ON DISCONJUGATE DIFFERENTIAL EQUATIONS

BY
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Introduction. In Wintner's [16] terminology (when $n=2$), a linear differential equation

$$(0.1) \qquad u^{(n)}+p_{n-1}^{(t)}u^{(n-1)}+\cdots+p_0(t)u = 0$$

with continuous coefficients on a t-interval I is said to be *disconjugate* on I if no solution ($\not\equiv 0$) has n zeros on I. For $n>2$, most known results giving conditions which assure that (0.1) is disconjugate concern perturbations of $u^{(n)}=0$ either for a fixed finite interval [as, e.g., in the theorem of de la Vallée Poussin (cf. [2, Exercise 5.3(d), p. 346])] or for large t. This paper deals with equations (1.1) which are perturbations of equations with constant coefficients, disconjugate on $-\infty<t<\infty$. As corollaries, we obtain theorems which are refinements of known results concerning perturbations of $u^{(n)}=0$, but we do not obtain the "best" constants occurring in some of these results ($n=2$).

The proofs depend on the technique introduced in [5] for discussing asymptotic behavior of solutions of perturbed linear systems with constant coefficients (cf. [2, Chapter X]). This technique is based on suitable changes of variables and arguments which have been subsumed by general theorems of Ważewski [15], (cf. [2, pp. 278–283]). §§1 and 2 use the simple Lemma 4.2, [2, p. 285]; §3 requires a generalization given as Theorem (*) in an Appendix below.

In addition to arguments from the theory of asymptotic integration, the proofs use a theorem of Pólya [14] characterizing equations (0.1) disconjugate on an open interval I in terms of Wronskians of subsets of solutions of (0.1); for a generalization, see [2, pp. 51–54] (also obtained in [6]). Theorems I_{**} and IV of Pólya [14] show that no solution ($\not\equiv 0$) of (0.1) on an open interval I has n distinct zeros if and only if no solution ($\not\equiv 0$) has n zeros counting multiplicities; cf. also [13]. (A generalization of this last fact, when the linear family of solutions of (0.1) is replaced by an arbitrary (not necessarily linear) interpolating family of functions, is given in [1].)

In §4, it is observed that the results of the previous sections, together with theorems and methods of Lasota and Opial [8], give criteria for the existence of solutions of certain nonlinear boundary value problems.

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1. **Distinct real roots.** We shall first consider linear differential equations

$$u^{(n)}+a_{n-1}u^{(n-1)}+\cdots+a_0u = q_{n-1}(t)u^{(n-1)}+\cdots+q_0(t)u, \tag{1.1}$$

which are perturbations of equations

$$u^{(n)}+a_{n-1}u^{(n-1)}+\cdots+a_0u = 0$$

with constant coefficients, having real distinct characteristic values.

ASSUMPTION (A_1). Let $a_0,\ldots,a_{n-1}$ be n real numbers such that the polynomial

$$\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_0 = 0 \tag{1.2}$$

has real distinct roots $\lambda_1<\lambda_2<\cdots<\lambda_n$ and let

$$c = \min(\lambda_2-\lambda_1,\ldots,\lambda_n-\lambda_{n-1}) > 0. \tag{1.3}$$

THEOREM 1.1. *Let Assumption* (A_1) *hold. Then there exists a positive number* η_0, *depending on* $\lambda_1,\ldots,\lambda_n$ *but independent of* T, *with the following properties: If* $q_0(t),\ldots,q_{n-1}(t)$ *are continuous functions on* $0\leqq t<T\,(\leqq\infty)$ [*or on* $0\leqq t\leqq T\,(<\infty)$] *such that*

$$q(t) = \left(\sum_{j=0}^{n-1}|q_j(t)|^2\right)^{1/2} \geqq 0 \tag{1.4}$$

satisfies

$$\int_0^T q(s)\,ds < \eta_0 \quad or \quad q(t) < c\eta_0$$

or, more generally,

$$\int_0^t q(s)e^{-c(t-s)}\,ds, \qquad \int_t^T q(s)e^{-c(s-t)}\,ds \leqq \eta < \eta_0. \tag{1.5}$$

Then (1.1) *is disconjugate on* $0\leqq t<T\,(\leqq\infty)$ [*or on* $0\leqq t\leqq T\,(<\infty)$].

Proof. Let $y=(u,u',\ldots,u^{(n-1)})$ and write (1.1) as a linear first order system

$$y' = (A+Q(t))y, \tag{1.6}$$

where $A=(a_{jk})$ is the constant $n\times n$ matrix with the first $n-1$ rows given by $a_{jk}=1$ or $a_{jk}=0$ for $1\leqq j\leqq n-1$, $k=j+1$ or $k\neq j+1$ and the last row is $(-a_0,\ldots,-a_{n-1})$, and $Q(t)$ is the matrix with 0 in the first $n-1$ rows and the last row is $(q_0(t),\ldots,q_{n-1}(t))$.

Let $\Lambda=\Lambda_n=(\lambda_{jk})$ be the constant $n\times n$ matrix with $\lambda_{jk}=\lambda_k^{j-1}$ for $j,k=1,\ldots,n$. Since the kth column $(1,\lambda_k,\ldots,\lambda_k^{n-1})$ of Λ is an eigenvector of A belonging to the eigenvalue λ_k, we have

$$\Lambda^{-1}A\Lambda = J \equiv \operatorname{diag}(\lambda_1,\ldots,\lambda_n). \tag{1.7}$$

We make the change of variables

$$y = \Lambda z, \quad \text{i.e.,} \quad y_j = \sum_{m=1}^{n} \lambda_m^{j-1} z_m, \tag{1.8}$$

in (1.6) to obtain

$$z' = (J + G(t))z, \tag{1.9}$$

where

$$G(t) = (g_{jk}(t)) = \Lambda^{-1} Q(t) \Lambda. \tag{1.10}$$

It is clear that $g_{jk}(t)$ is a linear combination of $q_0(t), \ldots, q_{n-1}(t)$ with coefficients depending on j, k and $\lambda_1, \ldots, \lambda_n$. (The explicit formulas for $g_{jk}(t)$ will not be needed below, but a simple calculation shows that

$$g_{jk}(t) = \sum_{m=0}^{n-1} q_m(t) \lambda_k^m / \pi_j, \tag{1.11}$$

where π_j is defined by

$$\pi_j = (-1)^n \prod_{h>j} (\lambda_h - \lambda_j);$$

cf., e.g., [2, pp. 318–319].) Thus there exists a constant $M_0 = M_0\ (\lambda_1, \ldots, \lambda_n) > 0$ with the property that

$$\left(\sum_{k=1}^{n} |g_{jk}(t)|^2 \right)^{1/2} \leqq M_0 q(t) \quad \text{for } j = 1, \ldots, n. \tag{1.12}$$

(In fact, by (1.11), M_0 can be chosen to be $\|\Lambda\|/\pi$, where $\|\Lambda\|$ is the bound of the matrix Λ as an operator from R^n to R^n and $\pi = \min(|\pi_1|, \ldots, |\pi_n|)$.)

Define the functions

$$\sigma(t) = M_0 \int_0^t q(s) e^{-c(t-s)}\, ds, \qquad \tau(t) = M_0 \int_t^T q(s) e^{-c(s-t)}\, ds$$

and assume that the bound η in (1.5) is so small that

$$7 M_0 \eta < 1. \tag{1.13}$$

Then, for each k, [2, Lemma 4.2, p. 285] with $\varepsilon = 0$ and $\psi(t) = M_0 q(t)$ and its proof imply that (1.9) has a solution $z = (z_{k1}(t), \ldots, z_{kn}(t)) \not\equiv 0$ satisfying, for $0 \leqq t < T$,

$$|z_{kj}(t)| \leqq 7\sigma(t) z_{kk}(t) \quad \text{for } 1 \leqq j < k,$$
$$|z_{kj}(t)| \leqq 7\tau(t) z_{kk}(t) \quad \text{for } k < j \leqq n;$$

cf. the Appendix below for a generalization. (Actually, the conditions of Lemma 4.2 in [2, p. 285] and its proof require that $\psi(t) = M_0 q(t) \geqq 0$ be positive, but it is clear that the validity of Lemma 4.2 with $\psi(t) > 0$ implies its validity for $\psi(t) \geqq 0$.)

