THE K-THEORY OF A CLASS OF HOMOGENEOUS SPACES

BY

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1. **Introduction.** We calculate the groups $K^*(G/U)$ modulo torsion (K^* denoting the Grothendieck-Atiyah-Hirzebruch ring based on complex vector bundles) for homogeneous spaces G/U satisfying the following conditions: G is a compact, connected, simply-connected simple Lie group, U is a closed subgroup which is totally nonhomologous to zero in G for rational coefficients, and τ is an automorphism of period 2 of G with U as fixed point set. Since G is simple, it must be SU (n), Spin $(2n)$, or E_6 (see §3) and the simplest example of such a "symmetric space" G/U is $S^{2n-1} = \text{Spin}\,(2n)/\text{Spin}\,(2n-1)$.

We use Atiyah's result [1] that $K^*(G)$ mod torsion is an exterior algebra $\bigwedge([\rho_1], \ldots, [\rho_l])$ where the ρ_i are the basic irreducible representations of G. Hodgkin's theorem that $K^*(G)$ is actually torsion free will be used only in a few places to sharpen the results (see Hodgkin [10], or a forthcoming paper of Araki).

For any space X let $K_f^*(X)$ denote $K^*(X)$ modulo torsion. G and U will be assumed to satisfy the conditions in the first paragraph unless otherwise stated. The first theorem describes $K_f^*(G/U)$ and the homomorphisms:

$$K_f^*(G/U) \xrightarrow{p^*} K_f^*(G) \xrightarrow{q^*} K_f^*(G/U)$$

where $p\colon G \to G/U$ is the natural projection, and $q\colon G/U \to G$ is given by $q(gU) = g\tau(g)^{-1}$ (τ is the automorphism of G). The second theorem uses the relation of G/U to a certain Jordan algebra to show that $K_f^*(G/U)$ as a ring with operations is generated by one element defined by means of the Jordan algebra (e.g., a half-spin representation for S^{2n-1}). The third theorem relates $K_f^*(G/U)$ to $K_f^*(\Omega(L/G))$ where L/G is again a homogeneous space and Ω denotes its loop space. The fourth theorem applies these results to fiber bundles with G/U as fiber.

We now state these theorems:

THEOREM 1. (a) *$K_f^*(G/U)$ is an exterior algebra $\bigwedge(x_1, \ldots, x_r)$ and $p^*\colon K_f^*(G/U) \to K_f^*(G)$ is injective, $q^*\colon K_f^*(G) \to K_f^*(G/U)$ is surjective.*

(b) $K_f^*(G) = \bigwedge([\rho_1], \ldots, [\rho_r], \tau^*[\rho_1], \ldots, \tau^*[\rho_r], [\rho_{r+1}], \ldots, [\rho_l])$ *where* $\tau^*[\rho_i] = [\rho_i]$ *for* $i > r$, *and* $K_f^*(G/U) = \bigwedge(q^*[\rho_1], \ldots, q^*[\rho_r])$ *while* $q^*[\rho_i] = 0$ *for* $i > r$.

(c) $p^*K_f^*(G/U) = \bigwedge([\rho_1] - \tau^*[\rho_1], \ldots, [\rho_r] - \tau^*[\rho_r]) \subset K_f^*(G)$.

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The proof of Theorem 1, with some similar results for more general homogeneous spaces, is given in §2. The proof is very short and does not require the classification theory.

The next theorem gives a more precise description of $K_f^*(G/U)$, including operations, for the class of symmetric spaces described in the first paragraph. With each such space we associate a central simple formally-real Jordan algebra J (see [5] and its references) such that U is the automorphism group of J, and G is the compact simply-connected form of the group $N(J)$ of linear transformations of J preserving the norm form (a kind of determinant) of J.

THEOREM 2. *Let R be a complex irreducible representation of the Jordan algebra $J \otimes_{\boldsymbol{R}} \boldsymbol{C}$, which is a special representation if J is a special Jordan algebra. Then R determines a complex irreducible representation ρ of G and a corresponding element $[\rho]$ of $K_f^{-1}(G)$, and*

$$K_f^*(G/U) = (q^*[\rho], q^*[\lambda^2\rho], \ldots, q^*[\lambda^r\rho])$$

where $q\colon G/U \to G$ is the imbedding, the λ^i are exterior powers and $r=\text{rank of } G-\text{rank of } U$.

