ON PRIMITIVE ELEMENTS IN DIFFERENTIALLY ALGEBRAIC EXTENSION FIELDS

BY
A. BABAKHANIAN

It is well known that if F is a field of characteristic zero and $K=F(\alpha_1,\dots,\alpha_n)$ is a finite algebraic extension of F, then K contains a primitive element, i.e. an element α such that $F(\alpha_1,\dots,\alpha_n)=F(\alpha)$. Moreover, by means of Galois theory, it is possible to characterize those elements of the extension field which are primitive.

In the case of finite differentially algebraic extensions the theorem without further restrictions is false. Let Q be the field of rational numbers and δ the usual derivation, i.e., $\delta q=0$ for every $q\in Q$. Let $c_1,\dots,c_n$ be algebraically independent complex numbers over Q. If $(Q\langle c_1,\dots,c_n\rangle,\delta)$ is the differentially algebraic extension of Q where $\delta c=0$ for every $c\in Q\langle c_1,\dots,c_n\rangle$, then the underlying set of $Q\langle c_1,\dots,c_n\rangle$ is identical with that of $Q(c_1,\dots,c_n)$, whence it is clear that there is no element $c\in Q\langle c_1,\dots,c_n\rangle$ such that $Q\langle c_1,\dots,c_n\rangle=Q\langle c\rangle$. Kolchin [2] (also [5, p. 52]) has shown the existence of primitive elements in the case where the differential field $F\langle\alpha_1,\dots,\alpha_n\rangle$ has one derivation operator and the field F has an element f such that $\delta f\neq 0$(1). The differential fields $(F\langle x_0,\dots,x_p\rangle, D)$ considered in this paper are differentially algebraic over F, but F does not contain nonconstant elements. We prove the existence of primitive elements in the case where the derivation operator satisfies the conditions

$$(1)\qquad \begin{aligned} Df &= 0 \quad \text{for every } f\in F, \qquad Dx_0 = 1,\\ x_0\cdot x_1\cdot \cdots \cdot x_{k-1}Dx_k &= 1 \quad \text{for } 0<k\leqq p. \end{aligned}$$

An example of such a differential field is $(C\langle e^z, z, \log z, \log\log z\rangle, \delta)$ where C is the field of complex numbers, $\delta=e^{-z}D$ and D is the usual derivation of functions of a complex variable, i.e., $\delta e^z=e^{-z}De^z=1$, $\delta\log z=e^{-z}D\log z=(e^z\cdot z)^{-1}$, $\delta\log\log z=e^{-z}D\log\log z=(e^z\cdot z\cdot\log z)^{-1}$.

In the sequel, for differentially algebraic extension fields which satisfy conditions (1) not only do we establish the existence of primitive elements, but we give explicit formulas for such elements. In §§7 and 8 we apply these formulas to the asymptotic theory of ordinary differential equations. More precisely, in [6] W. Strodt introduced the concept of the "principal monomials" and "principal solutions" for a certain class of differential equations whose coefficients belong to a logarithmic domain. In [8] Strodt characterized the principal monomials by the concept of

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(1) In the partial case (more than one derivation operator) F must contain a set of elements whose Jacobian does not vanish (Kolchin [2, §4]).

stability. S. Bank [1] investigated all the logarithmic monomials at which an nth order differential polynomial of a certain class is unstable; such logarithmic monomials were called "critical monomials" of the differential polynomial. The algorithm which produced the principal monomials in [6] and the critical monomials in [1] consists essentially of the repeated applications of the transformation $x=e^u$, $y=ve^{mu}$. The effectiveness of this transformation depends upon two crucial lemmas ([6, Lemma 61] and [1, Lemma 13]), to the effect that whenever a transformation $x=e^u$, $y=ve^{mu}$ is applied to a homogeneous, isobaric differential polynomial of positive weight W, with constant coefficients, the transformed differential polynomial always effectively involves at least one term whose weight is less than W, unless the differential polynomial is of the form $cY^{d-W}\cdot Y'^W$ and $m=0$. In this note we generalize these lemmas and prove them with the aid of a result on the transcendence degree of differential field extensions (Theorem 7.1 below)(2).

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1. **Preliminaries.** This section contains some elementary results on differential field extensions. All differential fields considered here are of characteristic zero. The notations are the same as in [3], [4], and [5]. The differential field defined over a field G by a derivation δ will be denoted by (G, δ).

