + |F_{aq}(\alpha)P_{aq}(\alpha) - Q(\alpha)|;$$

$$|F_{aq}(\alpha)P_{aq}(\alpha) - Q(\alpha)| = \Bigg| \sum_{\substack{r=1 \\ bq - ar \neq 0}}^{[\Delta]} \sum_{b}^{r}{}' F_{br}(\alpha)P_{br}(\alpha) \Bigg|$$

if $q \leqq [\Delta]$, and

$$|F_{aq}P_{aq}(\alpha)-Q(\alpha)| \leqq |F_{aq}(\alpha)P_{aq}(\alpha)|+\left|\sum_{\substack{r=1\\ bq-ar\neq 0}}^{[\Delta]}\sum_{b}^{r}{}' F_{br}(\alpha)P_{br}(\alpha)\right|$$

for $q>\Delta$, we have

LEMMA 1.

$$\int_0^1 |F(\alpha)P(\alpha)-Q(\alpha)|^2\, d\alpha = B\sum_{i=1}^{4}\Sigma_i(I)$$

where

$$\Sigma_i(I) = \sum_{\mathcal{M}_i}\int_{\Gamma_{aq}} |F(\alpha)P(\alpha)-F_{aq}(\alpha)P_{aq}(\alpha)|^2\, d\alpha \quad \textit{for} \quad i=1, 2,$$

$$\Sigma_3(I) = \sum_{\mathcal{M}}\int_{\Gamma_{aq}}\left|\sum_{\substack{r=1\\ bq-ar\neq 0}}^{[\Delta]}\sum_{b}^{r}{}' F_{br}(\alpha)P_{br}(\alpha)\right|^2 d\alpha,$$

and

$$\Sigma_4(I) = \sum_{\Delta<q\leqq\tau}\sum_{a}^{q}{}' |F_{aq}(\alpha)P_{aq}(\alpha)|^2\, d\alpha.$$

The next few lemmas deal with approximations for $F(\alpha)$ and $P(\alpha)$.

LEMMA 2. *If z is a positive integer then*

$$\sum_{m=1}^{z} e(m^2a/q) = \frac{z}{q}W(a,q)+Bq^{(1/2)+\varepsilon}$$

where ε is any fixed positive number and B is a constant that depends only on ε. Furthermore if $1\leqq z\leqq q$ then

$$\sum_{m=1}^{z} e(m^2a/q) = Bq^{(1/2)+\varepsilon}.$$

See Theorem 2, p. 10 of [6].

LEMMA 3. *If*

$$(a,q) = 1, \quad \left|\alpha-\frac{a}{q}\right| \leqq \frac{1}{\tau q}, \quad \tau = N\exp[-9(\log N)^{1/4}], \quad \textit{and} \quad y = [\sqrt{N}]$$

then

$$\sum_{m=1}^{y} e(m^2\alpha) = \frac{W(a,q)}{q}\sum_{m=1}^{y} e(m^2\beta)+B[q^{-(1/2)+\varepsilon}\exp[9(\log N)^{1/4}]+q^{(1/2)+\varepsilon}]$$

where $\beta=\alpha-(a/q)$.

Proof. Set

$$S(m) = \sum_{j=1}^{m} e(j^2a/q) \quad \text{and} \quad T = \sum_{m=1}^{y} e(m^2\alpha).$$

Then, for $\beta=\alpha-(a/q)$,

$$T = \sum_{m=1}^{y} e(m^2\beta)e(m^2a/q) = \sum_{m=1}^{y} e(m^2\beta)(S(m)-S(m-1)).$$

Rearranging in the usual way we get

$$T = \sum_{m=1}^{y} S(m)[e[m^2\beta]-e[(m+1)^2\beta]]+S(y)e[(y+1)^2\beta].$$

If we now apply Lemma 2 we find that T is equal to

$$\frac{W(a,q)}{q}\sum_{m=1}^{y} m[e[m^2\beta]-e[(m+1)^2\beta]]+\frac{y}{q}W(a,q)e[(y+1)^2\beta]$$
$$+Bq^{(1/2)+\varepsilon}\sum_{m=1}^{y}|e[m^2\beta]-e[(m+1)^2\beta]|+Bq^{(1/2)+\varepsilon}.$$

The main term of this last quantity is equal to

$$\frac{W(a,q)}{q}\sum_{m=1}^{y} e(m^2\beta).$$

As for the sum appearing in the error term, we have,

$$\sum_{m=1}^{y}|e[m^2\beta]-e[(m+1)^2\beta]| = \sum_{m=1}^{y}|1-e[(2m+1)\beta]|$$
$$\ll \sum_{m=1}^{y}(2m+1)\beta \ll y^2\beta \ll \frac{N}{\tau q} = \frac{\exp[9(\log N)^{1/4}]}{q}.$$

If we bring these results together we have Lemma 3.

LEMMA 4. *If*

$$|\alpha-a/q| \leqq 1/\tau q, \quad (a,q)=1, \quad (\log N)^{2t} \leqq q \leqq N^{1/4}, \quad \textit{and} \quad y=[\sqrt{N}]$$

then

$$\sum_{m=1}^{y} e(m^2\alpha) = B\sqrt{N}(\log N)^{-t/2}.$$

This follows from Lemmas 2 and 3 with $\varepsilon=\frac{1}{4}$.

LEMMA 5. *If*

$$|\alpha-a/q| \leqq 1/\tau q, \quad (a,q)=1, \quad N^{1/4}<q\leqq\tau, \quad \textit{and} \quad y=[\sqrt{N}]$$

then

$$\sum_{m=1}^{y} e(m^2\alpha) = B\sqrt{N}\exp(-(\log N)^{1/4}).$$

We shall first prove the following:

Let

P and Y be integral parameters with $1 \leqq Y < P$;

a and q be any two integers such that $0 \leqq a < q$, $(a, q) = 1$ and $1 \leqq q \leqq P^2$;

α_2 be any real number such that $|\alpha_2 - a/q| \leqq 1/q^2$,

and

α_1 be any real number.