The proof of Theorem 2, in §3, makes use of the classification. However, it is short and illustrates the concepts involved.

If G/U is a symmetric space as before, or under more general circumstances, then one can find Lie groups L containing G and Bott maps $B\colon G/U \to \Omega(L/G)$, Ω denoting the loop space. The composition of B and the natural map $\omega\colon \Omega(L/G) \to G$ is just $q\colon G/U \to G$ and we obtain

THEOREM 3. $K_f^*(\Omega(L/G)) \approx K_f^*(G/U) \oplus \operatorname{Ker} B^*$.

Finally, we make an application to fiber bundles. Let G, U be as in Theorems 1 and 2, and let ρ be the representation of G described in Theorem 2. Let $P \to P/G$ be a principal G-bundle and $G/U \to P/U \to P/G$ the associated G/U-bundle. Let (ρ) and $\tau^*(\rho) = (\rho \circ \tau)$ be the corresponding elements of $K^0(P/G)$. We then have:

THEOREM 4. *Let $(\rho) - \tau^*(\rho) = 0$ in $K^0(P/G)$. Then $K_f^*(P/U) \to K_f^*(G/U)$ is surjective. If further $K^*(G/U)$ is torsion-free then $K^*(P/U) \approx K^*(P/G) \otimes K^*(G/U)$ as algebras.*

The results stated for symmetric spaces actually hold more generally but in more complicated form, as shown in the following sections.

2. By [1], $K_f^*(G) = \bigwedge([\rho_1], \ldots, [\rho_l])$ where $\rho_1, \ldots, \rho_l$ are the basic irreducible representations $\rho_i\colon G \to U(N_i)$ and $[\rho_i]$ are the corresponding elements in $K_f^{-1}(G)$.

We recall that $\operatorname{ch}\colon K_f^*(G) \to H^*(G; \boldsymbol{Q})$ is injective and identifies $K^*(G) \otimes \boldsymbol{Q}$ with $H^*(G; \boldsymbol{Q})$. Elements x of $K_f^*(G)$ will be called primitive if $\operatorname{ch} x$ is primitive in the Hopf algebra $H^*(G; \boldsymbol{Q})$.

It is easy to see that the $[\rho_i]$ are a basis for the primitive elements $PK_f^*(G)$ in $K_f^*(G)$ (for, one may look at $K^{-1}(U(N))$ and the element $[R]$ corresponding to the standard identity representation; $[R]$ is primitive since ch $[R]$ is the suspension of an element in $H^*(B_{U(N)})$.

Let $p: G \to G/U$, $q: G/U \to G$ be given by $p(g)=gU$, $q(gU)=g\tau(g)^{-1}$. Thus $q \circ p(g)=g\tau(g)^{-1}$. An easy calculation (see [4]) shows that if $y \in H^*(G; \boldsymbol{Q})$ is primitive then $p^*q^*(y)=y-\tau^*y$. Thus

