UNIFORM DISTRIBUTION IN LOCALLY COMPACT ABELIAN GROUPS(1)

BY

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Introduction. The notion of a uniformly distributed sequence on a compact group was introduced and studied by Eckmann in [1], and has since been the object of much research. Recently, this notion was extended to locally compact groups by Rubel in [9]. Our principal interest here is in the following two questions. First, when does a locally compact Abelian group G possess a uniformly distributed sequence? Second, when is there an associated compact group $G^{\sim}$ and a continuous homomorphism $\varphi: G \to G^{\sim}$, such that $\varphi(G)$ is dense in $G^{\sim}$, with the property that $\{g_\nu\}$ is uniformly distributed in G if and only if $\{\varphi(g_\nu)\}$ is uniformly distributed in $G^{\sim}$? We call such a group $G^{\sim}$ a D-compactification of G. We obtain solutions to these and related problems.

In this introduction, we present various definitions and preliminary results and describe briefly the main results of the paper. In §1, we prove some structural lemmas. In §2, we consider the existence of uniformly distributed sequences on G. Finally, in §3, we consider the existence of D-compactifications of G.

By G, we will denote a locally compact Abelian group. We will, in general, assume that all groups are locally compact Abelian (abbreviated LCA) groups. We do not require that our groups be first-countable. Indeed, even when we start with first-countable groups G, we will be forced to consider certain "compactifications" which will rarely be first-countable. Throughout this paper, we shall assume the continuum hypothesis, denoting by c the cardinal number of the continuum. If A is a set, we denote the cardinal number of A by card A.

Characters of G are supposed to be continuous characters unless otherwise stated. We will denote the trivial character by 1. The dual group of G will be denoted by Γ or by $G^{\wedge}$.

If G is an LCA group, we will call a compact group H a *compactification* of G if there is a continuous homomorphism from G onto a dense subgroup of H. For example, $\bar{G}$, the Bohr compactification of G, (defined as the dual of Γ_d, where

Received by the editors January 24, 1967 and, in revised form, May 2, 1967.

(1) A preliminary report on this paper was presented as an address by the third author to the Österreichischen Mathematischen Gesellschaft on March 25, 1966.

(2) The research of the first two authors was partially sponsored by the National Science Foundation under grant NSF GP-5370. The research of the third author was sponsored in part by the Air Force Office of Scientific Research under grant number AFOSR 460-63, and also in part by a Senior Postdoctoral Fellowship from the National Science Foundation.

Γ_d is Γ in the discrete topology) is a compactification of G. The trivial group $\{0\}$ is a compactification of any LCA group G.

In order to give a canonical construction for all compactifications of an LCA group G, and to avoid isomorphism problems, we construct each compactification H as the compact dual of a subgroup of Γ_d. Hence when a character of H occurs we freely identify it with a character of G.

When an LCA group H is a quotient group of G, or if H contains such a quotient group, in such a way that there is a distinguished homomorphism from G into H, we will use the notation φ_H for this homomorphism, or simply φ if there is no possibility of confusion. Sometimes, by a slight abuse of notation, if H is a closed subgroup of G, we write φ_H rather than $\varphi_{G/H}$.

A subgroup H of G is said to be of *compact index* if G/H is compact. A function on G is said to be *periodic* if it is constant on the cosets of some subgroup of compact index. It is easy to see that a character χ is periodic if and only if $G/(\ker \chi)$ is compact. We denote the collection of periodic characters by Γ^p. It is not always the case (see §1) that Γ^p is a subgroup of Γ. By $\bar{G}^p$ we denote the compact dual of the subgroup of Γ_d generated by Γ^p, and call $\bar{G}^p$ the *periodic compactification* of G. It is easy to see that $\bar{G}^p$ is actually a compactification of G and that $\bar{G}^p$ is always a quotient of $\bar{G}$.

For example, if $G=R$ is the real numbers in the usual topology, we see that $\bar{G}^p=\bar{G}$ since every character of R is periodic. If, on the other hand, $G=R_d$, the reals in the discrete topology, then $\bar{G}^p=\{0\}$, since 1 is the only periodic character. However in this case, $\bar{G}$ is a decidedly nontrivial compact group, being the dual of the discrete group consisting of all characters of R, whether continuous or not in the usual topology. If $G=Z$, the integers in the discrete topology, then the periodic characters form a group. In this case $\bar{G}^p$ is smaller than the Bohr compactification —namely $\bar{G}^p$ is the so-called universal monothetic Cantor group. That is, $\bar{G}^p$ is the compact dual of the discrete group of rational numbers modulo 1.