Set

$$F(x) = \alpha_2 x^2 + \alpha_1 x$$

and

$$S = \sum_{x=1}^{P} e(F(x)).$$

Then

$$|S|^2 \ll P^3/Y^2 + P^3/Yq + Pq/Y + Y^2.$$

We can obtain Lemma 5 from this result by taking

$$P = [\sqrt{N}] \quad \text{and} \quad Y = [\sqrt{N} \exp(-(\log N)^{1/4})].$$

The scheme we shall use to obtain the stated bound for $|S|$ is a simple version of the proof of Lemma 5.10 of [6]. We begin by setting

$$S_0(y) = \sum_{x=1}^{P} e(F(x+y) - F(y)) = \sum_{x=1}^{P} e(\phi(x))$$

where

$$\phi(x) = F^{(1)}(y)x + \frac{F^{(2)}(y)}{2} x^2, \quad F^{(1)}(y) = 2\alpha_2 y + \alpha_1,$$

and

$$\frac{F^{(2)}(y)}{2} = \alpha_2.$$

Since we also have

$$S_0(y) = \sum_{m=y+1}^{P+y} e(F(m) - F(y))$$

it follows that

$$|S_0(y)| = |S| + 2\vartheta y$$

where $|\vartheta| \leqq 1$. Adding, we find that

$$|S| = \frac{1}{Y} \sum_{y=1}^{Y} |S_0(y)| + BY.$$

Thus

$$|S|^2 \ll \left[\frac{1}{Y} \sum_{y=1}^{Y} |S_0(y)|\right]^2 + Y^2 \ll \frac{1}{Y} \sum_{y=1}^{Y} |S_0(y)|^2 + Y^2.$$

Next, let

$$S_1(\beta) = \sum_{x=1}^{P} e(\beta x + \alpha_2 x^2)$$

and

$$\Omega(y) = \{\beta : 0 \leqq \beta \leqq 1, \langle \beta - F^{(1)}(y) \rangle \leqq Y/(2P^2)\}$$

where $\langle z \rangle$ is the distance from z to the nearest integer to z. Now if β is in $\Omega(y)$ then

$$\beta = I + F^{(1)}(y) + \vartheta Y/(2P^2)$$

where I is an integer and $|\vartheta| \leqq 1$. Hence, since $\alpha_2 = F^{(2)}(y)/2!$,

$$S_1(\beta) = \sum_{x=1}^{P} e[(F^{(1)}(y) + \vartheta Y/(2P^2))x + (F^{(2)}(y)/2!)x^2]$$

$$= \sum_{x=1}^{P} e(\phi(x)) + B \sum_{x=1}^{P} (Yx)/P^2 = S_0(y) + BY.$$

That is,

$$|S_0(y)|^2 \ll |S_1(\beta)|^2 + Y^2$$

if the β appearing in the definition of $S_1(\beta)$ is in $\Omega(y)$. If we integrate over $\Omega(y)$ we have

$$|S_0(y)|^2 \ll \frac{P^2}{Y} \int_{\Omega(y)} |S_1(\beta)|^2 \, d\beta + Y^2$$

since the measure of $\Omega(y)$ is greater than $Y/(2P^2)$.

At this point we have

$$|S|^2 \ll \frac{P^2}{Y^2} \sum_{y=1}^{Y} \int_{\Omega(y)} |S_1(\beta)|^2 \, d\beta + Y^2.$$

We must now find a bound for the number of times any point in the unit interval is covered by an $\Omega(y)$. Let y_0 be a fixed integer and suppose $\beta \in \Omega(y_0)$. Then if $\beta \in \Omega(y) \cap \Omega(y_0)$ we have

$$\langle \beta - F^{(1)}(y_0) \rangle \leqq Y/2P^2 \quad \text{and} \quad \langle \beta - F^{(1)}(y) \rangle \leqq Y/2P^2.$$

Thus, since $F^{(1)}(y) - F^{(1)}(y_0) = 2\alpha_2(y - y_0)$, it follows that

$$\langle \alpha_2 2(y - y_0) \rangle \leqq Y/P^2.$$

According to Lemma 5.7 of [6, p. 56] the number of y that satisfy this inequality does not exceed

$$2(Yq/P^2 + 1)(Y/q + 1).$$

In short, the number of $\Omega(y)$ that cover any given point of the unit interval is bounded above by a number of order

$$1 + Y/q + Yq/P^2.$$

The results of the preceding paragraph give us:

$$|S|^2 \ll \frac{P^2}{Y^2}\left(1+\frac{Y}{q}+\frac{Yq}{P^2}\right)\int_0^1 |S_1(\beta)|^2\,d\beta + Y^2.$$

But

$$\int_0^1 |S_1(\beta)|^2\,d\beta = \int_0^1 \left|\sum_{x=1}^{P} e(\beta x+\alpha_2 x^2)\right|^2 d\beta = P.$$

Consequently,

$$|S|^2 \ll P^3/Y^2+P^3/Yq+Pq/Y+Y^2.$$

This is the result we set out to prove.

Lemma 6. *If*

$$|\alpha-a/q| \leqq 1/\tau q, \quad (a,q)=1 \quad \textit{and} \quad 1\leqq q\leqq (\log N)^{2t}$$

then

$$\sum_{3\leqq p\leqq N} e(p\alpha) = \frac{\mu(q)}{\phi(q)}\sum_{u=3}^{N}\frac{e(u\beta)}{\log u}+BN\exp\left(-\gamma\sqrt{(\log N)}\right)$$

where $\beta=\alpha-(a/q)$ and γ is some positive constant.