$$p^*q^*[\rho_i] = [\rho_i]-\tau^*[\rho_i] = [\rho_i]-[\tau(\rho_i)].$$

The action of τ on the ρ_i, defined by $\tau(\rho_i)=\rho_i \circ \tau$, is easily described: the ρ_i are in 1-1 correspondence with the vertices of the Dynkin diagram of G, and τ (modulo inner automorphisms) is determined by a permutation of these vertices which is an automorphism of the diagram. Thus τ permutes the ρ_i, and τ^2 gives the identity permutation. Hence we may relabel the ρ_i as $\rho_1, \ldots, \rho_r$, $\tau\rho_1, \ldots, \tau\rho_r$, $\rho_1' \ldots, \rho_s'$ where $\tau\rho_i'=\rho_i'$, and $2r+s=l$. Thus $K_f^*(G)=\bigwedge'' \otimes \tau^* \bigwedge'' \otimes \bigwedge'$, $\bigwedge''=\bigwedge(\rho_1, \ldots, \rho_r)$, $\bigwedge'=\bigwedge(\rho_1', \ldots, \rho_s')$. The image of p^*q^* is $\bigwedge(\rho_1-\tau\rho_1, \ldots, \rho_r-\tau\rho_r)$, an abelian group *direct summand* of $K_f^*(G)$. We note that $\tau^*[\rho_i']=[\rho_i']$ implies $q^*[\rho_i']=0$: for τ operates on G/U so that $q\tau=\tau q$ and $\tau q(gK)=q(gK)^{-1}=u(q(gK))$ where $u(g)=g^{-1}$ in G; thus $q^*[\rho_i']=q^*\tau^*[\rho_i']=q^*u^*[\rho_i']=-q^*[\rho_i']$ since $u^*(x)=-x$ for any primitive x.

Thus $q^*=0$ on $\bigwedge'$, $q^*\bigwedge''=q^*\tau^*\bigwedge''=\operatorname{Im} q^*$, p^*q^* maps $\bigwedge''$ isomorphically onto a direct summand, and $K_f^*(G/U)=\operatorname{Im} q^* \oplus \operatorname{Ker} p^*$. However, we assumed U nonhomologous to zero in G (over $\boldsymbol{Q}$) so p^* is 1-1, and $K_f^*(G/U)=\operatorname{Im} q^*=q^*\bigwedge'' \approx \bigwedge''$. Theorem 1 is now proved.

The argument used to prove Theorem 1 also applies in more general situations: let U be a closed subgroup of G, with G compact, connected, simply connected, and let $\rho_1, \rho_1', \rho_2, \rho_2' \cdots$ be representations of G such that $\rho_i=\rho_i'$ on U. Then if $\rho_i, \rho_i': G \to U(N_i)$, we can define maps $q_i: G/U \to U(N_i)$ by $q_i(gU)=\rho_i(g)\rho_i'(g)^{-1}$ and corresponding elements $[q_i] \in K^{-1}(G/U)$, $[\rho_i] \in K^{-1}(G)$. An example of this occurs when ρ_i is a representation of G, τ an automorphism of G which is the identity on U, and $\rho_i'=\rho_i \circ \tau$: then $q_i=\rho_i \circ q$ where $q: G/U \to G$ is as before.

If $p: G \to G/U$ is the projection, then $p^*[q_i]=[\rho_i]-[\rho_i'] \in K^*(G)$. With this notation, we now have:

PROPOSITION 2.1. *Let M be the submodule of $K_f^{-1}(G)$ generated by the elements $[\rho_i]-[\rho_i']$. Then $K_f^*(G/U)$ contains a subalgebra A isomorphic under p^* to $\bigwedge(M)$.*

Proof. Pick a basis $\{m_j\}$ of M, let $a_j \in K_f^{-1}(G/U)$, $p^*a_j=m_j$. A is generated by the a_j.

PROPOSITION 2.2. *Let τ be an outer automorphism of G of prime order (modulo inner automorphisms) which is the identity on U. Then*

1. *$K_f^*(G)=\bigwedge(M) \otimes \bigwedge(N)$ where $M, N \subset K_f^{-1}(G)$, τ is the identity on N and $M=\boldsymbol{Z}[\tau] \otimes M'$ is a free module over the group algebra $\boldsymbol{Z}[\tau]$ of τ.*

2. *$K_f^*(G/U)=A\otimes B$ where*

$$p^*: A \xrightarrow{\approx} ((\tau-1)\boldsymbol{Z}(\tau)\otimes M')$$

and B is an ideal.

To prove Proposition 2.2, we pick a basic representation ρ_i of G from each orbit of τ (acting on basic representations) and define an element $x_{i,j}$ of $K^{-1}(G/U)$ by $x_{i,j}=q^*[\tau^j\rho_i]$ for $1\leqq j\leqq(\text{order of }\tau)-1$. Then $p^*x_{ij}=[(\tau-1)\tau^j\rho_i]$. The $x_{i,j}$ generate the exterior algebra A, while B is the kernel of p^* followed by projection on $p^*(A)$.