DEFINITION. Let G be a group, f a complex-valued function on G, and $\{g_\nu\}$ a sequence of elements of G. By $\langle f, \{g_\nu\}\rangle$ we denote the following limit, if it exists:

$$\langle f, \{g_\nu\}\rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} f(g_\nu).$$

Uniform distribution in *compact* groups has been the object of a great deal of study. The interested reader is referred to Volume 16 of Compositio Mathematica which is entirely devoted to uniform distribution, or our references [1] and [3], for access to the literature. We choose as our definition of uniform distribution on a compact group a characterization which is due to Eckmann. A standard alternative definition can be found on p. 253 of [9] where it is referred to as a proposition of Eckmann's.

DEFINITION (ECKMANN). Let G be a compact Abelian group. We say that the

sequence $\{g_\nu\}$ of elements of G is *uniformly distributed* in G if for each continuous complex-valued function f on G, $\langle f, \{g_\nu\}\rangle = \int f\, d\mu$, where μ is Haar measure on G.

If we write $\langle f, \{g_\nu\}\rangle = a$, it is understood that $\langle f, \{g_\nu\}\rangle$ exists. We use the abbreviation u.d. for *uniformly distributed.*

With this definition the following basic theorem, which can be found in [1], is immediate.

THEOREM (WEYL CRITERION). *If G is a compact Abelian group, then $\{g_\nu\}$ is uniformly distributed in G if and only if, for each nontrivial character χ, $\langle \chi, \{g_\nu\}\rangle = 0$.*

DEFINITION (RUBEL [9]). Let G be an LCA group. We say that the sequence $\{g_\nu\}$ of elements of G is *uniformly distributed* in G if for each subgroup H of compact index in G, $\{\varphi_H(g_\nu)\}$ is uniformly distributed in G/H.

The Weyl criterion now clearly assumes the following form:

THEOREM (WEYL CRITERION). *If G is an LCA group then $\{g_\nu\}$ is uniformly distributed in G if and only if for each nontrivial periodic character χ, $\langle \chi, \{g_\nu\}\rangle = 0$.*

If $G = R$, then $\{g_\nu\}$ is u.d. if and only if $\{\varphi(g_\nu)\}$ is u.d. in $\bar{R}$, the Bohr compactification of R. This is the case if and only if $\{tg_\nu\}$ is u.d. mod 1 for each $t \in R$, $t \neq 0$. If $G = R_d$ then, trivially, any sequence $\{g_\nu\}$ is u.d. If $G = Z$, then $\{g_\nu\}$ is u.d. if and only if $\{\varphi(g_\nu)\}$ is u.d. in $\bar{G}^p$. However, $\{g_\nu\}$ may be u.d. in Z even though $\{\psi(g_\nu)\}$ is not u.d. in $\bar{G}$, even though both φ and ψ are one-to-one. It is easy to see (in accordance with a definition of Niven [6]) that $\{g_\nu\}$ is u.d. in Z if and only if the g_ν fall into each arithmetic progression with limiting frequency equal to the arithmetic density of the progression.

We will often use, sometimes without special reference, the following two well-known results. The first is the structure theorem for LCA groups [10, Theorem 2.4.1] and the second is a special case of a theorem of Kakutani [5].

STRUCTURE THEOREM. *Every LCA group G has an open subgroup G_1 of the form $G_1 = H \times R^n$ where H is compact and R^n is Euclidean n-space in the usual topology; G_1 is also closed. Furthermore, G/G_1 is discrete.*

THEOREM A (KATUTANI). *A compact Abelian group G is separable (that is, there exists a dense sequence in G) if and only if* card $\Gamma \leqq c$.

DEFINITION. An LCA group G is *K-separable* if there exists a sequence $\{g_\nu\}$ in G such that, for every subgroup H of compact index in G, the sequence $\{\varphi_H(g_\nu)\}$ is dense in G/H.

We may now describe briefly the main results of this paper.

A necessary and sufficient condition that the periodic characters of G form a group is that either every discrete quotient of G be of bounded order (which happens precisely when *every* character of G is periodic) or G be totally disconnected.

Next, G admits a u.d. sequence if and only if G is K-separable. This is not an

entirely satisfactory result. The following result, although not quite complete, is perhaps more satisfactory: If card $\Gamma^p \leqq c$, then G admits a u.d. sequence. If Γ^p is a group and G admits a u.d. sequence, then card $\Gamma^p \leqq c$. For any cardinal n, there exists a group G such that card $\Gamma^p \geqq n$ yet such that G admits a u.d. sequence.

Finally, there is the question of the existence and uniqueness of a D-compactification of G. Let us make the mild assumption that G admits a u.d. sequence, since otherwise the subject of D-compactifications becomes trivial. Under this assumption, G has a D-compactification if and only if Γ^p is a group, and if Γ^p is a group then there is a unique D-compactification, namely $\bar{G}^p$. In this case, the suitability of $\bar{G}^p$ is clear; the proof of uniqueness requires some work.