See [8, Theorem 31, p. 180] for a proof.

From this point on the symbol γ will denote a positive constant that is bounded below by a positive number that is independent of N; its value will usually be different each time it appears.

Lemma 7. *Let $\Sigma_1(I)$ be defined as in Lemma* 1. *Then*

$$\Sigma_1(I) = BN^2\exp\left(-\gamma\sqrt{(\log N)}\right).$$

Proof. We have

$$|F(\alpha)P(\alpha)-F_{aq}(\alpha)P_{aq}(\alpha)|^2 \ll |P(\alpha)|^2\,|F(\alpha)-F_{aq}(\alpha)|^2+|F_{aq}(\alpha)|^2\,|P(\alpha)-P_{aq}(\alpha)|^2.$$

By Lemma 3, with $\varepsilon=\frac{1}{4}$,

$$\begin{aligned}\sum_{\mathscr{M}_1}\int_{\Gamma_{aq}} &|P(\alpha)|^2\,|F(\alpha)-F_{aq}(\alpha)|^2\,d\alpha\\ &\ll \sum_{\mathscr{M}_1}\int_{\Gamma_{aq}} |P(\alpha)|^2\left(q^{-1/4}\exp\left(9(\log N)^{1/4}\right)+q^{3/4}\right)^2 d\alpha\\ &\ll \exp\left(18(\log N)^{1/4}\right)\int_0^1 |P(\alpha)|^2\,d\alpha \ll \exp\left(18(\log N)^{1/4}\right)\frac{N}{\log N}.\end{aligned}$$

Moreover, by Lemma 6,

$$\begin{aligned}\sum_{\mathscr{M}_1}\int_{\Gamma_{aq}} &|F_{aq}(\alpha)|^2\,|P(\alpha)-P_{aq}(\alpha)|^2\,d\alpha\\ &\ll \sum_{\mathscr{M}_1} N\{N^2\exp[-2\gamma\sqrt{(\log N)}]\}\frac{1}{\tau q}\\ &\ll \frac{N^3}{\tau}\exp[-\gamma\sqrt{(\log N)}](\log N)^{2t} = N^2\exp[-\gamma\sqrt{(\log N)}].\end{aligned}$$

The lemma follows from these results.

LEMMA 8.

$$\Sigma_2(I) = BN^2(\log N)^{-t}.$$

Proof. We have

$$\Sigma_2(I) \ll \sum_{\mathscr{M}_2} \int_{\Gamma_{aq}} |F(\alpha)P(\alpha)|^2\, d\alpha + \sum_{\mathscr{M}_2} \int_{\Gamma_{aq}} |F_{aq}(\alpha)P_{aq}(\alpha)|^2\, d\alpha.$$

By Lemmas 4 and 5,

$$\sum_{\mathscr{M}_2} \int_{\Gamma_{aq}} |F(\alpha)P(\alpha)|^2\, d\alpha \ll \sum_{\mathscr{M}_2} \int_{\Gamma_{aq}} \frac{N}{(\log N)^t} |P(\alpha)|^2\, d\alpha$$
$$\ll N^2(\log N)^{-t}.$$

Furthermore, if $\beta = \alpha - a/q$ and $y = [\sqrt{N}]$,

$$\begin{aligned} S &= \sum_{\mathscr{M}_2} \int_{\Gamma_{aq}} |F_{aq}(\alpha)P_{aq}(\alpha)|^2\, d\alpha \\ &= \sum_{\mathscr{M}_2} \int_{\Gamma_{aq}} \left| \frac{W(a,q)}{q} \frac{\mu(q)}{\phi(q)} \sum_{m=1}^{y} e(m^2\beta) \sum_{u=3}^{N} \frac{e(u\beta)}{\log u} \right|^2 d\alpha \\ &\ll \sum_{\mathscr{M}_2} \frac{|W(a,q)|^2\, |\mu(q)|}{(q\phi(q))^2} \int_0^1 \sum_{m=1}^{y} \sum_{j=1}^{y} \sum_{u=3}^{N} \sum_{k=3}^{N} \frac{e[(m^2-j^2+u-k)\beta]}{\log u \log k}\, d\alpha \\ &\ll \sum_{\mathscr{M}_2} \frac{q^{1+2\varepsilon}\, |\mu(q)|}{(q\phi(q))^2} T(N) \end{aligned}$$

where ε is the constant of Lemma 2, and $T(N)$ is the number of solutions of the equation $m^2 - j^2 + u - k = 0$, subject to the conditions $1 \leqq m, j \leqq \sqrt{N}$ and $1 \leqq u, k \leqq N$. Since for any arbitrary choice of m, j and u there is at most one possible choice of k it is clear that $T(N) \leqq N^{1/2}N^{1/2}N = N^2$. Thus if $\varepsilon = 1/10$, then

$$S \ll N^2 \sum_{\mathscr{M}_2} \frac{1}{q^{4/5}} \frac{1}{\phi^2(q)}$$
$$\ll N^2 \sum_{q > (\log N)^{2t}} \frac{1}{q^{3/2}} \ll \frac{N^2}{(\log N)^t}.$$

This completes the proof of Lemma 8.