LEMMA 1.1. *Let (H, δ) be a subdifferential field of (G, δ). Suppose the subfield C of elements of G annihilated by δ is contained in H. If $\alpha \in G$ is such that $\delta\alpha \in H$, then either α is transcendental over H or $\alpha \in H$.*

Proof. Follows along the same lines as the proof of Lemma 3.9, Kaplansky [4].

LEMMA 1.2. *Let (H, δ) be a differential field and $(H\langle\alpha\rangle, \delta)$ a differential extension field of (H, δ) such that α is transcendental over H and $\delta\alpha=1$(3). Then there is no element in $H\langle\alpha\rangle$ whose derivative is α^{-1}.*

Proof. Since $\delta\alpha=1\in H$, the underlying sets of $H\langle\alpha\rangle$ and $H(\alpha)$ are identical. Thus every $g\in H\langle\alpha\rangle$, $g\neq 0$, can be written in the form $\alpha^n(P/Q)$ where n is an integer, and P and Q are polynomials in α with coefficients in H such that $P(0)\neq 0$, $Q(0)\neq 0$. The integer n is uniquely determined by g; by direct calculation $\delta(\alpha^n(P/Q))\neq\alpha^{-1}$.

We will need the following well-known lemma:

LEMMA 1.3. *Let $(C\langle\alpha\rangle, \delta)$ be a differential field where C is the subfield of elements of $C\langle\alpha\rangle$ annihilated by δ. If $C\langle\alpha\rangle$ has transcendence degree $p+1$ over C, then $\alpha, \delta\alpha, \ldots, \delta^p\alpha$ are algebraically independent over C. Moreover*

$$C\langle\alpha\rangle = C(\alpha, \delta\alpha, \ldots, \delta^{p+1}\alpha).$$

(2) A proof of the case $m\neq 0$ in the setting of graduated logarithmic fields, independently of ours, is given by Strodt [8].

(3) Such a differential field extension can always be constructed. See Corollary 1 of Theorem 39, page 124, Vol. 1 of [9].

2. **The logarithmic differential fields.** In this section K is a differential field with derivation D. K contains a distinguished sequence $x_0, x_1, \ldots$ of elements called a logarithmic sequence such that $Dx_0=1$ and $x_0 \cdot x_1 \cdot \cdots \cdot x_{p-1} Dx_p=1$ for $p=1, 2, \ldots$(4). $C=\{c : c \in K, Dc=0\}$. For any $p \geqq 0$ the subdifferential field

$$(C(x_0, x_1, \ldots, x_p), D)$$

will be called a *logarithmic differential field.*

LEMMA 2.1. *Let* $F_p=C(x_0, x_1, \ldots, x_p)$ *for each nonnegative integer* p, *and* $F_{-1}=C$. *Then for* $p=0, 1, 2, \ldots, x_p \notin F_{p-1}$.

Proof. By induction on p. Since $Dx_0=1 \neq 0$, for $p=0$ we have $x_p=x_0 \notin C=F_{p-1}$. Suppose for the nonnegative integer q, $x_q \notin F_{q-1}$. Since (F_{q-1}, D) is a differential field and $Dx_q=(x_0 \cdot x_1 \cdot \cdots \cdot x_{q-1})^{-1} \in F_{q-1}$, by Lemma 1.1, x_q is transcendental over F_{q-1}. Let $\delta_q=x_0 \cdot x_1 \cdot \cdots \cdot x_{q-1} D$, then (F_{q-1}, δ_q) is also a differential field. Since x_q is transcendental over F_{q-1} and $\delta_q x_q=1$, by Lemma 1.2, there is no element y in $(F_{q-1}\langle x_q \rangle, \delta_q)$ such that $\delta_q y=x_q^{-1}$. Since $\delta_q x_{q+1}=x_q^{-1}$, $x_{q+1} \notin F_{q-1}\langle x_q \rangle = F_q$. This completes the induction.

LEMMA 2.2. $x_0, x_1, \ldots$ *are algebraically independent over* C.

Proof. Follows from Lemmas 1.1 and 2.1.