We remark that Rajagopalan and Rotman [7], have characterized those discrete groups that admit a u.d. sequence of the form $\{ng\}$, $n=1, 2, 3, \ldots$, and that Rajagopalan [8] has recently extended this result to arbitrary LCA groups.

1. The structure of Γ^p.

LEMMA 1.1. *Let G be an LCA group and let G_0 denote the identity component of G. Let χ be a character of G and let H denote the kernel of χ. If χ does not annihilate G_0, then $G/H=T$, the circle group in the usual topology, and χ is periodic. If χ annihilates G_0 then G/H is discrete, and in this case χ is periodic if and only if the range of χ is finite.*

Proof. We first note that χ induces a one-to-one continuous homomorphism $\chi^\sim$ from G/H into T; indeed if $g^\sim$ is the image in G/H of g in G then we define $\chi^\sim(g^\sim) = \chi(g)$.

Suppose that $G_0 \nsubseteq H$. Then G/H has a nontrivial connected component of the identity, G_1. If G/H is connected then the range of $\chi^\sim$ is T since the image of G/H must be connected. Because G/H is connected it is easily seen to be σ-compact and therefore $\chi^\sim$ must be open [4, Theorem 5.29], and hence G/H is (homeomorphically) isomorphic to T. In general, if G/H is not totally disconnected then G_1 must map isomorphically onto a connected subset of T and hence $G/H=G_1=T$ as above. Therefore χ is periodic.

Now suppose that $G_0 \subseteq H$. Then G/H is totally disconnected; moreover since every neighborhood of 0 in G/H contains an open subgroup and since there is a neighborhood of 0 in T that contains no nontrivial subgroup it follows that G/H is discrete. Hence G/H is compact and χ is periodic if and only if the range of χ is finite.

This completes the proof of Lemma 1.1.

LEMMA 1.2. *Let G be an LCA group. Then all the characters of G are periodic if and only if every discrete quotient of G is of bounded order.*

Proof. We have seen that if there is a nonperiodic character χ in Γ, then G/H is discrete and infinite. Since G/H is algebraically a subgroup of T, there must be

elements of G/H of arbitrarily high (possibly infinite) order. On the other hand, if G has a discrete quotient K that is not of bounded order, then there is a character χ of K whose range is infinite. This is not difficult to see—it follows by a simple category argument. But we may consider, in the obvious way, χ as a character on G, and we see that $G/(\ker \chi)$ is therefore discrete and infinite, since $G/(\ker \chi)$ is the same as $K/(\ker \chi)$, so that χ is not periodic. This completes the proof of Lemma 1.2.

THEOREM 1. *Let G be an LCA group. Then the periodic characters of G form a group if and only if either every discrete quotient of G is of bounded order, or else G is totally disconnected.*

Proof. If G is totally disconnected, then by Lemma 1.1, Γ^p is the set of characters whose range is finite, and these clearly form a group. If every discrete quotient of G is of bounded order, then by Lemma 1.2, every character is periodic so that the periodic characters again form a group.

Suppose now that G has a nontrivial connected component of the identity, say G_0, and a nonperiodic character χ_1. Then by Lemma 1.1, χ_1 must annihilate G_0. Now, since the characters on G separate the elements of G, we may choose a character χ_2 which does not annihilate G_0 and therefore must be periodic. Then $\chi_1\chi_2$ does not annihilate G_0 and hence $\chi_1\chi_2$ is periodic. But then $(\chi_1\chi_2)(\chi_2)^{-1}$ is a nonperiodic product of periodic characters, and our proof of Theorem 1 is done.

2. **The existence of uniformly distributed sequences.** We begin with a computational lemma, which is somewhat similar to some results for compact groups; for example, Hlawka's Theorem 7 of [3].

LEMMA 2.1. *Let G be an LCA group and let Φ be a subset of $\Gamma = G^\wedge$. Let H be a countable subgroup of G that separates the nontrivial characters in Φ from 1. That is, for each nontrivial character χ in Φ there is an element h in H such that $\chi(h) \neq 1$. Then there is a sequence $\{k_\nu\}$ of elements of H such that for each nontrivial character χ in Φ, we have $\langle \chi, \{k_\nu\} \rangle = 0$.*