LEMMA 9.

$$\Sigma_4(I) = BN^2\Delta^{-1/2}.$$

Proof. We have, if $\beta = \alpha - a/q$,

$$\begin{aligned} \Sigma_4(I) &= \sum_{\Delta < q \leqq \tau} \sum_{a}^{q}{}' \int_{\Gamma_{aq}} |F_{aq}(\alpha)P_{aq}(\alpha)|^2\, d\alpha \\ &\ll \sum_{\Delta < q \leqq \tau} \sum_{a}^{q}{}' \frac{|W(a,q)|^2\, |\mu(q)|}{(q\phi(q))^2} N \int_0^1 \left| \sum_{u=3}^{N} \frac{e(u\beta)}{\log u} \right|^2 d\alpha \\ &\ll \sum_{\Delta < q \leqq \tau} \sum_{a}^{q}{}' \frac{1}{q^{1-2\varepsilon}(\phi(q))^2} N^2 \ll \frac{N^2}{\Delta^{1/2}}, \end{aligned}$$

provided we set $\varepsilon = 1/10$.

LEMMA 10. *If*

$$|\alpha-a/q| \leqq 1/\tau q, \quad a/q \neq b/r, \quad (a,q) = (b,r) = 1, \quad \textit{and} \quad r \leqq \Delta$$

then there is an integer b', which is equal to one of the numbers $b-r$, b, or $b+r$, such that $ar-b'q \neq 0$ and

$$P_{br}(\alpha) = B \log\log \Delta\, q/|ar-b'q|.$$

Proof. First of all there is a b' such that $P_{br}(\alpha)=P_{b'r}(\alpha)$ and

$$\tfrac{1}{2}|a/q-b'/r| < |\alpha-b'/r| \leqq \tfrac{1}{2}.$$

To see this set $b'=b$ if $|\alpha-b/r|\leqq\frac{1}{2}$. If $\alpha-b/r>\frac{1}{2}$ set $b'=b+r$. Then $|\alpha-b'/r|\leqq\frac{1}{2}$ and $a/q\neq b'/r$, for $a/q=b'/r$ implies that $a/q=b/r+1$, or $a/q\geqq 1$. Similarly if $b/r-\alpha>\frac{1}{2}$ set $b'=b-r$. Moreover, since $\alpha=a/q+\theta/\tau q$ where $|\theta|\leqq 1$, we have

$$\left|\alpha-\frac{b'}{r}\right| = \left|\frac{a}{q}-\frac{b'}{r}+\frac{\theta}{\tau q}\right| \geqq \left|\frac{a}{q}-\frac{b'}{r}\right|-\frac{1}{\tau q} > \frac{1}{2}\left|\frac{a}{q}-\frac{b'}{r}\right|.$$

Next, set $\beta=\alpha-b'/r$ and

$$S(u) = \sum_{j=3}^{u} e(j\beta).$$

Note that

$$|S(u)| \leqq \frac{1}{|\sin\pi\beta|} \leqq \frac{rq}{|ar-b'q|}$$

since, by the previous inequalities,

$$|\sin\pi\beta| \geqq 2\beta \geqq |a/q-b'/r|.$$

Applying these results, we have,

$$\begin{aligned} P_{b'r}(\alpha) &= \frac{\mu(r)}{\phi(r)}\sum_{u=3}^{N}\frac{e(u\beta)}{\log u} \\ &= \frac{\mu(r)}{\phi(r)}\sum_{u=3}^{N} S(u)\left(\frac{1}{\log u}-\frac{1}{\log(u+1)}\right)+\frac{\mu(r)}{\phi(r)}\frac{S(N)}{\log(N+1)} \\ &\ll \frac{r}{\phi(r)}\frac{q}{|ar-b'q|} \ll \log\log\Delta\,\frac{q}{|ar-b'q|}. \end{aligned}$$

LEMMA 11.

$$\Sigma_3(I) = B\Delta^3(\log\log\Delta)^2 N^2 \exp(-9(\log N)^{1/4}).$$

Proof. Set

$$K(\Delta) = \sum_{r=1}^{[\Delta]}\sum_{b}^{r}{}' |F_{br}(\alpha)|^2$$

and

$$M(\Delta,a,q) = \sum_{\substack{r=1 \\ ar-bq\neq 0}}^{[\Delta]}\sum_{b}^{r}{}' |P_{br}(\alpha)|^2.$$

Then, by the Cauchy-Schwarz inequality,

$$\Sigma_3(I) \leqq \sum_{\mathscr{M}} \int_{\Gamma_{aq}} K(\Delta)M(\Delta, a, q)\, d\alpha.$$

By Lemma 2,

$$K(\Delta) \ll \sum_{r=1}^{[\Delta]} \sum_{b}^{r}{}' \frac{|W(b, r)|^2}{r^2} N$$

$$\ll N \sum_{r=1}^{[\Delta]} \frac{\phi(r) r^{1+2\varepsilon}}{r^2} \ll N\Delta^{1+2\varepsilon} = N\Delta^2,$$

if we take $\varepsilon = \frac{1}{2}$. By Lemma 10,

$$\int_{\Gamma_{aq}} |P_{br}(\alpha)|^2\, d\alpha \ll (\log\log\Delta)^2 \frac{q^2}{|ar-b'q|^2} \frac{1}{\tau q}.$$

Consequently,

$$\Sigma_3(I) \ll \frac{N}{\tau} (\Delta \log\log\Delta)^2 \sum_{q=1}^{\tau} \sum_{\substack{r=1 \\ ar-bq\neq 0}}^{[\Delta]} \sum_{a}^{q}{}' \sum_{b}^{r}{}' \frac{q}{|ar-b'q|^2}.$$

Fix q and r. Then since $1 \leqq a < q$, $(a, q) = 1$, $|b'| < 2r$, and since the set $\{b'\}$ forms a reduced residue class modulo r we have

$$\sum_{a}^{q}{}' \sum_{b}^{r}{}' \frac{1}{|ar-b'q|^2} \ll 1.$$

Hence

$$\Sigma_3(I) \ll \frac{N}{\tau} (\Delta \log\log\Delta)^2 \sum_{q=1}^{\tau} q \sum_{r=1}^{[\Delta]} 1$$

$$\ll N\tau\Delta^3 (\log\log\Delta)^2$$

$$= N^2\Delta^3 (\log\log\Delta)^3 \exp(-9(\log N)^{1/4}).$$

This completes the proof of Lemma 11.