By (1.8), the corresponding solution $y=(y_{k1}(t), \ldots, y_{kn}(t))=\Lambda z$ of (1.6) satisfies

$$|y_{kj}(t)-z_{kk}(t)\lambda_k^{j-1}| \leqq 7M_0\eta z_{kk}(t) \sum_{m\neq k} |\lambda_m|^{j-1} \tag{1.14}$$

for $0\leqq t<T$. The solution $y(t)$ of (1.6) corresponds to a solution $u=u_k(t)$ of (1.1) with the properties that if $e_{kj}(t)$ is defined by

$$e_{kj}(t)z_{kk}(t) = u_k^{(j-1)}(t)-\lambda_k^{j-1}z_{kk}(t),$$

then

$$|e_{kj}(t)| \leqq C\eta, \quad \text{where } C = 7M_0 \max_j \sum_{m=1}^{n} |\lambda_m|^{j-1}.$$

Let $1\leqq m\leqq n$ and let $W_m(t)$ be the Wronskian determinant of the m solutions $u_1(t), \ldots, u_m(t)$ of (1.1). Then

$$W_m(t) = z_{11}(t)\cdots z_{mm}(t) \det (\lambda_k^{j-1}+e_{kj})$$

and, consequently, there exists a constant $K=K(\lambda_1, \ldots, \lambda_n)$ with the property that

$$W_n(t) = z_{11}(t)\cdots z_{mm}(t)(\det \Lambda_m+\delta), \qquad |\delta| \leqq K\eta(1+K\eta)^{n-1}$$

and $\det \Lambda_m=\prod (\lambda_k-\lambda_j)$, $1\leqq j<k\leqq m$. Hence, there exists an $\eta_0>0$ with the property that

$$W_m(t) \neq 0 \quad \text{for } 0 \leqq t < T, m = 1, \ldots, n \quad \text{if } 0 \leqq \eta < \eta_0.$$

It follows from a theorem of Pólya [12, p. 317] that (1.1) is disconjugate on $0<t<T$.

In order to see that (1.1) is disconjugate on $[0, T)$, that is, on every interval $[0, b)$, $0<b<T$, it is only necessary to extend the definition of $q_k(t)$ to an interval $[-\varepsilon, T)$, for a suitably small $\varepsilon=\varepsilon(b)>0$, and apply the statement already proved with $(0, T)$ replaced by $(-\varepsilon, b+\varepsilon)$.

COROLLARY 1.1. *Let Assumption* (A_1) *hold. There exists a constant* $\eta_0=\eta_0(\lambda_1, \ldots, \lambda_n)>0$ *with the property that if* $q_0(t), \ldots, q_{n-1}(t)$ *are continuous for* $-\infty<t<\infty$ *and* $q(t)$ *in* (1.4) *satisfies* $q(t)<c\eta_0$ *for all* t, *then* (1.1) *is disconjugate on* $-\infty<t<\infty$.

This is a consequence of Theorem 1.1 which shows that the condition $q(t)<c\eta_0$ implies that (1.1) is disconjugate on every interval $[a, b)$, $-\infty<a<b<\infty$.

COROLLARY 1.2. *Let Assumption* (A_1) *hold. Let* $q_0(t), \ldots, q_{n-1}(t)$ *be continuous for* $0\leqq t<\infty$ *and* $q(t)$ *be defined by* (1.4). *Let* $u(t)\not\equiv 0$ *be a solution of* (1.1) *and* $N=N(T)$ *the number of zeros of* $u(t)$, *counting multiplicities, on* $0\leqq t<T$. *Then*

$$\eta_0 N(T) \leqq (n-1)\int_0^T q(t)\, dt+\eta_0(n-1). \tag{1.15}$$

Proof. Let the nonnegative zeros (if any) of $u(t)$ be $0 \leqq t_0 \leqq t_1 \leqq \cdots$. Then

$$\int_s^t q(\rho)\, d\rho \geqq \eta_0 \quad \text{if } s = t_j,\ t = t_{j+n-1}.$$

Thus if $t_{N-1} < T \leqq t_N$,

$$\int_0^T q(\rho)\, d\rho \geqq \eta_0[(N-1)/(n-1)] \geqq \eta_0\{(N-1)/(n-1)-(n-2)/(n-1)\}.$$

This gives (1.15).

2. **Coincident real roots.** We now consider the case when Assumption (A_1) does not hold and, in fact, the extreme opposite case, $\lambda_1 = \cdots = \lambda_n$, holds. We suppose that the value of $\lambda_1 = \cdots = \lambda_n$ is 0; cf. the Remark at the end of this section.

THEOREM 2.1. *There exists a number $\eta_1 > 0$, independent of T, with the following property: In the differential equation*

$$u^{(n)} = q_{n-1}(t)u^{(n-1)} + \cdots + q_0(t)u, \tag{2.1}$$

let $q_0(t), \ldots, q_{n-1}(t)$ be continuous for $0 \leqq t < T\ (\leqq \infty)$ [or for $0 \leqq t \leqq T\ (<\infty)$] and such that

$$q(t) = \left(\sum_{k=0}^{n-1} |q_k(t)t^{n-k-1}|^2\right)^{1/2} \geqq 0 \tag{2.2}$$

satisfies

$$t^{-1}\int_0^t sq(s)\, ds, \qquad t\int_t^T s^{-1}q(s)\, ds \leqq \eta < \eta_1 \tag{2.3}$$

(*in particular, let*

$$\int_0^T q(t)\, dt < \eta_1 \quad \textit{or} \quad tq(t) < \eta_1 \tag{2.4}$$

hold), *then* (2.1) *is disconjugate on* $0 \leqq t < T\ (\leqq \infty)$ [*or on* $0 \leqq t \leqq T\ (<\infty)$].

If $n=2$ and $q_1(t) \equiv 0$, then the first criterion in (2.4) is known with $\eta_1 = 1$ (cf. [4, Theorem 5.1, p. 345] with $m(t) = t-a$) and the second is an analogue of A. Kneser's criterion $t^2 q_0(t) \leqq 1/4$. If $n \geqq 2$, $q_1(t) \equiv \cdots \equiv q_{n-1}(t) \equiv 0$ and $q_0(t)$ is of constant sign and monotone, then the first criterion in (2.4) has been proved by Nehari [12]. (For a related result of de la Vallée Poussin and generalizations, see Levin [9], [10], Nehari [11], and Hukuhara [7].) According to a theorem of Dunkel (cf. [2, Theorem 17.1, p. 315]), the convergence of the integral in (2.4) for $T=\infty$ implies asymptotic formulae for the solutions of (2.1) for large t. But these formulae do not imply a disconjugacy criterion for (2.1) on $t \geqq 0$.