An example for Proposition 2.1 is the Stiefel manifold SU $(n+k)$/SU (n): more generally,

PROPOSITION 2.3. *Let $G\supset U$ both be compact connected and simply-connected, and let the map of representation rings $R(G)\to R(U)$ be surjective. Then $K^*(G/U)$ is torsion free and is an exterior algebra generated by elements $[q_i]$ constructed from pairs (ρ_i,ρ_i') of representations of G which coincide on U.*

Proof. By Hodgkin's theorem $K^*(G)$ and $K^*(U)$ are both torsion free. The relation between $R(G)$ and $K^{-1}(G)$ is the following: if $\bar{R}(G)$ is the kernel of the rank homomorphism then $\bar{R}(G)/\bar{R}(G)^2$ is isomorphic to $PK^{-1}(G)$ under the sequence of homomorphisms:

$$\bar{R}(G)\xrightarrow{\alpha}\tilde{K}^0(B_G)\longrightarrow\tilde{K}^0(S(G))\xrightarrow{\delta^{-1}}K^{-1}(G)$$

using the standard map of the suspension $S(G)$ into B_G. The same things hold for U. Hence if $i^*: R(G)\to R(U)$ is surjective then so is $K^*(G)\to K^*(U)$. It follows that $K^*(G/U)\to K^*(G)$ is injective and has as image the exterior algebra on $M=\operatorname{Ker} i^*: PK^*(G)\to PK^*(U)$. Thus the proposition reduces to proving that if $m\in M$ then we can find representations ρ,ρ' of G which coincide on U such that $m=[\rho]-[\rho']$.

Let $m=\sum n_j[\rho_j]$, ρ_j being basic, and let

$$\mu=\sum\{n_j\rho_j\mid n_j>0\},\qquad \mu'=\sum\{n_j\rho_j\mid n_j<0\}.$$

By adding to μ or μ' suitably many copies of the trivial one-dimensional representation we may assume $\mu-\mu'\in\bar{R}(G)$. Also, since $i^*m=0$, we have $i^*\mu-i^*\mu'\in\bar{R}(U)^2$. Since $i^*R(G)=R(U)$, we also have $i^*\bar{R}(G)^2=\bar{R}(U)^2$, so we can find $\gamma\in\bar{R}(G)^2$ with $i^*(\gamma)=i^*(\mu-\mu')$. Writing $\gamma=\gamma_2-\gamma_1$ where the γ_i are representations of G and letting $\rho=\mu+\gamma_1$, $\rho'=\mu'+\gamma_2$ we have $m=[\rho]-[\rho']$ and $i^*\rho=i^*\rho'$, as required.

3. We shall prove Theorem 2 by looking at the classification of the symmetric pairs (G, U) such that G is simple, compact, and simply-connected, and U is totally nonhomologous to zero in G with rational coefficients. We will find in each case a central simple Jordan algebra J over $\boldsymbol{R}$ such that

(a) the Lie algebra D of derivations of J is the Lie algebra of U and consists of all linear transformations of the form $\sum_i [Ra_i, Rb_i]$ where $Ra_i(x)=xa_i$ all $x \in J$, and a_i, b_i are elements of J of trace 0,

(b) the Lie algebra $\mathfrak{G}_0$ of linear transformations of J of the form R_a+d, a of trace zero in J, $d \in D$, is a noncompact real form of the Lie algebra $\mathfrak{G}$ of G: i.e., $\mathfrak{G}=\{iR_a+d\} \subset \mathfrak{G}_0 \otimes_{\mathbf{R}} \mathbf{C}$. $\mathfrak{G}$ is also the Lie algebra of norm-preserving linear transformations in J.