3. **Partial order in** $(C(x_0, x_1, \ldots, x_p), D)$. Let K, C, and the sequence $x_0, x_1, \ldots$ be as in §2. We will introduce a partial order in the subfield $F_p=C(x_0, x_1, \ldots, x_p)$ of K as follows. Let

$$V_p = \{x_0^{i_0} \cdot x_1^{i_1} \cdot \cdots \cdot x_p^{i_p} : (i_0, i_1, \ldots, i_p) \in Z^p\}$$

where Z is the ring of integers. V_p is a subgroup of the multiplicative group of F_p. Let $\mathcal{N}_p=\{cv : c \in C-\{0\}, v \in V_p\}$. Let $M=ax_0^{m_0} \cdot x_1^{m_1} \cdot \cdots \cdot x_p^{m_p}$ and $N=bx_0^{n_0} \cdot x_1^{n_1} \cdot \cdots \cdot x_p^{n_p}$ be elements of $\mathcal{N}_p$. We will write $M<N$ if $m_0<n_0$, or for some natural number q, $0<q \leqq p$, $m_k=n_k$ for $k=0, 1, \ldots, q-1$ and $m_q<n_q$. If $(m_0, m_1, \ldots, m_p) = (n_0, n_1, \ldots, n_p)$ we write $M \approx N$. If $f \in C[x_0, x_1, \ldots, x_p]$, $f \neq 0$, it can be written in the form $f=\sum_{i=1}^n c_i M_i$ where $c_i \in C-\{0\}$ and $M_i \in V_p$ such that $M_i \neq M_j$ if $i \neq j$. For some positive integer s, $1 \leqq s \leqq n$, $c_i M_i < c_s M_s$ for $i \neq s$. We will call $c_s M_s$ the *dominating monomial* of f. If $g \in F_p-\{0\}$ then g can be written $g=\sum c_i M_i / \sum b_j N_j$ where $c_i, b_j \in C-\{0\}$ and $M_i, N_j \in V_p$. If $c_s M_s$ is the dominating monomial of $\sum c_i M_i$ and $b_t N_t$ is the dominating monomial of $\sum b_j N_j$ we say $g^*=(c_s M_s)(b_t N_t)^{-1}$ is the dominating monomial of g. If $g, h \in F_p-\{0\}$ we write $g<h$ if $g^*<h^*$, and $g \sim h$ if $g^*=h^*$. We extend this definition of order to F_p by setting $0<g$ for every $g \in F_p-\{0\}$(5). It is clear now that if $f, g, h, k \in F_p$,

(4) The notion of logarithmic sequences was introduced by Strodt (cf. [6] and [8]).

(5) F_p with the partial order $<$ is a field with asymptotic order (cf. Strodt [7]).

(i) if $f<g$, then $f\neq g$;
(ii) $0<1$;
(iii) if $f<g$ and $h\neq 0$, then $fh<gh$;
(iv) if $f<g$ and $h<k$, then $fh<gk$;
(v) if $f<g$ and $h<g$, then $f+h<g$;
(vi) if $f<g$ and $h<g$, then $f<g+h$;
(vii) if $f\sim g$ then $f-g<g$ and conversely;
(viii) if $f\sim g$ and $g\sim h$, then $f\sim h$;
(ix) if $f\sim g$ and $h\sim k$, then $fh\sim gk$;
(x) if $f\sim g$ then $f\neq 0$ and $g\neq 0$.

DEFINITION 3.1. *Let*

$$f=\frac{\sum b_i x_0^{i_0}\cdot x_1^{i_1}\cdot \cdots \cdot x_p^{i_p}}{\sum c_j x_0^{j_0}\cdot x_1^{j_1}\cdot \cdots \cdot x_p^{j_p}}\in F_p.$$

Define $Ef\in F_{p+1}$ *to be*

$$Ef=\frac{\sum b_i x_1^{i_0}\cdot x_2^{i_1}\cdot \cdots \cdot x_{p+1}^{i_p}}{\sum c_j x_1^{j_0}\cdot x_2^{j_1}\cdot \cdots \cdot x_{p+1}^{j_p}}.$$

LEMMA 3.1. *Let* $f, g\in F_p$, *then*
(i) $f<g$ *implies* $Ef<Eg$.
(ii) $f\sim g$ *implies* $Ef\sim Eg$.

Proof. Follows from the definitions.

4. **The functions $S_{i,j}(m)$.**

DEFINITION 4.1. *Let m be a variable; define*

$$S_{k,k}(m) = m(m-1)(m-2)\cdots(m-k+1) \quad \text{if } k > 0,$$

and

$$S_{k,k-j}(m) = (1/j!)S_{k,k}^{(j)}(m) \quad \text{if } j > 0,$$

where $S_{k,k}^{(j)}(m)$ *is the jth derivative of* $S_{k,k}(m)$. *We will make the convention that* $S_{0,0}(m)=1$.