Proof. Under the change of independent variables

$$s = -\log t \quad \text{or} \quad t = e^{-s}, \tag{2.5}$$

the interval $0<t<T$ is changed to $-\log T<s<\infty$. Note that $u'=-e^sDu$, where $D = d/ds$. Thus

$$u^{(k)} = (-t)^{-k}D(D+1)\cdots(D+k-1)u.$$

Write this relation as

$$u^{(k)} = (-t)^{-k}\sum_{j=1}^{k}\beta_{kj}D^j, \quad \text{where } \beta_{kk} = 1,$$

so that

$$\sum_{j=1}^{k}\beta_{kj}\lambda^j = \lambda(\lambda+1)\cdots(\lambda+k-1). \tag{2.6}$$

Thus (2.1) becomes

$$D^n u+\sum_{j=1}^{n-1}\beta_{nj}D^j u = Q_{n-1}(s)D^{n-1}u+\cdots+Q_0(s)u, \tag{2.7}$$

where $t=e^{-s}$, $Q_0(s)=(-t)^n q_0(t)$, and

$$Q_j(s) = \sum_{k=j}^{n-1}(-t)^{n-k}q_k(t)\beta_{kj} \quad \text{for } j = 1,\ldots,n-1. \tag{2.8}$$

Hence, there is a constant $C=C(n)$ such that

$$Q(s) = \left(\sum_{j=0}^{n-1}|Q_j(s)|^2\right)^{1/2} \leqq Cq(t)t, \tag{2.9}$$

where $q(t)$ is defined by (2.2).

It is clear from (2.6) that the roots of the characteristic polynomial belonging to the left side of (2.7) are $0, -1,\ldots, -(n-1)$. In view of (2.9), Theorem 2.1 follows from Theorem 1.1 after an obvious change of integration variables.

Theorem 2.1 will be combined with the method of [3] for the case $n=2$, $q_1(t)\equiv 0$ (cf. [2, pp. 346–347]) to obtain an analogue of Corollary 1.2.

COROLLARY 2.1. *There exist positive constants $M_0,\ldots,M_{n-1}$ with the following property: Let $q_0(t),\ldots,q_{n-1}(t)$ be continuous for $t\geqq 0$, $u(t)\not\equiv 0$ a solution of* (2.1), *and $N=N(T)$ the number of zeros of $u(t)$ on $0\leqq t<T$ $(<\infty)$, counting multiplicities. Then $N\leqq n-1$ or N satisfies the inequality*

$$\sum_{k=0}^{n-1}M_k\{N/(n-1)-1\}^{k-n}\left\{T^{n-1}\int_0^T|q_k(t)|^{n/(n-k)}\,dt\right\}^{1-k/n} > 1. \tag{2.10}$$

Since $k-n<0$, (2.10) can be used to estimate $N=N(T)$ from above. An admissible choice for $M_0,\ldots,M_{n-1}$ is given by $M_k=2^{k-n}$, $k=0,\ldots,n-1$. This follows from the proof of Corollary 2.1 and a result of Nehari [11]. (Another choice of the constants may be given in the paper by Hukuhara [7] which is not available to me; the pertinent (last) result stated in the review in the Mathematical Reviews does not seem correct.) Still another choice is, e.g., $M_0=(n-2)^{n-2}/(n-1)^{n-1}(n-1)!$ and $M_k=1/(n-k-1)!$ for $k=1,\dots,n-1$; cf. the argument in [2, Exercise 5.3, p. 570]. See also Levin [9], [10].

Proof. Let $N>n-1$ and $(0\leqq)\ t_0\leqq t_1\leqq\cdots$ be the nonnegative zeros of $u(t)$; so that $t_{N-1}<T\leqq t_N$. By Theorem 2.1, there exist positive constants $M_0,\dots,M_{n-1}$ such that

$$\sum_{k=0}^{n-1} M_k(t-s)^{-k}\int_s^t |q_k|\,d\rho > (t-s)^{1-n} \quad \text{if } s=t_j,\ t=t_{j+n-1};$$

e.g., by (2.2) and (2.4), any set of constants $M_k>1/\eta_1$ for $k=0,\dots,n-1$ is admissible. By [11], one can choose $M_k=2^{-k}$ for $k=0,\dots,n-1$.

Letting $s=t_j$, $t=t_{j+n-1}$ and adding for $j=0,\dots,m$ gives

$$\sum_{k=0}^{n-1} M_k \sum_{j=0}^{m} (t-s)^{-k}\int_s^t |q_k|\,d\rho > \sum_{j=0}^{m}(t-s)^{1-n}. \tag{2.11}$$

Note that if $1<\alpha<\infty$,

$$(t-s)^{-k}\int_s^t |q_k|\,d\rho \leqq (t-s)^{1-k-1/\alpha}\left(\int_s^t |q_k|^\alpha\,d\rho\right)^{1/\alpha}.$$

Thus, if $1/\alpha+1/\beta=1$,

$$\sum_{j=0}^{m}(t-s)^{-k}\int_s^t |q_k|\,d\rho \leqq \left(\sum_{j=0}^{m}(t-s)^{\beta(1-k-1/\alpha)}\right)^{1/\beta}\left(\int_{t_0}^{t_{m(n-1)}} |q_k|^\alpha\,d\rho\right)^{1/\alpha}.$$

For $1\leqq k\leqq n-1$, choose $\alpha=n/(n-k)$ and $\beta=n/k$, so that $\beta(1-k-1/\alpha)=1-n$ and the last product is

$$\left(\sum_{j=0}^{m}(t-s)^{1-n}\right)^{k/n}\left(\int_{t_0}^{t_{m(n-1)}} |q_k|^{n/(n-k)}\,d\rho\right)^{1-k/n}.$$

Thus, by (2.11),

$$\sum_{k=0}^{n-1} M_k\left(\sum_{j=0}^{m}(t-s)^{1-n}\right)^{-1+k/n}\left(\int_{t_0}^{t_{m(n-1)}} |q_k|^{n/(n-k)}\,d\rho\right)^{1-k/n} > 1.$$

From the inequality for the harmonic mean,

$$\left[\sum_{j=0}^{m}(t-s)^{-(n-1)}\right]^{-1/(n-1)} \leqq (m+1)^{-n/(n-1)}\sum_{j=0}^{m}(t-s).$$

The last sum is $t_{m(n-1)}-t_0<T$ if $t_{m(n-1)}\leqq t_{N-1}<T$. Thus we obtain

$$\sum_{k=0}^{n-1} M_k\left[(m+1)^{-n}T^{n-1}\int_0^T |q_k|^{n/(n-k)}\,ds\right]^{1-k/n} > 1.$$

If we choose $m+1=[(N-1)/(n-1)]\geqq(N-1)/(n-1)-(n-2)/(n-1)$, then $m+1\geqq N/(n-1)-1$. This, together with the last inequality, gives (2.10) and proves Corollary 2.1.

REMARK. The analogues of Theorem 2.1 and its corollary in which (2.1) is replaced by

$$(d/dt-\lambda)^n u = q_{n-1}(t)u^{(n-1)}+\cdots+q_0(t)u, \tag{2.12}$$

and (2.2) by

$$q(t) = \left(|q_0(t)t^{n-1}|^2+\sum_{k=1}^{n-1}|q_k(t)t^{n-k-1}(1+|\lambda t|^k)|^2\right)^{1/2}, \tag{2.13}$$

are valid (and the corresponding number η_1 does not depend on λ).

In fact, the change of dependent variables

$$v = e^{-\lambda t}u$$

transforms (2.12) into an equation of the form

$$v^{(n)} = \rho_{n-1}(t)v^{(n-1)}+\cdots+\rho_0(t)v,$$

where

$$\rho_j(t) = \sum_{k=j}^{n-1} C_{kj}\lambda^{k-j}q_k(t), \quad \text{where } C_{kj} = k!/j!(k-j)!.$$

Hence, (2.13) satisfies

$$\sum_{k=0}^{n-1}|\rho_k(t)t^{n-k-1}|^2 \leqq C^2|q(t)|^2$$

for a suitable constant C, independent of λ.