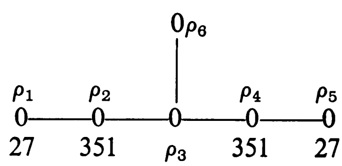
Any complex irreducible representation of $J \otimes_{\mathbf{R}} \mathbf{C}$ then determines a complex irreducible representation of $\mathfrak{G}$. We shall now list these:

(i) $G=\mathrm{SU}(2n+1)$, $U=\mathrm{SO}(2n+1)$. Here J is the Jordan algebra of all real symmetric matrices of $2n+1$ rows. $J \otimes_{\mathbf{R}} \mathbf{C}$ is defined in the same way but with *complex* matrices, and the representation is the obvious one on $\mathbf{C}^{2n+1}$. The representation ρ of $\mathrm{SU}(2n+1)$ is the standard one, and it is known that the basic representations can be taken as $\rho_i=\lambda^i\rho$, $1 \leqq i \leqq 2n$.

(ii) $G=\mathrm{SU}(2n)$, $U=\mathrm{Sp}(n)$. Here J is the Jordan algebra of all n-rowed quaternion hermitian matrices, $J \otimes_{\mathbf{R}} \mathbf{C}$ consists of all $2n$-rowed symplectic-symmetric matrices with complex coefficients. The representation of G obtained is the standard one on $\mathbf{C}^{2n}$, and the concluding remark of case (i) again applies.

(iii) $G=\mathrm{Spin}\,2n$, $U=\mathrm{Spin}\,2n-1$. J is the Jordan algebra of the vector space $\mathbf{R}^{2n-1}$ with the Euclidean inner product. The universal associative enveloping algebra $U_{\mathfrak{S}}(J)$ is the Clifford algebra $C_{2n-1}=C(\mathbf{R}^{2n-1})$ and the complex irreducible representation of Spin $2n$ is a half-spin representation.

(iv) $G=E_6$, $U=F_4$. Here J is the Jordan algebra of 3-rowed Hermitian matrices with Cayley number coefficients: its dimension over $\mathbf{R}$ is 27. The representation ρ of G obtained is the obvious representation on $J \otimes_{\mathbf{R}} \mathbf{C}$. Here the "rank" r is 2, and we will show that ρ, $\lambda^2\rho$, $\tau^*\rho$ and $\tau^*\lambda^2\rho$ are four of the basic irreducible representations of E_6, the remaining two representations being fixed under τ. We look at the Dynkin diagram of E_6, with dimensions written next to the vertices:



$\rho=\rho_1$ corresponds to the leftmost vertex. A simple observation of Dynkin's ([3, p. 347]) is that the highest weight vector occuring in $\lambda^2\rho_1$ is the highest weight vector of the next representation, ρ_2. Thus ρ_2 occurs in the decomposition of $\lambda^2\rho_1$ into irreducible parts; however, the dimensions of ρ_2 and of $\lambda^2\rho_1$ coincide, so $\rho_2=\lambda^2\rho_1$.

τ being a symmetry of the Dynkin diagram, it is clear that $\tau^*\rho_1=\rho_5$, $\tau^*\rho_2=\rho_4$, $\tau^*\rho_3=\rho_3$, $\tau^*\rho_6=\rho_6$ (ρ_6 is the adjoint representation).

This concludes the proof of Theorem 2.

It seems likely that a proof of Theorem 2 without use of the classification could be carried out by considering the Dynkin diagram of $\mathfrak{G}$, identifying the representation ρ as associated with an end vertex (e.g., ρ_1 for E_6), and, finally, discussing the exterior powers $\lambda^i\rho$ and the automorphism τ of the Dynkin diagram.

The classification used in the proof of Theorem 2 can be used to show that the map of representation rings $R(G)\to R(U)$ is surjective: this is easy to see if U is a classical group, whereas if $G=E_6$, $U=F_4$ then one can show that the representation ring of F_4 is generated by the 26 dimensional representation on J and the adjoint representation, and both of these are in the image of $R(E_6)\to R(F_4)$. Since (Sp n), (Spin $2n-1$), and F_4 are simply-connected groups their K^* groups are torsion free by Hodgkin's theorem, so that by Proposition 2.3, $K^*(G/U)$ is torsion free in cases (ii), (iii), and (iv) of the classification; similarly one can see that $K^*(\mathrm{SU}\,(2n+1)/\mathrm{SO}\,(2n+1))$ has no p-torsion for p odd, but it probably has 2-torsion.