Proof. Let $H = \{h_\nu\}$, $\nu = 1, 2, \ldots$, and consider the following sequence $\{k_\nu\}$:

$$h_1, h_1h_2, h_1^2h_2, h_1h_2^2, h_1^2h_2^2, h_1h_2^3, h_1^2h_2^3, h_1h_2^4, h_1^2h_2^4, h_1h_2h_3, h_1^2h_2h_3,$$
$$h_1^3h_2h_3, h_1h_2^2h_3, h_1^2h_2^2h_3, h_1^3h_2^2h_3, h_1h_2^3h_3, h_1^2h_2^3h_3, h_1^3h_2^3h_3, h_1h_2^4h_3,$$
$$h_1^2h_2^4h_3, h_1^3h_2^4h_3, \ldots, h_1^3h_2^9h_3, \ldots, h_1^3h_2^9h_3^{27}, h_1h_2h_3h_4, \ldots,$$
$$h_1^nh_2^{n^2} \cdots h_n^{n^n}, \ldots.$$

To describe this construction, consider the block that begins with $h_1h_2 \cdots h_n$ and ends with $h_1^nh_2^{n^2} \cdots h_n^{n^n}$. The exponent of h_1 runs in order through the numbers from 1 to n and then cycles. The exponent of h_2 runs through the numbers from 1 to n^2 in blocks of n and then cycles. In general, for $j = 1, \ldots, n$, the exponent of h_j runs through the numbers from 1 to n^j in blocks of n^{k_j}, with $k_j = j(j-1)/2$, and then cycles.

We give a formula for finding the pth term of this block where p runs from 0 to $n^{n(n-1)/2}-1$. We write

$$\begin{aligned} p &= p_1 n+(q_1-1) && \text{where } 1 \leqq q_1 \leqq n, \\ p_1 &= p_2 n^2+(q_2-1) && \text{where } 1 \leqq q_2 \leqq n^2, \\ &\vdots \\ p_{n-1} &= p_n n^n+(q_n-1) && \text{where } 1 \leqq q_n \leqq n^n. \end{aligned}$$

Then the pth term of this block is $h_1^{q_1} h_2^{q_2} \cdots h_n^{q_n}$. We then begin the next block with $h_1 h_2 \cdots h_{n+1}$.

We will show that this sequence has the required properties. Let χ be a fixed nontrivial character in Φ, and let h_ν be the first element of H on which $\chi \neq 1$. After a finite number of terms, we can group the above sequence in successive blocks, in each term of which all the factors with indices exceeding ν are fixed. In each such block, the exponents of h_ν run from 1 to p^ν for appropriate p. Let us observe that for $p>\nu$, there are several such blocks corresponding to p. We will designate such a block by B_p. Note that p is nondecreasing. Each exponent is repeated $p \cdot p^2 \cdot \cdots \cdot p^{\nu-1}$ times in each B_p for a total block length of p^λ terms, where $\lambda = \nu(\nu+1)/2$. Over this block B_p, we have

$$\frac{1}{p^\lambda} \sum_{j \in B_p} \chi(k_j) = \frac{1}{p^\lambda} \sum_{r=1}^{p^\nu} (p \cdot p^2 \cdot \cdots \cdot p^{\nu-1}) \chi(h_\nu^r) \chi(m)$$

for some fixed m. Hence

$$\lim_{p \to \infty} \frac{1}{p^\lambda} \sum_{j \in B_p} \chi(k_j) = 0.$$

Since the length of B_p divided by the number of terms preceding B_p approaches 0 as the block lengths grow larger, and since $|\chi|=1$, we see that $\langle \chi, \{k_j\} \rangle = 0$.

This completes the proof of Lemma 2.1. We will not use the next lemma until the proof of Theorem 4 of §3.

LEMMA 2.2. *Let G be an LCA group, let Φ be a subgroup of $\Gamma = G^\wedge$, and let $\chi \in \Gamma \backslash \Phi$. That is, $\chi \in \Gamma$ but $\chi \notin \Phi$. Suppose there is a sequence $\{x_\nu\}$ in G such that for each nontrivial $\gamma \in \Phi$ we have $\langle \gamma, \{x_\nu\} \rangle = 0$. Then if Ψ is the subgroup of Γ generated (algebraically) by Φ and χ, there is a sequence $\{y_\nu\}$ in G such that for each nontrivial $\gamma \in \Psi$, we have $\langle \gamma, \{y_\nu\} \rangle = 0$.*

Proof. First, we note that for each nontrivial $\gamma \in \Phi$ there is an x_ν such that $\gamma(x_\nu) \neq 1$. Since Φ is a group, for each γ_1 and γ_2 in Φ there is an x_ν such that $\gamma_1(x_\nu) \neq \gamma_2(x_\nu)$. Hence for all $\gamma \in \Phi$ except at most one, there is an x_ν such that $(\gamma\chi)(x_\nu) \neq 1$. By adjoining one more element w_1, say, to $\{x_\nu\}$ we obtain a set separating each character of the form $\gamma\chi$ from 1. That is, for each $\gamma \in \Phi$, either there is some x_ν so that $(\gamma\chi)(x_\nu) \neq 1$ or else $(\gamma\chi)(w_1) \neq 1$. Proceeding similarly by adjoining elements w_n, we separate all characters of the form $\gamma\chi^n$ from 1, where $n=0, \pm 1, \pm 2, \ldots$. By