3. **Arbitrary real roots.** We now consider the differential equation

$$u^{(n)}+a_{n-1}u^{(n-1)}+\cdots+a_0u = q_{n-1}(t)u^{(n-1)}+\cdots+q_0(t)u, \tag{3.1}$$

when the following holds:

ASSUMPTION (A_2). Let $a_0,\ldots,a_{n-1}$ be real numbers such that the polynomial

$$\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_0 = 0 \tag{3.2}$$

has only real roots, say, $\lambda(1),\ldots,\lambda(g)$, with the respective multiplicities $h(1),\ldots,h(g)$; so that $h(j)\geqq 1$ and $h(1)+\cdots+h(g)=n$. Let

$$h_* = \max(h(1),\ldots,h(g)). \tag{3.3}$$

THEOREM 3.1. *Let Assumption* (A_2) *hold. Then there exist numbers* $C\geqq 1$ *and* $\eta_2>0$, *depending on* $a_0,\ldots,a_{n-1}$ *but independent of* T, *with the following property: If* $q_0(t),\ldots,q_{n-1}(t)$ *are continuous on* $0\leqq t<T$ $(\leqq\infty)$ [*or on* $0\leqq t\leqq T\,(<\infty)$] *and*

$$q(t) = (1+t)^{h_*-1}\sum_{k=0}^{n-1}|q_k(t)| \tag{3.4}$$

satisfies

$$\int_0^t q(s)[(s+C)/(t+C)]\,ds,\ \int_t^T q(s)[(t+C)/(s+C)]\,ds \leqq \eta < \eta_2 \tag{3.5}$$

or, in particular,

$$\int_0^T q(t)\,dt < \eta_2 \quad or \quad (t+C)q(t) < \eta_2, \tag{3.6}$$

then (3.1) *is disconjugate on* $0 \leqq t < T\,(\leqq \infty)$ [*or on* $0 \leqq t \leqq T\,(<\infty)$].

The proof depends on a method of Hartman and Wintner [5] for the asymptotic integration of linear systems with nearly constant coefficients. This method involves a change of variables which enables us to obtain rather precise information on the asymptotic behavior of each component of the solution vector.

Proof. We shall deal only with disconjugacy for $0<t<T$. The passage to $0 \leqq t < T$ can be made as in the proof of Theorem 1.1.

(a) Let $C = C(a_0, \ldots, a_{n-1}) \geqq 1$ be a number to be fixed below and introduce the abbreviation

$$s = t + C \geqq 1 \quad \text{if} \quad 0 \leqq t < T. \tag{3.7}$$

Assume, that for each j, there exist $h(j)$ solutions $u = u_{j1}(t), \ldots, u_{jh(j)}(t)$ of (3.1) with the properties that $v_{j\kappa}(t) = u_{j\kappa}(t)e^{-\lambda(j)s}$ and its derivatives satisfy

$$\begin{aligned} |D^{i-1}v_{j\kappa}(t) - w^j_\kappa(t)s^{\kappa-i}/(\kappa-1)!| &\leqq \gamma\eta|w^j_\kappa(t)|s^{\kappa-i} \quad \text{for } i = 1, \ldots, \kappa, \\ |D^{i-1}v_{j\kappa}(t)| &\leqq \gamma\eta|w^j_\kappa(t)|s^{\kappa-i} \quad \text{for } i = \kappa+1, \ldots, h(j), \\ |D^{i-1}v_{j\kappa}(t)| &\leqq \gamma\eta|w^j_\kappa(t)|s^{\kappa-h_*} \quad \text{for } h(j) < i \leqq n, \end{aligned} \tag{3.8}$$

for $0 \leqq t < T$, where $w^j_\kappa(t)$ is a continuous nonvanishing function; γ denotes a constant, not always the same, depending only on $a_0, \ldots, a_{n-1}$; and $D = d/dt$.

It will now be shown that if $\eta > 0$ is sufficiently small and the n solutions $u_{11}, \ldots, u_{1h(1)}, u_{21}, \ldots, u_{gh(g)}$ are numbered as $u_1, \ldots, u_n$, then the Wronskian determinant

$$W_m(t) = \det(D^{i-1}u_j), \quad \text{where } i, j = 1, \ldots, m\,(\leqq n),$$

does not vanish for $0 \leqq t < T$.

Let $1 \leqq m \leqq n$; say, $m = h(1) + \cdots + h(\mu-1) + \nu$, where $1 \leqq \nu \leqq h(\mu)$. Let $h'(j) = h(j)$ or ν according as $j \leqq \mu - 1$ or $j = \mu$. The entry $D^{i-1}u_{j\kappa}$ in $W_m(t)$ is

$$D^{i-1}u_{j\kappa} = D^{i-1}e^{\lambda(j)s}v_{j\kappa} = e^{\lambda(j)s}\sum_{r=1}^{i} C_{i-1,r-1}\lambda^{i-r}(j)D^{r-1}v_{j\kappa}. \tag{3.9}$$

If $i \leqq h(j)$, let $e_{ih\kappa}(t) = 0$. If $i > h(j)$, let

$$e_{ij\kappa}(t) = \sum_{r=h(j)+1}^{i} C_{i-1,r-1}\lambda^{i-r}(j)D^{r-1}v_{j\kappa}, \tag{3.10}$$

so that, by (3.8),

$$|e_{ij\kappa}(t)| \leqq \gamma|w^j_\kappa(t)|\eta s^{\kappa-h_*} \quad \text{for } 0 \leqq t < T;\ \kappa = 1, \ldots, h(j);\ i = 1, \ldots, n. \tag{3.11}$$

The relations (3.9), (3.10) give

$$D^{i-1}u_{j\kappa} = e^{\lambda(j)s}[f_{ij\kappa}(t)+e_{ij\kappa}(t)], \tag{3.12}$$

where

$$f_{ij\kappa} = \sum_{r=1}^{\min(i,h(j))} C_{i-1,r-1}\lambda^{i-r}(j)D^{r-1}v_{j\kappa}.$$

The $m\times m$ matrix $(f_{ij\kappa}(t))$, corresponding to $(D^{i-1}u_j)$, $i, j=1,\ldots,m$, can be written as the product $\Lambda_{(m)}W^m(t)$ of two $m\times m$ matrices. The matrix $\Lambda_{(m)}$ is a constant matrix which is the Wronskian determinant at $t=0$ of the first m functions of $\omega_{11},\ldots,\omega_{1h(1)},\omega_{21},\ldots,\omega_{gh(g)}$, where $\omega_{jk}=e^{\lambda(j)t}t^{k-1}/(k-1)!$. By (3.9),

$$D^{i-1}\omega_{jr}(0) = \sum_k C_{i-1,k-1}\lambda^{i-k}(j)[t^{r-k}/(r-k)!]_{t=0},$$

and the sum is over the range $1\leqq k\leqq \min(i, r)$; so that

$$D^{i-1}\omega_{jr}(0) = C_{i-1,r-1} \quad \text{or 0, according as } r\leqq i \text{ or } r > i.$$

Choose $W^m(t)$ to be the matrix

$$W^m(t) = \operatorname{diag}(W_{(1)}(t),\ldots,W_{(\mu)}(t)), \tag{3.14}$$

where $W_{(j)}(t)$ is an $h'(j)\times h'(j)$ matrix,

$$W_{(j)}(t) = (D^{i-i}v_{j\kappa}) \quad \text{for } i, \kappa = 1,\ldots,h'(j).$$

Thus, we have $(f_{ij\kappa}(t))=\Lambda_{(m)}W^m(t)$.