Since $\Delta = [\exp(\log N)^{1/4}] + \frac{1}{2}$, Lemma A follows from (4) and Lemmas 1, 7, 8, 9, and 11.

Suppose now that $N(\log N)^{-A_2} < n \leqq N$ and that

$$|\Omega(n, N) - \Psi(n, N)| > \sqrt{n}\,(\log n)^{-A_1}. \tag{5}$$

Then

$$|\Omega(n, N) - \Psi(n, N)|^2 > n(\log n)^{-2A_1} > N(\log N)^{-2A_1-A_2}. \tag{6}$$

Let $E(N)$ be the number of integers n, with $N(\log N)^{-A_2} < n \leqq N$, for which (5) holds. Then by Lemma A and (6)

$$E(N) \leqq BN/(\log N)^{t-2A_1-A_2}.$$

Hence if we set $t=2A_1+2A_2$ we can conclude that the equation

$$\Omega(n, N) = \Psi(n, N)+B\sqrt{n}/(\log n)^{A_1} \tag{7}$$

holds for all but $BN(\log N)^{-A_2}$ integers $n \leqq N$.

3. The purpose of this section is to show that if A_3 is any fixed positive number then

$$\Psi(n, N) = \mathscr{P}(n)\int_1^{\sqrt{(n-3)}} \frac{dx}{\log(n-x^2)}+B\sqrt{n}\exp(-\gamma(\log n)^{1/8}) \tag{8}$$

for all but $BN(\log N)^{-A_3}$ integers $n \leqq N$.

By definition,

$$\Psi(n, N) = L(n, N)H(n, N)$$

where

$$L(n, N) = \sum_{1\leqq m\leqq y, 3\leqq u\leqq N;\, m^2+u=n} \frac{1}{\log u}$$

and

$$H(n, N) = \sum_{q=1}^{[\Lambda]} \sum_{a}^{q}{}' \frac{W(a, q)}{q}\frac{\mu(q)}{\phi(q)}e(-na/q).$$

As for $L(n, N)$, since $u=n-m^2 \geqq 3$, we have

$$L(n, N) = \sum_{1\leqq m<\sqrt{(n-3)}} \frac{1}{\log(n-m^2)} = \int_1^{\sqrt{(n-3)}} \frac{dx}{\log(n-x^2)}+B. \tag{9}$$

A large part of the balance of this paper is devoted to proving:

LEMMA B. *Let A_3 be any fixed positive number. Then there is a positive constant γ such that the relation*

$$H(n, N) = \mathscr{P}(n)+B\exp(-\gamma(\log n)^{1/8}) \tag{10}$$

holds for all but $BN(\log N)^{-A_3}$ positive integers $n \leqq N$.

We begin with

LEMMA 12. *If n is any fixed integer then*

$$G(n, q) = \frac{\mu(q)}{q\phi(q)}\sum_{a}^{q}{}' W(a, q)e(-na/q)$$

is a multiplicative function of q; that is, if $(q, r)=1$ then $G(n, qr)=G(n, q)G(n, r)$.

This is a straightforward consequence of the Chinese remainder theorem.

LEMMA 13. *Let p be a prime and $\omega(p, n)$ be the number of solutions of the congruence $x^2 \equiv n \bmod p$. Then*

$$G(n, p) = (1-\omega(p, n))/(p-1).$$

Proof. By the definition of $W(a, p)$

$$G(n,p) = \frac{-1}{p\phi(p)} \sum_{m=1}^{p} \sum_{a=1}^{p-1} e[(m^2-n)a/p]$$

$$= \frac{-1}{p\phi(p)} \sum_{m=1}^{p} \begin{Bmatrix} p-1 \text{ if } m^2 \equiv n \bmod p \\ -1 \ \text{ otherwise} \end{Bmatrix} = \frac{1-\omega(p,n)}{\phi(p)}.$$

Set

$$a_q = \mu(q) \prod_{p|q} (\omega(p,n)-1).$$

Then, by the definition of $H(n, N)$ and Lemmas 12 and 13, we have

$$H(n,N) = \sum_{q=1}^{[\Delta]} G(n,q) = \sum_{q=1}^{[\Delta]} \frac{a_q}{\phi(q)}.$$

We shall evaluate $H(n, N)$ by considering the properties of the function

$$Z(s) = Z(s,n) = \sum_{m=1}^{\infty} \frac{a_m}{\phi(m)m^{s-1}}.$$

LEMMA 14. *Suppose that $n=(n^*)(n')^2$ where n^* is square free and $n^*>1$. Set*

$$d = d(n) = n^* \ \textit{if} \ n^* \equiv 1 \bmod 4,$$
$$= 4n^* \ \textit{if}\ n^* \equiv 2 \textit{ or } 3 \bmod 4,$$

$\chi_d(m)=(d/m)$ where (d/m) is the Kronecker symbol, and let $L(s, \chi_d)$ be the L-function defined by the character $\chi_d(m)$. Let

$$J(s,n) = \prod_{p|2n} \left(1-\frac{\chi_d(p)}{p^s}\right)^{-1} \prod_{(p,2n)=1} \left(1-\frac{\chi_d(p)}{\phi(p)p^{s-1}}\right)\left(1+\frac{\chi_d(p)}{p^s}\right)\left(1-\frac{1}{p^{2s}}\right)^{-1}.$$