Finally, one can show by a general argument without case considerations that $K^*(G/U)$ in all cases has no p-torsion for p odd (using again Hodgkin's theorem): one starts with the result from [11] or [6] that the composite map $p\circ q$:

$$G/U \xrightarrow{q} G \xrightarrow{p} G/U$$

induces an automorphism in cohomology with coefficients $\boldsymbol{Z}_p$, or, equivalently, an automorphism on $H^*(G/U;\boldsymbol{Z}(\tfrac{1}{2}))$, $\boldsymbol{Z}(\tfrac{1}{2})$ denoting the subring of Q generated by $\tfrac{1}{2}$. By the spectral sequence leading from $E_2=H^*$ to $E_\infty=\mathrm{Gr}\,(K^*\otimes \boldsymbol{Z}(\tfrac{1}{2}))$ one sees that p^* is 1-1 (and q^* is onto) on $K^*(G/U)\otimes \boldsymbol{Z}(\tfrac{1}{2})$, and since $K^*(G)\otimes \boldsymbol{Z}(\tfrac{1}{2})$ is torsion free, so is $K^*(G/U)\otimes \boldsymbol{Z}(\tfrac{1}{2})$.

4. Bott [8], [9] has described maps $B\colon G/U\to \Omega(L/G)$ if G/U and L/G are suitably related symmetric spaces. For our purposes we will need only the more elementary considerations of [4], which we repeat with a slight change in notation.

We will assume the following data: $U\subset G\subset L$ are compact Lie groups, with G and L connected. τ is an automorphism of G, and $v(t)$, $t\in \boldsymbol{R}$, is a one-parameter subgroup of L satisfying:

(i) $v(1)^{-1}gv(1)=\tau(g)$, all $g\in G$,

(ii) $v(t)^{-1}uv(t)=u$, all $u\in U$.

Define $B\colon G/U\to \Omega(L/G)$ by $B(gU)(t)=v(t)^{-1}gv(t)G$, $0\leqq t\leqq 1$. Now consider the fiber bundles

$$G\xrightarrow{j} L\xrightarrow{l} L/G:$$

from general principles we have maps

$$\begin{array}{ccccc}
\Omega(L/G) & \longrightarrow & P(L/G) & \longrightarrow & L/G\\
{\scriptstyle\omega}\downarrow & & \downarrow & & \|\downarrow\\
G & \longrightarrow & L & \longrightarrow & L/G
\end{array}$$

where the first row is the path space fibration on L/G. We wish to show that the composition $\omega \circ B\colon G/U \to \Omega(L/G) \to G$ coincides with q. It will suffice to construct maps $Q\colon \mathrm{c}(G/U) \to L$ (c denotes "cone") $b\colon s(G/U) = \mathrm{c}(G/U)/(G/U) \to L/G$ with b the adjoint of B, giving a commutative diagram:

$$\begin{tikzcd} G/U \arrow[r] \arrow[d, "q"'] & \mathrm{c}(G/U) \arrow[r] \arrow[d, "Q"'] & s(G/U) \arrow[d, "b"'] \\ G \arrow[r, "j"] & L \arrow[r, "l"] & L/G. \end{tikzcd}$$

The homotopy class of b determines that of q. Similarly we have:

$$\begin{tikzcd} G/U \arrow[r] \arrow[d, "B"'] & \mathrm{c}(G/U) \arrow[r] \arrow[d] & s(G/U) \arrow[d, "b"'] \\ \Omega(L/G) \arrow[r] \arrow[d, "\omega"'] & P(L/G) \arrow[r] \arrow[d] & L/G \arrow[d, "\|"'] \\ G \arrow[r, "j"] & L \arrow[r, "l"] & L/G. \end{tikzcd}$$

The homotopy class of b determines that of $\omega \circ B$, hence $\omega \circ B$ is homotopic to q.

We define

$$Q(gU, t) = v(t)^{-1}gv(t)\tau(g)^{-1}, \qquad b(gU, t) = v(t)^{-1}gv(t)G.$$

Note that

$$\begin{aligned} Q(gU, 0) &= q(gU), \\ Q(gU, 1) &= v(1)^{-1}gv(1)v(1)^{-1}g^{-1}v(1) = e, \\ Q(eU, t) &= e. \end{aligned}$$

Thus Q, b satisfy the requirements. We now have: $q^* = B^*\omega^*\colon K_f^*(G) \to K_f^*(\Omega(L/G)) \to K_f^*(G/U)$. If G, U are as in Theorem 1, then $K_f^*(G)$ contains a subalgebra mapped isomorphically by q^*, and Theorem 3 follows.