The matrix $\Lambda_{(m)}$ is nonsingular, since its determinant is the Wronskian determinant of m linearly independent solutions of an mth order linear equation $u^{(m)}+\cdots=0$ with constant coefficients. Thus, by (3.12), the matrix $(D^{i-1}u_j)$ can be written

$$(D^{i-1}u_{j\kappa}) = \Lambda_{(m)}[W^m(t)+\Lambda_{(m)}^{-1}(e_{ij\kappa})]D_0,$$

where $D_0=\operatorname{diag}(e^{\lambda(1)s},\ldots,e^{\lambda(\mu)s})$ in which $e^{\lambda(j)s}$ occurs $h'(j)$ times. Thus, the assertion $W_m(t)\neq 0$ is equivalent to

$$\det W^{(m)}(t) \neq 0, \quad \text{where } W^{(m)} = W^m(t)+\Lambda_{(m)}^{-1}(e_{ij\kappa}). \tag{3.15}$$

From (3.11), the absolute values of the elements in the columns of $\Lambda_{(m)}^{-1}(e_{ij\kappa})$ corresponding to $u_{j\kappa}$ are majorized by $\gamma|w_\kappa^j(t)|\eta s^{\kappa-h_*}$. Divide the corresponding column of $W^{(m)}(t)$ by $s^{\kappa-1}|w_\kappa^j(t)|$ and multiply the $[h(1)+\cdots+h(j-1)+i]$th row by s^{i-1}. In the resulting matrix, the elements which are not in the blocks corresponding to $W_{(1)},\ldots,W_{(\mu)}$ (cf. (3.14)) are majorized by $\gamma\eta s^{i-h_*}$, where $i-h_*\leqq h(j)-h_*\leqq 0$. Since $s\geqq 1$ for $t\geqq 0$, $\gamma\eta s^{i-h_*}\leqq\gamma\eta$. By the same argument, the elements below the diagonals on the blocks $W_{(1)},\ldots,W_{(\mu)}$ are majorized by $\gamma\eta s^{i-\kappa}\leqq\gamma\eta$ for $i\leqq\kappa$. The elements on and above the diagonal become $1/(i-\kappa)!$ $+$error, where $|\text{error}|\leqq\gamma\eta$; in particular, the diagonal elements differ from 1 by at most $\gamma\eta$. Thus, it is clear that if η_2 is sufficiently small and $\eta<\eta_2$, then (3.15) holds.

(b) In order to complete the proof of Theorem 3.1, it is necessary to prove the existence of the solutions $u=u_{j\kappa}(t)$ assumed in part (a). Let j be fixed, $1\leqq j\leqq g$. For the sake of definiteness let $j=1$, $\lambda=\lambda(1)$, $h=h(1)$ and put $\mu(j)=\lambda(j)-\lambda$ for $j=1,\dots,g$. In particular $\mu(1)=0$. In (3.1), introduce the new dependent variable

$$v = e^{-\lambda t}u. \tag{3.16}$$

Then (3.1) takes the form

$$v^{(n)}+b_{n-1}v^{(n-1)}+\cdots+b_hv^{(h)} = \sum_{i=0}^{n-1} \rho_i(t)v^{(i)}. \tag{3.17}$$

Here $b_h,\dots,b_{n-1}$ are constants such that the roots of

$$\tau^n+b_{n-1}\tau^{n-1}+\cdots+b_h\tau^h = 0$$

are $0=\mu(1),\mu(2),\dots,\mu(g)$ with the respective multiplicities $h(1),\dots,h(g)$. Also

$$\rho_i(t) = \sum_{k=i}^{n-1} C_{k,i}\lambda^{k-i}q_k(t), \quad \text{where } C_{k,i} = k!/i!(k-i)!. \tag{3.18}$$

Write (3.17) as a first order system

$$y' = By+P(t)y \tag{3.19}$$

for the vector $y=(v,v',\dots,v^{(n-1)})$; cf. (1.1) and (1.6).

For a given j, let J_j denote the $h(j)\times h(j)$ Jordan matrix with the diagonal elements $\mu(j)$ and, if $h(j)>1$, superdiagonal elements 1. The first h columns of B constitute an $h\times n$ matrix with J_1 in the upper portion and the zero matrix $0_{h\times(n-h)}$ below. Thus, there exists a nonsingular constant matrix of the form

$$\Lambda = \begin{pmatrix} I_h & \cdots \\ 0 & \cdots \end{pmatrix}, \qquad \Lambda^{-1} = \begin{pmatrix} I_h & \cdots \\ 0 & \cdots \end{pmatrix}, \tag{3.20}$$

such that I_h is the unit $h\times h$ matrix and

$$\Lambda^{-1}B\Lambda = J \equiv \operatorname{diag}(J_1,\dots,J_g). \tag{3.21}$$

The linear change of variables

$$y = \Lambda z \tag{3.22}$$

sends (3.19) into

$$z' = Jz+\Lambda^{-1}P(t)\Lambda z. \tag{3.23}$$

Write $z=(z^1,\dots,z^g)$, where $z^j=(z^j_1,\dots,z^j_{h(g)})$ is a vector of dimension $h(g)$, and (3.23) as

$$z^{j\prime} = J_jz^j+\sum_{k=1}^{g} P^{jk}(t)z^k \quad \text{for } j = 1,\dots,g, \tag{3.24}$$

where $P^{jk}(t)$ is an $h(j)\times h(k)$ matrix.

Since $P(t)$ is an $n\times n$ matrix with 0 entries except for the last row $(-\rho_0(t),\dots,-\rho_{n-1}(t))$, it follows from (3.20), (2.23) that the ith component of $P^{j1}(t)z^1$ is of the form

$$(P^{j1}(t)z^1)_i = \alpha_i^1 \sum_{m=0}^{h-1} \rho_m(t) z_{m+1}^1, \tag{3.25}$$

where $(-\alpha_1^1,\dots,-\alpha_{h(g)}^g)$ is the last column of Λ^{-1}. Furthermore the elements of $P^{jk}(t)$ are linear combinations of $\rho_0(t),\dots,\rho_{n-1}(t)$ with constant coefficients.

Let $C\geqq 1$, $\varepsilon>0$ be constants to be specified below and introduce the abbreviation $s=t+C$, as in (a) above. (Note that t is still the independent variable.) Make the linear change of variables $z\to w$:

$$z^1 = (\exp J_1 s)w^1 \quad\text{and}\quad z_i^j = \varepsilon^{h(j)-i} w_i^j \tag{3.26}$$

for $j=2,\dots,g$, to obtain

$$w^{1\prime} = \sum_{k=1}^{g} Q^{1k}(t)w^k \quad\text{and}\quad w^{j\prime} = J_{j\varepsilon}w^j + \sum_{k=1}^{g} Q^{jk}(t)w^k, \tag{3.27}$$

where $J_{j\varepsilon}$ is obtained by replacing the superdiagonal elements 1 in J_j by ε. If (3.26) is written as

$$z = Q_0(t)w,\ Q_0(t) = \operatorname{diag}(\exp J_1 s, D_1),$$

where $D_1=D_1(\varepsilon)$ is a diagonal matrix, then (3.27) is

$$w' = \operatorname{diag}(0, J_{2\varepsilon},\dots,J_{g\varepsilon})w + Q_0^{-1}\Lambda^{-1}P\Lambda Q_0 w.$$

From this, we can write (3.27) in detail as

$$w_i^{1\prime} = \left(\sum_{r=i}^{h} c_{ir}s^{r-i}\right)\sum_{k=1}^{h}\sum_{m=0}^{k-1} d_{km}\rho_m(t)s^{k-1-m}w_k^1 + \sum_{l=2}^{g}\sum_{m=1}^{h(1)}\sum_{r=i}^{h} L_{1i,lmr}(t)s^{r-i}w_m^l, \tag{3.28}$$

$i=1,\dots,h$, and

$$w_i^{j\prime} = \mu_j w_i^j + \varepsilon_{ji}w_{i+1}^j + \sum_{k=1}^{h}\sum_{m=0}^{k-1} d_{ji,km}\rho_m(t)s^{k-1-m}w_k^1 + \sum_{l=2}^{g}\sum_{m=1}^{h(1)} L_{ji,lm}(t)w_m^l, \tag{3.29}$$

for $j=2,\dots,g$ and $i=1,\dots,h(j)$, where c_{ir}, d_{km}, $d_{ji,km}$ are constants; $L_{1i,lmk}(t)$ and $L_{ji,lm}(t)$ are linear functions of $\rho_0(t),\dots,\rho_{n-1}(t)$ with constant coefficients; and $\varepsilon_{ji}=\varepsilon$ or 0 according as $i<h(j)$ or $i=h(j)$.