Lemma 2.1, we may arrange a subset of the group generated by $\{x_\nu\}$ and $\{w_n\}$ into a sequence $\{y_\nu\}$ such that for each nontrivial γ in Ψ, we have $\langle\gamma, \{y_\nu\}\rangle = 0$, and Lemma 2.2 is proved.

THEOREM 2. *The LCA group G admits a uniformly distributed sequence if and only if G is K-separable.*

Proof. Suppose first that G admits the u.d. sequence $\{g_\nu\}$. Then $\{g_\nu\}$ is K-dense in G, in the sense that $\{\varphi_H(g_\nu)\}$ is dense in G/H for every subgroup H of compact index, since $\{\varphi_H(g_\nu)\}$ is u.d. in G/H and we recall that in a compact group a u.d. sequence must be dense. Next, suppose that G admits a K-dense sequence $\{g_\nu\}$. Now apply Lemma 2.1 with the choices $\Phi = \Gamma^p$ and H as the subgroup of G generated by $\{g_\nu\}$. The sequence $\{k_\nu\}$ whose existence is asserted by that lemma is u.d. in G by the Weyl criterion. This completes the proof of Theorem 2.

The next result is perhaps more interesting. We recall that Theorem 1 gives necessary and sufficient conditions that Γ^p be a group.

THEOREM 3. *Let G be a locally compact Abelian group. If* card $\Gamma^p \leqq c$ *then G admits a uniformly distributed sequence. If Γ^p is a group and G admits a uniformly distributed sequence, then* card $\Gamma^p \leqq c$. *There exists, for any cardinal number n, an LCA group G that admits a uniformly distributed sequence, but such that* card $\Gamma^p \geqq n$.

Before we begin the proof of Theorem 3 some remarks may be appropriate. We will first show that if card $\Gamma^p \leqq c$, then G admits a u.d. sequence. Our proof is based on the structure theorem. At first inspection, the following approach seems promising. From the hypotheses, and Theorem A, we know that there is a dense sequence in $\bar{G}^p$. If we could choose this dense sequence as the image of a sequence in G, we could then apply Lemma 2.1 and the Weyl criterion to get a sequence that is u.d. in G. Unfortunately this promising line does not seem to work, since it is possible for a dense subgroup of a separable group to be nonseparable.

Expanding a little on this theme, suppose we let $G = \prod Z_2$ be the product of continuum many copies of the two-element group Z_2 in the product topology, and let $H = \sum Z_2$ be the sum in the discrete topology. Now H plays two roles for us. First, we consider H as a group in its own right, embedded densely in G in the obvious manner; that is, G is a compactification of H. Second, we regard H as the dual of G and write $H = G^\wedge$. Now G has only c characters, yet there is no sequence of elements in H that separates from 1 all the nontrivial characters of G. In particular, there is no sequence in H that is u.d. in G. So we see that our result depends on the particular set of characters chosen, namely the periodic ones. Our actual proof uses the next lemma.

LEMMA 2.3. *Let G be discrete, and suppose that* card $\Gamma^p \leqq c$. *Then G admits a uniformly distributed sequence.*

Leaving the proof until later, we give the proof of Theorem 3.

Proof of Theorem 3. We first show that if card $\Gamma^p \leqq c$ then G admits a u.d. sequence. By the structure theorem, G contains a closed open subgroup $H = K \times R^n$, where K is compact and $G/H = M$ is discrete. We first note that card $(M^\wedge)^p \leqq c$ since a periodic character on M has finite range, and hence considered as a character on G is periodic. By Lemma 2.3 and the Weyl criterion, there is therefore a sequence $\{m_\nu\}$ in M that separates from 1 the nontrivial periodic characters of M. We choose $\{x_\nu\}$ in G so that $\varphi(x_\nu) = m_\nu$: then $\{x_\nu\}$ separates from 1 those nontrivial periodic characters of G that annihilate H.