Then, if $\mathrm{Re}(s)=\sigma>\frac{1}{2}$ *and* $L(s,\chi_d)\neq 0$,

$$Z(s) = J(s,n)/L(s,\chi_d).$$

Proof. Since $\omega(2,n)=1$ for all n, $\omega(p,n)=1$ if $p|n$, $\omega(p,n)=2$ or 0 if $(p,2n)=1$, and since a_m is a multiplicative function we have, for $\sigma>1$,

$$Z(s) = \prod_{(p,2n)=1} \left(1-\frac{(\omega(p,n)-1)}{\phi(p)p^{s-1}}\right).$$

Moreover if $(p, 2n)=1$ then [1, Chapter V],

$$(\omega(p,n)-1) = (n/p) = (n^*/p) = (d/p) = \chi_d(p).$$

Hence, if $\sigma>1$

$$Z(s) = J(s,n)(L(s,\chi_d))^{-1}.$$

This equation also holds at the points in the half-plane $\sigma>\frac{1}{2}$ for which $L(s,\chi_d)\neq 0$ since the product representing $J(s, n)$ converges for any $\sigma>\frac{1}{2}$.

LEMMA 15. *If* $b>1$ *and* $T>1$ *then*

$$\sum_{m\leqq\Delta}\frac{a_m}{\phi(m)}=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{Z(s)\Delta^{s-1}}{(s-1)}\,ds+E(\Delta,T)$$

where

$$E(\Delta,T)=B\left[\frac{\Delta^{b-1}}{T(b-1)}+\frac{(\log\Delta)^2}{T}\right].$$

The methods employed to prove this type of result are well known; see, for example, the proof of Theorem 3.1 in the appendix of [8].

LEMMA 16. *There are absolute positive constants* c_1, c_2, *and* c_3 *such that*

$$J(1,n)>c_1\exp(-c_2\log\log\log n)$$

for $n\geqq c_3$. *Furthermore, if* $v(n)\leqq A_4\log\log n$, *where* $v(n)$ *is the number of distinct prime divisors and* A_4 *is any fixed positive constant, and if* $\mathrm{Re}(s)=\sigma\geqq 3/4$ *then there are positive constants* B_1 *and* B_2, *which depend on* A_4, *such that*

$$|J(s,n)|\leqq B_1\exp(B_2(\log\log n)^{1/4}).$$

Proof. First of all,

$$\prod_{(p,2n)=1}\left(1-\frac{\chi_d(p)}{\phi(p)}\right)\left(1+\frac{\chi_d(p)}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1}\geqq\prod_{p\geqq 2}\left(1-\frac{2}{p(p-1)}\right)\left(1-\frac{1}{p^2}\right)^{-1}.$$

Secondly, if q_i is the ith prime,

$$\prod_{p|2n}\left(1-\frac{\chi_d(p)}{p}\right)\leqq\exp\sum_{p|2n}\log\left(1+\frac{1}{p}\right)$$

$$\leqq\exp\sum_{i=1}^{v(n)}\frac{1}{q_i}\leqq\exp(c_2\log\log\log n),$$

for $n\geqq c_3$. Hence

$$J(1,n)>c_1\exp(-c_2\log\log\log n).$$

The proof of the second part of the lemma is similar.

From this point on the symbols $c_4, c_5,\dots$ will denote absolute positive constants.

The evaluation of $H(n,N)$ will be based on

LEMMA 17. *Let* χ_d *and* $L(s,\chi_d)$ *be defined as in Lemma* 14. *Then the number of* d, *with* $1<d\leqq 4N$, *for which* $L(s,\chi_d)$ *has a zero in the rectangle*

$$1-1/\log\log N\leqq\sigma\leqq 1,\quad |t|\leqq\exp(\log N)^{\delta},\tag{12}$$

where $0<\delta<1$, *does not exceed* $c_4N^{3/8}$.

The proof of this lemma is based on Bombieri's recent density theorem. Several definitions are needed before his result can be stated. Let Q be any finite set of positive integers. Set

$$M = M(Q) = \max_{q \in Q} q$$

and

$$D = D(Q) = \max_{q \in Q} d(q)$$

where $d(q)$ is the number of divisors of q. Let χ be a character modulo q and

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a)e(a/q).$$

If χ is a primitive character we have $|\tau(\chi)|^2 = q$ [1, Chapter V, §4]. Finally, let $N(\alpha, T; \chi)$ denote the number of zeros of $L(s, \chi)$ in the region $\sigma \geqq \alpha$, $|t| \leqq T$. We then have

LEMMA 18.

$$\sum_{q \in Q} \frac{1}{\phi(q)} \sum_{\chi} |\tau(\chi)|^2 N(\alpha, T; \chi) \ll DT(M^2 + MT)^{4(1-\alpha)/(3-2\alpha)} \log^{10}(M+T).$$

See [2] for a proof.