Let $1\leqq\kappa\leqq h$. Make the last change of variables $w\to x$:

$$x_i^1 = s^{i-\kappa}w_i^1 \quad\text{and}\quad x_i^j = s^{h_*-\kappa}w_i^j, \qquad j>1. \tag{3.30}$$

Then (3.28)–(3.29) becomes

$$x_i^{1\prime} = (i-\kappa)s^{-1}x_i^1 + \sum\sum\sum c_{ir}d_{km}\rho_m(t)s^{r-m-1}x_k^i + \sum\sum\sum L_{1i,lmr}(t)s^{r-h_*}x_m^l, \tag{3.31}$$

$$\begin{aligned} x_i^{j\prime} &= [\mu(j) + (h_*-\kappa)s^{-1}]x_i^j + \varepsilon_{ji}x_{i+1}^j \\ &\quad + \sum\sum d_{ji,km}\rho_m(t)s^{h_*-m-1} + \sum\sum L_{ji,lm}(t)x_m^l, \end{aligned} \tag{3.32}$$

where the ranges for the sums are the same as in (3.28), (3.29), respectively.

Choose $C \geqq 1$ so large and $\varepsilon > 0$ so small that

$$\text{(3.33)} \qquad [\operatorname{sgn} \mu(j)] J_{j\varepsilon} x^j \cdot x^j - (h_* - \kappa) s^{-1} |x^j|^2 \geqq s^{-1} |x^j|^2 \quad \text{for } j > 1.$$

Since $s = t + C \geqq 1$ when $t \geqq 0$, it is clear that there exists a constant

$$C_1 = C_1(a_0, \ldots, a_{n-1})$$

such that if

$$\text{(3.34)} \qquad \psi(t) = C_1 \sum_{m=0}^{n-1} |\rho_m(t)| s^{h_* - 1},$$

then

$$|\{\cdots\}| \leqq \psi(t)|x| \quad \text{if } |x|^2 = \sum_{j=1}^{g} |x^j|^2 = \sum_{j=1}^{g} \sum_{k=1}^{h(j)} |x_k^j|^2$$

and $\{\cdots\}$ denotes either the sum of the two triple sums in (3.31) or the sum of the two double sums in (3.32). Thus, (3.31)–(3.32) imply that

$$\sum_{i=1}^{\kappa-1} x_i^1 x_i^{1\prime} \leqq -s^{-1} \sum_{i=1}^{\kappa-1} |x_i^1|^2 + \psi(t)|x| \left(\sum_{i=1}^{\kappa-1} |x_i^1|^2 \right)^{1/2},$$

$$x^j \cdot x^{j\prime} \leqq -s^{-1}|x^j|^2 + \psi(t)|x| \cdot |x^j| \quad \text{if } \mu(j) < 0,$$

$$|x_\kappa^1 x_\kappa^{1\prime}| \leqq \psi(t)|x| \cdot |x_\kappa^1|,$$

$$\sum_{i=\kappa+1}^{h} x_i^1 x_i^{1\prime} \geqq s^{-1} \sum_{i=\kappa+1}^{h} |x_i^1|^2 - \psi(t)|x| \left(\sum_{i=\kappa+1}^{h} |x_i^1|^2 \right)^{1/2},$$

$$x^j \cdot x^{j\prime} \geqq s^{-1}|x^j|^2 - \psi(t)|x| \cdot |x^j| \quad \text{if } \mu(j) > 0.$$

We can now apply Theorem (*) of the Appendix below (with y^κ corresponding to the 1-dimensional x_κ^1 with $\alpha^\kappa = \alpha_\kappa = 0$; the vectors $y^1, \ldots, y^{\kappa-1}$ correspond to $y^1 = (x_1^1, \ldots, x_{\kappa-1}^1)$ and the x^j for which $\mu(j) < 0$ with the respective $\alpha_1 = \alpha_2 = \cdots = \alpha_{\kappa-1} = -s^{-1}$; the vectors $y^{\kappa+1}, \ldots, y^n$ correspond to $y^{\kappa+1} = (x_{\kappa+1}^1, \ldots, x_h^1)$ and the x^j for which $\mu(j) > 0$ with the respective $\alpha_{\kappa+1} = \alpha_{\kappa+2} = \cdots = s^{-1}$.

Thus, we have to examine the functions

$$\sigma_1 = \int_0^t \psi(\rho)(\rho + C)/(t + C)\, d\rho, \qquad \sigma_{\kappa+1} = \int_t^T \psi(\rho)(t + C)/(\rho + C)\, d\rho.$$

According to Theorem (*), if σ_1 and $\sigma_{\kappa+1}$ exist and have a sufficiently small bound η, then (3.31)–(3.32) has a solution $x(t) \not\equiv 0$ such that

$$|x_i^1| \leqq \gamma\eta |x_\kappa^1| \quad \text{for } i \neq \kappa, \qquad |x^j| \leqq \gamma\eta |x_\kappa^1| \quad \text{for } j \neq 1,$$

$0 \leqq t < T$, where γ denotes a constant independent of η, but not always the same. In view of (3.30), this means that (3.28)–(3.29) has a solution $w(t) \not\equiv 0$ such that

$$|w_i^1(t)| \leqq \gamma\eta s^{\kappa - i} |w_\kappa^1(t)| \quad \text{and} \quad |w^j(t)| \leqq \gamma\eta s^{\kappa - h_*} |w_\kappa^1(t)|$$

for $i \neq \kappa, j \neq 1$, respectively. By (3.26), (3.24) has a solution $z(t) \not\equiv 0$ satisfying

$$|z_i^1(t) - w_\kappa^1(t)s^{\kappa-i}/(\kappa-i)!| \leqq \gamma\eta |w_\kappa^1(t)| s^{\kappa-i} \quad \text{for } i = 1, \ldots, \kappa,$$
$$|z_i^1(t)| \leqq \gamma\eta |w_\kappa^1(t)| s^{\kappa-i} \quad \text{for } i = \kappa+1, \ldots, h,$$
$$|z^j(t)| \leqq \gamma\eta |w_\kappa^1(t)| s^{\kappa-h_*} \quad \text{for } j = 2, \ldots, g.$$

Finally, by (3.20) and (3.22), (3.19) has a solution $y = (v, v', \ldots, v^{(n-1)})$ satisfying (3.8) if $j=1$ and $v = v_{j\kappa}$. This completes the proof of Theorem 3.1.

4. **Nonlinear interpolation problems.** The results and methods of Lasota and Opial [8], combined with those above, make it possible to treat nonlinear boundary value or interpolation problems associated with certain nonlinear differential equations

$$u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_0 u = f(t, u, u', \ldots, u^{(n-1)}), \tag{4.1}$$

$$u^{i-1}(t_k) = c_{ik} \quad \text{for } i = 1, \ldots, m(k) \text{ and } k = 1, \ldots, \nu, \tag{4.2}$$

$$(0=)t_1 < \cdots < t_\nu (=T) \quad \text{and} \quad m(1) + \cdots + m(\nu) = n. \tag{4.3}$$

Assumption (B_1). Let $a_0, \ldots, a_{n-1}$ be real numbers satisfying Assumption (A_1) of Theorem 1.1, or the assumption $a_0 = \cdots = a_{n-1} = 0$ of Theorem 2.1, or Assumption (A_2) of Theorem 3.1. Correspondingly, let $0 < T < \infty$ and let $q_0(t), \ldots, q_{n-1}(t)$ be nonnegative, continuous functions on $0 \leqq t \leqq T$ such that $q(t)$ defined in (1.4) satisfies (1.5), or $q(t)$ defined in (2.2) satisfies (2.3), or $q(t)$ defined by (3.4) satisfies (3.5).