One might ask whether there is an analogue for K^* of the following result on the cohomology of the spaces defined above:

Proposition 4.1. *Let p be an odd prime. Then $H^*(\Omega(L/G); Z_p) \approx H^*(\Omega(L/U); Z_p) \otimes H^*(G/U; Z_p)$ as algebras.*

Proof. Consider the principal fiber space

$$\Omega(L/U) \longrightarrow \Omega(L/G) \xrightarrow{\pi} G/U$$

here π is the composition of $\omega\colon \Omega(L/G) \to G$ and the natural map $G \to G/U$. We also have the map $B\colon G/U \to \Omega(L/G)$. Then $\pi \circ B$ is just the composition $q\colon G/U \to G \to G/U$ which induces an isomorphism in cohomology with coefficients Z_p for p odd [6, p. 489].

In the case that G and U are associated to a Jordan algebra J as in §3, a specific group L and a suitable one-parameter subgroup $v(t)$ can be described in terms of J, as shown by Kumpel [6] using a construction due to Tits [7].

5. We shall now apply the preceding to fiber bundles.

As always, G will be compact, connected and simply-connected, U a closed connected subgroup. We will say that $K_f^*(G/U)$ is generated by representations if there are pairs (ρ_j, ρ_j') of representations of G that coincide on U, such that the corresponding elements $[q_j] \in K^{-1}(G/U)$ (constructed as in §2) together with elements of $K^0(G/U)$ associated with representations of U, together generate the algebra $K_f^*(G/U)$.

If $E \to E/G$ is a principal G-bundle we denote by α the usual homomorphism $R(G) \to K^0(E/G)$.

Proposition 5.1. *Let $E \to E/G$ be a principal G-bundle over a compact base space and let*

$$G/U \xrightarrow{i} E/U \xrightarrow{\pi} E/G$$

be the associated G/U-bundle. Suppose that $K_f^*(G/U)$ is generated by representations and that for each of the generators $[q_j]$ defined by (ρ_j, ρ_j') we have $\alpha(\rho_j)-\alpha(\rho_j')=0$ in $K^0(E/G)$. Then $i^*: K_f^*(E/U) \to K_f^*(G/U)$ is surjective. If further $K^*(G/U)$ is torsion-free then $K^*(E/U) \approx K^*(E/G) \otimes K^*(G/U)$ as left $K^*(E/G)$-module.

Proof. Clearly, it suffices to show that i^* is surjective, and since the generators of $K_f^*(G/U)$ obtained from representations of U are in the image of i^*, we only have to prove the same thing for the $[q_j] \in K_f^{-1}(G/U)$. We follow the ideas of [2, p. 121]. Let B be the mapping cylinder of $\pi: E/U \to E/G$, and $B_1 = E/U \subset B$. Call the inclusion $B_1 \subset B$ again π. Similarly, let C be the mapping cylinder of $\pi_0: G/U \to$ pt. (projection into a single point), and $C_1 = G/U \subset C$. Call the inclusion $C_1 \subset C$ again π_0:

$$\begin{array}{ccc} C_1 = G/U & \xrightarrow{\pi_0} & C = C(G/U) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle C(i)} \\ B_1 = P/U & \xrightarrow{\pi} & B \end{array}$$

is a commutative diagram, giving rise to:

$$\begin{array}{ccccccc} \tilde{K}^0(C_1) & \longleftarrow & \tilde{K}^0(C) & \xleftarrow{j^*} & K^0(C, C_1) & \xleftarrow{\delta} & K^{-1}(C_1) \\ \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle i^*} \\ \tilde{K}^0(B_1) & \longleftarrow & \tilde{K}^0(B) & \xleftarrow{j^*} & K^0(B, B_1) & \xleftarrow{\delta} & K^{-1}(B_1). \end{array}$$