In order to derive Lemma 17 from Lemma 18 let X be any number such that $\exp(\log N)^{\delta} < X \leqq 4N$; let Q be the set of integers d with $X < d \leqq 2X$. Let $E(X)$ denote the number of d in Q for which $L(s, \chi_d)$ has a zero in (12). We apply Lemma 18 with $M = 2X$, $T = \exp(\log N)^{\delta}$ and $\alpha = 1 - (\log\log N)^{-1}$. As for D and $|\tau(\chi_d)|^2$: since $d(q) \ll q^{\varepsilon}$ for any $\varepsilon > 0$ we can take $D \ll X^{\varepsilon}$; since χ_d, the character defined by the Kronecker symbol, is a primitive character we have $|\tau(\chi_d)|^2 = d$. Thus it follows that

$$E(X) \ll X^{\varepsilon} \cdot T(X^2 + X \cdot T)^{\beta} \log^{10}(X+T),$$

where

$$\beta = \frac{4(1-\alpha)}{3-2\alpha} \leqq \frac{4}{\log\log N}.$$

Since we are assuming that $T = \exp(\log N)^{\delta} < X$ we have

$$E(X) \ll TX^{\varepsilon} X^{8/\log\log N} \log^{10} X \ll X^{2\varepsilon} \exp(\log N)^{\delta}.$$

If we take

$$X = 2^j \exp(\log N)^{\delta},$$

where $0 \leqq j \leqq (\log N)/\log 2$, it follows that there are at most

$$\sum_{j \leqq \log N/\log 2} 2^{2\varepsilon j} \exp(\varepsilon + 1)(\log N)^{\delta} \ll N^{3\varepsilon}$$

integers d in the interval $(\exp(\log N)^{\delta}, 4N)$ for which $L(s, \chi_d)$ has a zero in (12).

If we take $\varepsilon=1/8$ those integers $d \leqq \exp(\log N)^{\delta}$ can be absorbed by the bound $N^{3/8}$. This completes the proof of Lemma 17.

LEMMA 19. *Let χ be a nonprincipal character modulo k and suppose that $L(s,\chi)\neq 0$ for $\sigma>1-\beta$, $|t|\leqq T$, where $0<\beta<1/2$ and $T>2/\beta$. Let $\log L(s,\chi)$ be that branch of the logarithm of $L(s,\chi)$ that is zero at $s=\sigma=+\infty$. Let η be any number such that $2/T<\eta<\beta$. Then for*

$$1-\beta+\eta \leqq \sigma \leqq 1+\eta \quad \text{and} \quad |t| \leqq T/2$$

we have

$$\log L(s,\chi) = (B_1/\eta^2)(\log k(1+|t|))^a$$

where

$$a = (1-\sigma)/\beta + B_2(\eta/\beta)$$

and B_1 and B_2 are constants that are independent of k, T, β, and η.

This lemma is a refinement of Theorem 14.2 of [10].

Proof. Set $\sigma_0=1/\eta$; then $2\leqq\sigma_0<T/2$. Let, for $i=1,2,3,4$, C_i be the circle centered at $s_0=\sigma_0+it$ of radius r_i where $r_1=\sigma_0-(1+\eta)$, $r_2=\sigma_0-\sigma$, $r_3=\sigma_0-(1-\beta)-\eta$ and $r_4=\sigma_0-(1-\beta)-\eta/2$.

Since χ is not a principal character we have

$$L(s,\chi) = Bk(1+|t|)$$

for $\sigma\geqq 1/2$ [8, Chapter IV, Theorem 5.4]. Consequently on C_4,

$$\operatorname{Re}\log L(s,\chi) = \log|L(s,\chi)| \leqq c_5 \log k(1+|t|).$$

Hence on C_3 [8, A., Theorem 4.2]

$$|\log L(s,\chi)-\log L(s_0,\chi)| = \frac{2\{B\log k(1+|t|)-\log|L(s_0,\chi)|\}\{\sigma_0-(1-\beta)-\eta\}}{(\sigma_0-(1-\beta)-\eta/2)-(\sigma_0-(1-\beta)-\eta)}.$$

That is, since $\sigma_0\geqq 2$

$$|\log L(s,\chi)| = (B\sigma_0/\eta)\log k(1+|t|)$$

for s on C_3. On C_1

$$|\log L(s,\chi)| = \left|\sum_p \log\left(1-\frac{\chi(p)}{p^s}\right)\right| \leqq \sum_{n=1}^{\infty}\frac{1}{n^{1+\eta}} = \frac{B}{\eta}.$$

Now, let M_i be the maximum of $|\log L(s,\chi)|$ on C_i. We have

$$M_1 = B/\eta \quad\text{and}\quad M_3 = (B\sigma_0/\eta)\log k(1+|t|).$$

Thus, by the three-circle theorem

$$M_2 \leqq (B/\eta)^{1-a}((B\sigma_0/\eta)\log k(1+|t|))^a = (B/\eta^2)(\log k(1+|t|))^a$$

where

$$a = \frac{\log(r_2/r_1)}{\log(r_3/r_1)} = \frac{1-\sigma}{\beta}+B\left(\frac{\eta}{\beta}\right).$$

This proves the lemma.

LEMMA 20. *Suppose that* $v(n) \leqq A_4 \log\log N$ *and that the* d *associated with* n *by Lemma* 14 *is not an exception to Lemma* 17. *Then*

$$H(n, N) = \mathscr{P}(n) + B\exp(-\gamma(\log N)^{1/8})$$

where γ *is a positive constant that depends on* A_4.

Proof. By Lemmas 14 and 15 we have

$$H(n, N) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \frac{J(s, n)}{L(s, \chi_d)} \frac{\Delta^{s-1}}{s-1}\, ds + E(\Delta, T)$$

where

$$E(\Delta, T) \ll \frac{\Delta^{b-1}}{T(b-1)} + \frac{(\log \Delta)^2}{T}$$

and

$$\Delta = [\exp(\log N)^{1/4}] + 1/2.$$

According to our assumptions we have $L(s, \chi_d) \neq 0$ for

$$\sigma \geqq 1 - 1/\log\log N, \quad |t| \leqq \exp(\log N)^{\delta}.$$

Thus if we set

$$\beta = 1/\log\log N \quad \text{and} \quad \eta = 1/(\log\log N)^2$$

in Lemma 19 we have

$$\log L(s, \chi_d) \ll (\log\log N)^4 (2\log N)^h,$$

where $h = (\log\log N)(1-\sigma)$, in the region

$$1 - \frac{1}{2\log\log N} \leqq \sigma \leqq 1 + \frac{1}{(\log\log N)^2}, \quad |t| \leqq \frac{\exp(\log N)^{\delta}}{2}.$$