Assumption (B_2). Let $f(t, y_0, \ldots, y_{n-1})$ be a continuous function for $0 \leqq t \leqq T$ and arbitrary $(y_0, \ldots, y_{n-1})$ satisfying

$$|f(t, y_0, \ldots, y_{n-1})| \leqq \sum_{k=0}^{n-1} q_k(t)|y_k|. \tag{4.4}$$

Theorem 4.1. *Let* $a_0, \ldots, a_{n-1}$ *and* $q_0(t), \ldots, q_{n-1}(t)$ *satisfy Assumption* (B_1) *and* $f(t, y_0, \ldots, y_{n-1})$ *satisfy Assumption* (B_2). *Then the interpolation problem* (4.1)–(4.3), *in which* $c_{11}, \ldots, c_{m(\nu)\nu}$ *are arbitrary constants, has at least one solution. If, in addition,* $f(t, y_0, \ldots, y_{n-1})$ *satisfies*

$$|f(t, y_0, \ldots, y_{n-1}) - f(t, z_0, \ldots, z_{n-1})| \leqq \sum_{k=0}^{n-1} q_k(t)|y_k - z_k|, \tag{4.5}$$

then the solution of (4.1)–(4.3) *is unique.*

Proof. If $\nu = n$ and $m(1) = \cdots = m(n) = 1$, then this theorem follows from [8]. In fact, since T is finite and the main conditions in (1.5), (2.3), (3.5) involve a strict inequality, it is clear that these conditions hold whenever $q_k(t)$ is replaced by a continuous function $Q_k(t)$ satisfying $|Q_k(t)| < q_k(t) + (\eta_* - \eta)/nT$, where $\eta_* = \eta_0$, η_1, η_2, respectively. The proof for the general case (4.1)–(4.3) is similar.

APPENDIX

Let $y^k=(y_1^k,\ldots,y_{m(k)}^k)$ be a real Euclidean vector of dimension $m(k)$ and $y=(y^1,\ldots,y^N)=(y_1^1,\ldots,y_{m(N)}^N)$ the corresponding Euclidean vector of dimension $M=m(1)+\cdots+m(N)$. Consider a linear differential equation

$$y' = A(t)y, \tag{1}$$

where $A(t)$ is an $M\times M$ matrix with real continuous entries for $0\leqq t<T\,(\leqq\infty)$. Let κ be fixed, $1\leqq\kappa\leqq N$. Assume that there exists continuous functions $\psi_1(t)\geqq 0,\ldots,\psi_n(t)\geqq 0$ and $\alpha_1(t),\ldots,\alpha_\kappa(t)$ and $\alpha^\kappa(t),\ldots,\alpha^N(t)$ such that (1) implies

$$y^k\cdot y^{k'} \leqq \alpha_k(t)|y^k|^2+\psi_k(t)|y^k|\cdot|y| \quad \text{for } k=1,\ldots,\kappa, \tag{2}$$

$$y^k\cdot y^{k'} \geqq \alpha^k(t)|y^k|^2-\psi_k(t)|y^k|\cdot|y| \quad \text{for } k=\kappa,\ldots,N. \tag{3}$$

Assume that the integrals

$$\sigma_k(t) = \int_0^t [\psi_k(s)+\psi_\kappa(s)]\left\{\exp\int_s^t [\alpha_k(\rho)-\alpha^\kappa(\rho)]\,d\rho\right\}ds \quad \text{for } k=1,\ldots,\kappa-1, \tag{4}$$

$$\sigma_k(t) = \int_t^T [\psi_k(s)+\psi_\kappa(s)]\left\{\exp-\int_t^s (\alpha^k(\rho)-\alpha_\kappa(\rho)]\,d\rho\right\}ds \quad \text{for } k=\kappa+1,\ldots,N, \tag{5}$$

exist and are bounded for $0\leqq t<T$. Suppose finally that there exist positive constants γ, η with the properties that

$$\gamma\sigma_k(t) \leqq \eta \quad \text{for } 0\leqq t<T,\ k\neq\kappa, \tag{6}$$

$$[1+(N-1)\eta^2]^{1/2}\max(1,\eta) < \gamma. \tag{7}$$

THEOREM (*). *Under the conditions enumerated above, there exists an $m(1)+\cdots+m(\kappa)$ parameter family of solutions of* (1) *such that, for* $0\leqq t<T$, *we have* $y^\kappa(t)\neq 0$ *and*

$$|y^k(t)| \leqq \gamma\sigma_k(t)|y^\kappa(t)| \quad \textit{for } k\neq\kappa. \tag{8}$$

REMARK 1. The proof will show that one obtains a solution of (1) satisfying (8) for which one can assign an arbitrary partial set of initial conditions $y^k(0)$, $k=1,\ldots,\kappa$ such that $y^\kappa(0)\neq 0$ and $|y^k(0)|/|y^\kappa(0)|$ is small for $k=1,\ldots,\kappa-1$.

REMARK 2. Theorem (*) has an analogue if the components of y and the entries of $A(t)$ are complex-valued. It is only necessary to replace $y^k\cdot y^{k'}$ by $\operatorname{Re}\bar{y}^k\cdot y^{k'}$ in (2), (3).

REMARK 3. Note that there is no assumption about the signs of the functions α_k, α^k, $\alpha^\kappa-\alpha_k$, $\alpha^k-\alpha_\kappa$.

For particular cases of this theorem, see [2, pp. 284–290; especially, Lemmas 4.1, 4.2 and Exercise 4.3].

Proof. Theorem (*) will be deduced from a result of Ważewski [15]. We shall use the formulation and notation of [2, Theorem 3.1, p 282]. In order to be able to use this theorem, we shall suppose that $\psi_k(t)>0$ for $0<t<T$ and prove the existence of solutions satisfying $y^\kappa(t)\neq 0$ and (8) for $t_0\leqq t<T$, where t_0 is an arbitrary point of $(0, T)$. The passage to the case $\psi(t)\geqq 0$ and $t_0=0$ will be clear.

Let Ω be the open (t, y)-set $\Omega=\{(t, y) : 0<t<T, y\neq 0 \text{ arbitrary}\}$. Define an open subset by

$$\Omega^0 = \{(t, y)\in\Omega : u_j(t, y) < 0, v_k(t, y) < 0\} \tag{9}$$

for $1\leqq k<\kappa<j\leqq N$, where

$$u_b(t, y) = |y^b|^2-\gamma^2\sigma_b^2(t)|y^\kappa|^2 \quad \text{for } b = \kappa+1, \ldots, N,$$
$$v_a(t, y) = |y^a|^2-\gamma^2\sigma_a^2(t)|y^\kappa|^2 \quad \text{for } a = 1, \ldots, \kappa-1.$$

Correspondingly, define the subsets of Ω

$$U_b = \{u_b(t, y) = 0, u_j(t, y) \leqq 0, v_k(t, y) \leqq 0 \quad \text{for } 1 \leqq k < \kappa < j \leqq N\},$$
$$V_a = \{v_a(t, y) = 0, u_j(t, y) \leqq 0, v_k(t, y) \leqq 0 \quad \text{for } 1 \leqq k < \kappa < j \leqq N\},$$

where $a=1, \ldots, \kappa-1$ and $b=\kappa+1, \ldots, N$.