Now consider Q, the discrete group of characters of H generated by the images in $H^\wedge$ of members of Γ^p, and note that card $Q \leqq c$. Hence $Q^\wedge$, the compact dual of Q, is separable by Theorem A. Each character on H has the form $\chi_1 \cdot \chi_2$, where $\chi_1 \in K^\wedge$ and $\chi_2 \in (R^n)^\wedge = R^n$; it follows that $Q^\wedge$ is a quotient group of $K \times B^n$, where B^n is the Bohr compactification of R^n. We now note that the rational points in R^n are dense in B^n. Hence we may choose the separating sequence in $Q^\wedge$ to be the images of elements of the form $k \times r$, where $k \in K$ and $r \in R^n$. Thus there is a sequence $\{h_\nu\}$ in H which separates from 1 all the nontrivial periodic characters on G that do not annihilate H. Hence we have a countable set in G that separates Γ^p from 1, and so by Lemma 2.1 and the Weyl criterion, we can construct from it a u.d. sequence in G.

Suppose now that Γ^p is a group and that G admits a u.d. sequence. Then this sequence maps into a u.d. sequence in $\bar{G}^p$ since $\bar{G}^p$ has the same periodic characters as G. In particular, since $\bar{G}^p$ is compact, the image of the sequence is a dense sequence in $\bar{G}^p$, and by Theorem A, we see that card $(\bar{G}^p)^\wedge \leqq c$. Since $(\bar{G}^p)^\wedge = \Gamma^p$, this proves the second assertion.

For the required example, we choose the group H as a product of so many copies of R_d, the reals in the discrete topology, that card $H^\wedge$ is as great as we please and so that H has no nontrivial periodic characters. Let T be the circle group. Then $G = T \times H$ has as periodic characters those characters of the form $e^{in\theta} \times \chi$, where $n \in Z$, $n \neq 0$, and χ is any character of H. By Lemma 1.1, these characters are periodic because they do not annihilate T, the connected component of the identity of G. It is easy to see that no other characters are periodic.

Any character of the form $1 \times \chi$ is not periodic, so the periodic characters do not form a group since $1 \times \chi = (e^{i\theta} \times \chi)(e^{-i\theta} \times 1)$. Now choose $\{y_\nu\}$ a sequence that is u.d. in T, and let $x = y \times 0$. Then $\{x_\nu\}$ is u.d. in G since all nontrivial periodic characters of G restrict to nontrivial characters of T. Thus G has as many periodic characters as required and yet admits a u.d. sequence. This completes the proof of Theorem 3.

We now must prove Lemma 2.3 which was used in the proof of Theorem 3. We remark that the converse of Lemma 2.3 also holds, since the periodic characters of a discrete group form a group, and by Theorem 3 the cardinality of this group of characters does not exceed c.

Proof of Lemma 2.3. First, $\chi \in \Gamma^p$ if and only if $G/(\ker \chi)$ is finite. If $G/(\ker \chi)$

has n elements, then χ may be considered as a character of G/nG. Now if card $G/nG \geqq c$ then there are at least 2^c characters on G/nG ([5]), each of which, by Lemma 1.2, represents a distinct periodic character on G. Since this is impossible, we have that card $(G/nG) \leqq \aleph_0$. Now let $(g_{n,1}, g_{n,2}, \ldots)$ be elements in G that map into $(\varphi(g_{n,1}), \varphi(g_{n,2}), \ldots,)$, the distinct elements of G/nG, and let H be the group generated by $\{g_{n,r} : r=1, 2, \ldots; n=1, 2, \ldots\}$. Then $\varphi(H)$ separates from 1 the nontrivial characters in $\bar{G}^p$, and hence by Lemma 2.1 and the Weyl criterion, we may construct a sequence $\{h_\nu\}$ from H which is u.d. in G. This completes the proof of Lemma 2.3.

S. Hartman has introduced in [2] the following notion of uniform distribution in an LCA group G; we call it *Hartman uniform distribution* and use the abbreviation H.u.d.

DEFINITION. The sequence $\{g_\nu\}$ of elements of G is uniformly distributed in the sense of Hartman if $\langle \gamma, \{g_\nu\}\rangle = 0$ for each nontrivial $\gamma \in \Gamma$.

It is immediate that a sequence is H.u.d. in G if and only if its image in $\bar{G}$, the Bohr compactification of G, is u.d. there. There are certain disadvantages to this notion of uniform distribution. In case $G=R$, the notions of H.u.d. and u.d. are easily seen to coincide, but in case $G=Z$, they are different. In view of Niven's paper [6], the notion of u.d. seems more natural. We observe in passing that Lemma 1.2 gives necessary and sufficient conditions on a group G that the notions of u.d. and H.u.d. coincide, namely that every discrete quotient of G be of bounded order. In particular, by Lemma 1.1, they coincide for all connected groups.

Hartman proves (Theorem 3 of [2]) the following result. (We recall that the weight of a topological space is defined to be the smallest cardinal number W such that the space has a neighborhood basis of cardinality W.)