Let R be the rectangle with vertices $a \pm iT$, $b \pm iT$ where:

$$a = 1 - \frac{1}{(\log\log N)^2}, \quad b = 1 + \frac{1}{(\log N)^{1/4}}, \quad T = \frac{\exp(\log N)^{\delta}}{2}.$$

Then we have

$$\frac{1}{2\pi i}\int_{b+iT}^{a+iT} \frac{J(s, n)}{L(s, \chi_d)} \frac{\Delta^{s-1}}{(s-1)}\, ds \ll \exp(-\gamma(\log N)^{\delta}),$$

$$\frac{1}{2\pi i}\int_{a+iT}^{a-iT} \frac{J(s, n)}{L(s, \chi_d)} \frac{\Delta^{s-1}}{(s-1)}\, ds \ll \exp\left(-\gamma \frac{(\log N)^{1/4}}{(\log\log N)^2}\right)$$

and

$$E(\Delta, T) \ll \exp(-\gamma(\log N)^{\delta}).$$

If we take $\delta=1/8$ we have Lemma 20 since the residue of the integrand at $s=1$ is $\mathscr{P}(n)$.

The exceptions to Lemma 20 occur if $v(n)>A_4 \log\log N$ or if the d associated with n is an exception to Lemma 17. It is known [8, I, Theorem 5.3] that the number of positive integers $n \leqq N$ for which $v(n)>A_4 \log\log N$ does not exceed $BN(\log N)^g$ where $g=-A_4 \log 2+1=-A_3$. As for the second type, if $n=n^*(n')^2$ then $d=n^*$ or $4n^*$. If we take $d=n^*$ we have $n=d(n')^2 \leqq N$; that is there are at most $(N/d)^{1/2} \leqq N^{1/2}$ integers n associated with any given d. Since there are at most $c_4N^{3/8}$ exceptional integers d we have at most $c_4N^{7/8}$ exceptions to Lemma 17. This completes the proof of Lemma B.

According to (9) and (10)

$$\Psi(n, N) = \mathscr{P}(n) \int_1^{\sqrt{(n-3)}} \frac{dx}{\log(n-x^2)} + B\mathscr{P}(n) + B\sqrt{n} \exp(-\gamma(\log n)^{1/8}).$$

By Lemma 14

$$\mathscr{P}(n) = Z(1) = J(1, n)/L(1, \chi_d).$$

By Lemma 16,

$$J(1, n) = B_1 \exp(B_2 (\log\log N)^{1/4}).$$

Furthermore if d is not an exception to Lemma 20 we have, from the proof of Lemma 20,

$$\log L(1, \chi_d) = B(\log\log N)^4.$$

Hence, for all but $N(\log N)^{-A_3}$ positive integers $n \leqq N$,

$$\Psi(n, N) = \mathscr{P}(n) \int_1^{\sqrt{(n-3)}} \frac{dx}{\log(n-x^2)} + B[\sqrt{n} \exp(-\gamma(\log n)^{1/8})].$$

This proves (8).

If we set $A_3=A_2$ then our theorem follows from (7) and (8). Finally, since

$$\int_1^{\sqrt{(n-3)}} \frac{dt}{\log(n-t^2)} > \frac{\sqrt{(n-3)}-1}{\log n},$$

$$\mathscr{P}(n) = J(1, n)(L(1, \chi_d))^{-1}$$

$$0 < L(1, \chi_d) < c_6 \log d < c_7 \log n \qquad [8, IV, 8.1]$$

and

$$J(1, n) > c_1 \exp(-c_2 \log\log\log n)$$

it follows that

$$\mathscr{P}(n) \int_1^{\sqrt{(n-3)}} \frac{dt}{\log(n-t^2)} > c_8 \frac{\sqrt{n}}{(\log n)^2(\log\log n)^{c_2}}.$$

That is, the main term in (1) dominates the error term, provided that $A_1 \geqq 3$.

Bibliography

1. R. Ayoub, *An introduction to the analytic theory of numbers*, Math. Surveys No. 10, Amer. Math. Soc., Providence, R. I., 1963.

2. E. Bombieri, *On the large sieve*, Mathematika **12** (1965), 201–225.

3. G. H. Hardy and J. E. Littlewood, *Some problems of partitio numerorum*: III: *On the expression of a large number as a sum of primes*, Acta Math. **44** (1923), 1–70.

4. H. Halberstam, *On the representation of large numbers as sums of squares, higher powers, and primes*, Proc. London Math. Soc. (2) **53** (1951), 363–380.

5. C. Hooley, *On the representation of a number as a sum of two squares and a prime*, Acta Math. **97** (1957), 189–210.

6. L. K. Hua, *Additive theory of prime numbers*, Transl. Math. Monos., Amer. Math. Soc., Providence, R. I., 1965.

7. Ju. V. Linnik, *The dispersion method in binary additive problems*, Transl. Math. Monos., Amer. Math. Soc., Providence, R. I., 1963.

8. K. Prachar, *Primzahlverteilung*, Springer, Berlin, 1957.

9. G. K. Stanley, *On the representation of numbers as a sum of squares and primes*, Proc. London Math. Soc. (2) **29** (1928), 122–144.

10. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Clarendon Press, Oxford, 1951.

11. N. G. Tschudakoff, *On the density of the set of even numbers which are not representable as a sum of two primes*, Izv. Akad. Nauk SSSR Ser. Mat. **1** (1938), 25–39. (Russian)

University of California,
Los Angeles, California