If $f(t, y)$ is a function of class C^1, let $\dot{f}(t, y)$ be the trajectory derivative of f relative to (1); i.e.,

$$\dot{f}(t, y) = \partial f/\partial t+(\text{grad}_y f)\cdot A(t)y.$$

Then we can verify

$$\dot{v}_a(t, y) > 0 \quad \text{for } (t, y)\in V_a,\ a = 1, \ldots, \kappa-1, \tag{10}$$

$$\dot{u}_b(t, y) < 0 \quad \text{for } (t, y)\in U_b,\ b = \kappa+1, \ldots, N. \tag{11}$$

For example, in order to obtain (11), note that

$$\dot{u}_b = 2(y^b\cdot y^{b'}-\gamma^2\sigma_b^2 y^\kappa\cdot y^{\kappa'}-\gamma^2\sigma_b\sigma_b'|y^\kappa|^2).$$

Using (2) and (3), it is seen that

$$\tfrac{1}{2}\dot{u}_b \geqq \alpha^b|y^b|^2-\psi_b(t)|y^b|\cdot|y|-\gamma^2\sigma_b^2(\alpha_\kappa|y^\kappa|^2+\psi_\kappa(t)|y^\kappa|\cdot|y|)-\gamma^2\sigma_b\sigma_b'|y^\kappa|^2.$$

For $(t, y)\in U_b$, we have $|y^b|=\gamma\sigma_b|y^\kappa|$ and

$$|y| \leqq |y^\kappa|\left(1+\sum_{j\neq\kappa}\gamma^2\sigma_j^2\right)^{1/2} \leqq c|y^\kappa|,$$

where, by (6), $c=[1+(N-1)\eta^2]^{1/2}>0$. Thus we have

$$\tfrac{1}{2}\dot{u}_b \geqq \gamma^2\sigma_b|y^\kappa|^2\{(\alpha^b-\alpha_\kappa)\sigma_b-\sigma_b'-(c/\gamma)[\psi_b+\psi_\kappa\gamma\sigma_b]\}.$$

Note that, by (5),

$$\sigma_b' = -(\psi_b+\psi_\kappa)+(\alpha^b-\alpha_\kappa)\sigma_b;$$

hence, for $(t, y)\in U_b$,

$$\tfrac{1}{2}\dot{u}_b \geqq \gamma\sigma_b|y^\kappa|^2\{(\gamma-c)\psi_b+\psi_\kappa(\gamma-c\gamma\sigma_b)\}.$$

Thus $\tfrac{1}{2}\dot{u}_b>0$ if $\gamma>c$ and $\gamma>c\gamma\sigma_b$, while $\gamma>c\gamma\sigma_b$ if $\gamma>c\eta$. (Note that $y^\kappa\neq 0$, for otherwise $(t, y) \in U_b \subset \Omega$ implies that $y=0$, which cannot hold for $(t, y) \in \Omega$.) Similarly (10) is verified.

Thus, in the terminology of [2, pp. 281–282], Ω^0 is a (u, v)-subset of Ω and, by a theorem of Ważewski [13] (cf. [2, Lemma 3.1, p. 281]), the set of egress points Ω_e^0 of Ω^0 consists only of strict egress points and

$$\Omega_e^0 = \bigcup_{b=\kappa+1}^{N} U_b - \bigcup_{a=1}^{\kappa-1} V_a. \tag{12}$$

If $0 < t_0 < T$, let $y^1(t_0), \ldots, y^\kappa(t_0)$ be arbitrary vectors satisfying $y^\kappa(t_0) \neq 0$, $|y^a(t_0)| < \gamma\sigma_a(t_0)|y^\kappa(t_0)|$ for $a=1, \ldots, \kappa-1$. Let S be the set of points $(t, y) = (t_0, y^1(t_0), \ldots, y^\kappa(t_0), y^{\kappa+1}, \ldots, y^N) \in \Omega$ satisfying $|y^b| \leqq \gamma\sigma_b(t_0)|y^\kappa(t_0)|$ for $b=\kappa+1, \ldots, N$. Topologically, S is a ball of dimension $m(\kappa+1)+\cdots+m(N)$ and $S \cap \Omega_e^0$ is its boundary. Thus $S \cap \Omega_e^0$ is not a retract of S. But $S \cap \Omega_e^0$ is a retract of Ω_e^0. In fact, a retraction is given by the map $\pi\colon \Omega_e^0 \to S \cap \Omega_e^0$ defined by $\pi(t, y) = (t_0, z)$, where $z^a = y^a(t_0)$ for $a=1, \ldots, \kappa$ and $z^b = y^b\sigma_b(t_0)|y^\kappa(t_0)|/\sigma_a(t)|y^\kappa|$ for $b=\kappa+1, \ldots, N$. The map π is continuous (since $y \neq 0$, hence $y^\kappa \neq 0$, on Ω_e^0 and $\sigma_b(t) > 0$), $\pi(\Omega_e^0) \subset S \cap \Omega_e^0$ and $\pi \mid S \cap \Omega_e^0$ is the identity.

By a theorem of Ważewski [13] (cf. [2, Theorem 3.1, p. 282]), it follows that there exists a point $(t_0, y(t_0)) \in S$ such that the solution of (1) determined by this initial condition is in Ω^0 for $t_0 \leqq t < T$, i.e., satisfies (8). This proves Theorem (*).

Remark. It is clear from the proof of this theorem that the conclusion is valid if the functions (4)–(5) and conditions (6)–(7) are replaced by a set of functions $\sigma_a(t)$ and $\sigma_b(t)$, $1 \leqq a < \kappa < b \leqq N$, which are nonnegative, continuously differentiable for $0 \leqq t < T$, and satisfy the following system of differential inequalities

$$\sigma_b' = [\alpha^b(t) - \alpha_\kappa(t)]\sigma_b - \chi_b(t), \qquad \sigma_a' = [\alpha_a(t) - \alpha^\kappa(t)]\sigma_a + \chi_a(t),$$

where χ_a, χ_b are continuous functions satisfying

$$\gamma\chi_k > \left(1 + \gamma^2 \sum_{j \neq \kappa} \sigma_j^2\right)^{1/2} [\psi_k + \psi_\kappa \gamma\sigma_k], \qquad k \neq \kappa.$$

References

1. P. Hartman, *Unrestricted n-parameter families*, Rend. Circ. Mat. Palermo (2) **7** (1958), 123–142.

2. ———, *Ordinary differential equations*, Wiley, New York, 1964.

3. P. Hartman and A. Wintner, *A criterion for the non-degeneracy of the wave equation*, Amer. J. Math. **71** (1949), 206–213.

4. ———, *On an oscillation criterion of Liapounoff*, Amer. J. Math. **73** (1951), 885–890.

5. ———, *Asymptotic integrations of linear differential equations*, Amer. J. Math. **77** (1955), 45–87.

6. D. B. Hinton, *Disconjugate properties of a system of differential equations*, J. Differential Equations **2** (1966), 420–437.

7. M. Hukuhara, *On the zeros of solutions of linear ordinary differential equations*, Sûgaku **15** (1963), 108–109 (Japanese). Cf. Math. Reviews **29** (1965), 709, #3704, and reference there to M. Nagumo, T. Sato and M. Tumura.

8. A. Lasota and Z. Opial, *L'existence et l'unicité des solutions du problèmes d'interpolation pour l'équation différentielle ordinaire d'ordre n*, Ann. Polon. Math. **15** (1964), 253–271.

9. A. Ju. Levin, *On some boundary value problems*, Naučn. Dokl. Vysš. Školy Fiz.-Mat. Nauki **1958**, No. 5, 34–37. (Russian)

10. ———, *Some estimates of a differentiable function*, Dokl. Akad. Nauk SSSR **138** (1961), 37–38 = Soviet Math. Dokl. **2** (1961), 523–524.

11. Z. Nehari, *On an inequality of Lyapunov*, pp. 256–261, Studies in Mathematical Analysis and Related Topics, Stanford Univ. Press, Stanford, Calif., 1962.

12. ———, *Disconjugacy criteria for linear differential equations*, Carnegie Inst. Tech. Rep., Pittsburgh, Pa., 1967.

13. Z. Opial, *On a theorem of O. Aramă*, J. Differential Equations **3** (1967), 88–91.

14. G. Pólya, *On the mean value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.

15. T. Ważewski, *Sur un principe topologique de l'examen de l'allure asympotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. **20** (1947), 279–313.

16. A. Wintner, *On the non-existence of conjugate points*, Amer. J. Math. **73** (1951), 368–380.

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