THEOREM (HARTMAN). *If G is an LCA connected group with weight not exceeding c, then G admits a Hartman uniformly distributed sequence.*

The following result generalizes Hartman's theorem. To see that it implies his theorem, observe that a connected LCA group can be written as $K \times R^n$, where K is compact. Since by Theorem 2 of [5], the weight of a compact group is the same as the cardinality of its dual, we see that for such a group G, the weight of G does not exceed c if and only if the cardinality of $G^\wedge$ does not exceed c.

PROPOSITION 2.4. *Let G be an LCA group. A necessary and sufficient condition that G admit a Hartman uniformly distributed sequence is that* card $\Gamma \leqq c$.

Proof. Necessity is immediate since such a sequence must be dense in $\bar{G}$ because $\bar{G}$ is compact. Hence by Theorem A, card $(\bar{G})^\wedge \leqq c$ and since $(\bar{G})^\wedge = G^\wedge$ as a group, we are done. In the other direction, we show first, extending part of Theorem A, that if card $\Gamma \leqq c$ then G is separable. This is surely known, but we give the proof for completeness. Once we know that G is separable, we are done, since we can use Lemma 2.1 with $\Phi = G^\wedge$ and H the subgroup generated by the dense sequence in G to produce an H.u.d. sequence. Now suppose that card $\Gamma \leqq c$ and apply the

structure theorem to G, writing $G/M=Q$. Any discrete quotient Q of G must have cardinality not exceeding $\aleph_0$. For otherwise, let $Q^\wedge$ be its compact dual. By a theorem of Kakutani [5, Theorem 1], we would have card $Q^\wedge > c$. But $Q^\wedge$ is a subgroup of Γ, which leads to a contradiction. Also, any compact open subgroup K of G must have no more than c characters since $K^\wedge$ is isomorphic to a quotient group of Γ. Thus any such K must be separable, by Theorem A, and of course R^n is separable. It follows that G is separable since it is the union of the countably many cosets of $M=K\times R^n$, and each of these cosets is open and separable. This completes the proof of Proposition 2.4.

3. **The existence and uniqueness of D-compactifications.** We recall that a D-compactification $G^\sim$ of the LCA group G is a compactification such that a sequence is u.d. in G if and only if its image is u.d. in $G^\sim$.

If G admits no u.d. sequence then G trivially admits many D-compactifications. Indeed, if G admits no u.d. sequence then, by Theorem 3, card $\Gamma^p > c$. If we then let Λ be any subgroup of Γ such that card $\Lambda > c$ and if we let H be the compact dual of Λ then H will serve vacuously as a D-compactification of G since, by Theorem 3, H admits no u.d. sequence.

If G admits a u.d. sequence, we show in Theorem 4 that G has a D-compactification if and only if Γ^p is a group, and that if Γ^p is a group then there is a unique D-compactification, namely $\bar{G}^p$.

We wish to thank R. Doss who simplified the authors' original proof of the following computational lemma.

LEMMA 3.1. *Let G be a compact Abelian group such that* card $\Gamma \leqq c$ *and suppose given a character χ_1 in Γ. Then there exists a sequence $\{x_\nu\}$ in G such that for each character χ other than χ_1, χ_1^-, and* 1, *we have $\langle \chi, \{x_\nu\}\rangle = 0$ but $\langle \chi_1, \{x_\nu\}\rangle \neq 0$.*

Proof. First, we construct a sequence $\{y_\nu\}$ in G such that for each nontrivial character χ we have $\langle \chi, \{y_\nu\}\rangle = 0$. This is possible by Theorem 3 and the Weyl criterion. Now let $\Psi = 1 + (\chi_1 + \chi_1^-)/2$. We will construct the desired sequence $\{x_\nu\}$ by rearranging and repeating terms of the sequence $\{y_\nu\}$ so that for each continuous complex-valued function f on G we have $\langle f, \{x_\nu\}\rangle = \int f\, dm$ where $dm = \Psi\, d\mu$ and μ is Haar measure on G. This will prove our lemma, since if $\chi \in \Gamma$, then $\int_G \chi\, dm = 0$ unless χ is one of χ_1, χ_1^-, or 1, whereas if χ is one of these three characters then $\int_G \chi\, dm \neq 0$.