Let ρ, ρ' be representations of G that coincide on U. Let $\alpha(\rho)$, $\alpha(\rho')$ be the associated vector bundles over B (using the homotopy equivalence of B with $P(G)$). The restrictions to $B_1=E/U$ of these bundles are isomorphic, and by the clutching construction define an element $\alpha(\rho, \rho')$ of $K^0(B, B_1)$ which satisfies

$$j^*\alpha(\rho, \rho') = [\alpha(\rho)]-[\alpha(\rho')] \in \tilde{K}^0(B).$$

We will shortly make this clutching function more explicit.

If furthermore the right-hand side of this last equation vanishes, then

$$\alpha(\rho, \rho') = \delta\xi \quad \text{for some} \quad \xi \in K^{-1}(B_1).$$

Restricting to C and $C_1=G/U$, $i^*\xi \in K^{-1}(G/U)$ satisfies

$$\delta i^*\xi = i^*\delta\xi = i^*\alpha(\rho, \rho') \in K^0(c(G/U), G/U).$$

It remains to show that $i^*\alpha(\rho, \rho')$ is determined by the clutching function q on G/U, where $q(gU)=\rho(g)\rho'(g)^{-1}$: if we know this, then under the isomorphism $\delta\colon K^{-1}(G/U)\to K^0(c(G/U), G/U)$, $[q]$ is sent into $i^*\alpha(\rho, \rho')$ so $i^*\xi=[q]$. Let $\rho_1=\rho$, $\rho_2=\rho'$ be the representations of G on a vector space V.

Consider now

$$\begin{array}{ccc} G/U & \xrightarrow{i} & P/U \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi} \\ \text{pt.} & \xrightarrow{i} & P/G. \end{array}$$

The vector bundles $\alpha(\rho_j)$ over P/G can be written as $P\times_{\rho_j} V\to P/G$. Lifting to P/U by π^*, $\pi^*(\alpha(\rho_j))$ has as total space the set

$$\{(p_1U, p_2\times_{\rho_j} v) \mid p_1G = p_2G; p_k \in P, v\in V\}.$$

A specific isomorphism ϕ_j of $P\times_U V$ with this vector bundle:

$$P\times_U V \xrightarrow{\phi_j} \pi^*(\alpha(\rho_j))$$

is given by

$$(p\times_U v) \xrightarrow{\phi_j} (pU, p\times_{\rho_j} v)$$

with inverse:

$$(p_1\times_U \rho_j(g)v) \xleftarrow{\phi_j^{-1}} (p_1U, p_1g\times_{\rho_j} v).$$

We fix p_0 in P and identify $i\colon G/U\to P/U$ with the map $gU\to p_0gU$. $\pi_0^*i^*\alpha(\rho_j)$ $=i^*\pi^*\alpha(\rho_j)$ has the total space

$$\{(g_1U, g_2\times_{\rho_j} v)\} = (G/U)\times V$$

since $G\times_{\rho_j} V=V$, $(g\times_{\rho_j} v)=\rho_j(g)v$, and ϕ_j^{-1} restricted to this space

$$i^*\pi^*\alpha(\rho_j) \xrightarrow{\phi_j^{-1}} G\times_U V$$

is given by

$$(gU, v) \xrightarrow{\phi_j^{-1}} (g \times_U \rho_j(g)^{-1}v)$$

and

$$(gU, \rho_j(g)v) \xleftarrow{\phi_j} (g \times_U v).$$

The clutching isomorphism $\pi^*\alpha(\rho_2) \to \pi^*\alpha(\rho_1)$ over P/U is $\phi_1\phi_2^{-1}$. The restriction of this to G/U is then

$$(G/U) \times V \xrightarrow{\phi_2^{-1}} G \times_U V \xrightarrow{\phi_1} (G/U) \times V,$$

$$(gU, v) \to (g \times_U \rho_2(g)^{-1}v) \to (gU, \rho_1(g)\rho_2(g)^{-1}v) = (gU, q(gU)v).$$

This concludes the proof of the proposition, and Theorem 4 is an immediate consequence.

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