We construct our sequence $\{x_\nu\}$ by blocks, with the nth block having T_n terms. For fixed n, let $p_{\nu n} = [n\Psi(y_\nu)]$, where $[t]$ denotes the integral part of t. Then

$$|\Psi(y_\nu)/n - p_{\nu n}/n^2| < 1/n^2$$

and consequently

$$\sum_{\nu=1}^{n} \left| \frac{\Psi(y_\nu)}{n} - \frac{p_{\nu n}}{n^2} \right| < \frac{1}{n}.$$

The nth block of our sequence $\{x_\nu\}$ will be $y_1, \ldots, y_1, y_2, \ldots, y_2, \ldots, y_n, \ldots, y_n$, where y_j occurs p_{jn} times as j runs through $1, 2, \ldots, n$. This block has T_n terms, where

$$T_n = \sum_{\nu=1}^{n} p_{\nu n}.$$

Let us label the terms of this block as $z_1, z_2, \ldots, z_N$, where $N = T_n$. Since $\langle \Psi, \{y_\nu\} \rangle = 1$, we see that $\lim n^{-2} T_n = 1$. Also we see that for continuous functions f,

$$\left| \frac{1}{n^2} \sum_{j=1}^{T_n} f(z_j) - \frac{1}{n} \sum_{\nu=1}^{n} \Psi(y_\nu) f(y_\nu) \right|$$

$$= \left| \frac{1}{n^2} \sum_{\nu=1}^{n} p_{\nu n} f(y_\nu) - \frac{1}{n} \sum_{\nu=1}^{n} \Psi(y_\nu) f(y_\nu) \right| \leqq \frac{1}{n} \sup |f(y_\nu)|,$$

and it follows directly from the uniform distribution of $\{y_\nu\}$ that

$$\lim_{n \to \infty} \frac{1}{T_n} \sum_{j=1}^{T_n} f(z_j) = \int_G f(x) \Psi(x)\, d\mu(x).$$

But the averages that terminate in the middle of a block behave the same way, since

$$\lim_{n \to \infty} \frac{T_{n+1}}{\sum_{\nu=1}^{n} T_\nu} = 0.$$

This completes the proof of Lemma 3.1.

THEOREM 4. *Let G be an LCA group which admits a uniformly distributed sequence. Suppose Γ^p is a group. Then a sequence $\{x_\nu\}$ of elements of G is uniformly distributed if and only if $\{\varphi(x_\nu)\}$, the image of $\{x_\nu\}$ under the natural homomorphism from G into $\bar{G}^p$, is uniformly distributed in $\bar{G}^p$. Moreover, $\bar{G}^p$ is the only compactification of G with this property. That is, $\bar{G}^p$ is the unique D-compactification of G. If Γ^p is not a group, then there is no D-compactification.*

Proof. Assume first that Γ^p is a group. By Theorem 3, card $\Gamma^p \leqq c$. Then since G has as periodic characters precisely the characters of $\bar{G}^p$, we see by the Weyl criterion that $\{x_\nu\}$ is u.d. in G if and only if $\{\varphi(x_\nu)\}$ is u.d. in $\bar{G}^p$. Suppose now that H is another compactification of G, so that $H^\wedge$ is a subgroup of Γ. If $\chi_1 \in H^\wedge \backslash \Gamma^p$, then by Lemma 2.2 and the Weyl criterion we can find a sequence $\{g_\nu\}$ in G so that, if Q is the compact dual of the discrete group generated by Γ^p and χ_1, then $\{\varphi_Q(g_\nu)\}$ is u.d. in Q. Then by Lemma 3.1 we may rearrange (with repetitions) $\{g_\nu\}$ in a new sequence $\{k_\nu\}$ such that $\langle \chi, \{k_\nu\} \rangle = 0$ for all χ in $Q^\wedge$ other than χ_1, χ_1^-, and 1, but such that $\langle \chi_1, \{k_\nu\} \rangle \neq 0$. So we see that $\{k_\nu\}$ is u.d. in G but $\{\varphi(k_\nu)\}$ is not u.d. in H. If, on the other hand, there is a character $\chi_2 \in \Gamma^p \backslash H$, then we apply Lemma 3.1 to get a sequence $\{g_\nu\}$ such that $\langle \chi, \{g_\nu\} \rangle = 0$ for all χ in Γ^p other than χ_2, χ_2^-, and 1,

and such that $\langle \chi_2, \{g_\nu\}\rangle \neq 0$. Then $\{\varphi(g_\nu)\}$ is u.d. in H but not in G. Hence, if H is a D-compactification then $H = \Gamma^p$.

If Γ^p is not a group, then for any compactification H, either $H^\wedge$ has a character not in Γ^p or else $H^\wedge$ lacks a character of Γ^p. If H were a D-compactification, then H would have to admit a u.d. sequence that was the image of a u.d. sequence in G. But then applying Lemma 3.1 and Lemma 2.2, we would obtain either a u.d. sequence in G whose image is not u.d. in H or else a non u.d. sequence in G whose image is u.d. in H. Hence H cannot be a D-compactification. This completes the proof of Theorem 4.

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