It is clear that $S_{i,j}(m)$ is the elementary symmetric function of degree j in $m, m-1, \ldots, m-i+1$. Thus

$$\sum_{j=0}^{k} x^j S_{k,k-j}(m) = (x+m)(x+m-1)\cdots(x+m-k+1) \tag{1}$$

(cf. [6, §58]). We remark that $S_{k,0}(m)=1$ and $S_{k,j}(m)=0$ if $j<0$.

LEMMA 4.1. *If* $k\geq 1$, $m=(-1)^{k+1}\cdot k\cdot S_{k,k}(m)-(-1)^k\cdot m\cdot \sum_{j=1}^{k-1}(-1)^j S_{k,k-j}(m)$.

Proof. By (1) above $\sum_{j=0}^{k}(-1)^j S_{k,k-j}(m)=(m-1)(m-2)\cdots(m-k)$. Therefore

$$\begin{aligned} m\sum_{j=1}^{k-1}(-1)^j S_{k,k-j}(m) &= m[(m-1)(m-2)\cdots(m-k)-S_{k,k}(m)-(-1)^k S_{k,0}(m)] \\ &= m[(m-1)\cdots(m-k)-m(m-1)\cdots(m-k+1)+(-1)^{k+1}] \\ &= m[(m-1)\cdots(m-k+1)(m-k-m)+(-1)^{k+1}] \\ &= -kS_{k,k}(m)+(-1)^{k+1}m. \end{aligned}$$

5. **The elements** $(x_0\cdot x_1\cdot\cdots\cdot x_k)^m$ **of** $(C(x_0, x_1, \ldots, x_p), D)$. Here and in the next section m is a nonzero integer.

DEFINITION 5.1. *Let* $E_0=E$ *where* E *is as in Definition* 3.1, *and for the positive integer* p, $E_p=[(m-p+1)+x_0D]E_{p-1}$.

LEMMA 5.1. *Let* $V=V(x_0, x_1, \ldots, x_k)\in C(x_0, x_1, \ldots, x_p)$, $k<p$. *Then* $x_0DEV=EDV$.

Proof.

$$\begin{aligned} x_0DEV(x_0, x_1, \ldots, x_k) &= x_0DV(x_1, x_2, \ldots, x_{k+1}) \\ &= x_0\left[\frac{1}{x_0}\frac{\partial EV}{\partial x_1}+\frac{1}{x_0x_1}\frac{\partial EV}{\partial x_2}+\cdots+\frac{1}{x_0x_1\cdots x_k}\frac{\partial EV}{\partial x_{k+1}}\right] \\ &= \frac{\partial EV}{\partial x_1}+\frac{1}{x_1}\frac{\partial EV}{\partial x_2}+\cdots+\frac{1}{x_1\cdot x_2\cdot\cdots\cdot x_k}\frac{\partial EV}{\partial x_{k+1}} \\ &= EDV. \end{aligned}$$

Let $T_k=(x_0\cdot x_1\cdot\cdots\cdot x_k)^m$ where m is a nonzero integer, then:

COROLLARY.

$$\begin{aligned} E_pT_k &= (m-p+1+x_0D)(m-p+2+x_0D)\cdots(m-1+x_0D)(m+x_0D)ET_k \\ &= \sum_{i=0}^{p} S_{p,i}(m)ED^{p-i}T_k. \end{aligned}$$

Proof. By straightforward calculation and Lemma 5.1.

LEMMA 5.2([6]). $D^pT_0=S_{p,p}(m)x_0^{m-p}$ *and for positive integers* p *and* k, $D^pT_k=x_0^{m-p}E_pT_{k-1}$.

Proof. By induction on p. $D^0T_k=T_k=x_0^mE_0T_{k-1}$ by the definitions of T_k and E_0. Suppose $D^{p-1}T_k=x_0^{m-p+1}\cdot E_{p-1}T_{k-1}$. Then

$$D^pT_k = x_0^{m-p}[(m-p+1)+x_0D]E_{p-1}T_{k-1} = x_0^{m-p}E_pT_{k-1}$$

by the definition of E_pT_{k-1}.

([6]) This is a special case of [8, Lemma 66(c)].

LEMMA 5.3. *For any pair of integers* (p, q) *such that* $0 \leqq p < q$, $D^q T_k < D^p T_k$ *if* $D^p T_k \neq 0$.

Proof. If $D^q T_k = 0$, then $D^q T_k < D^p T_k$. Suppose $D^q T_k \neq 0$. By Lemma 5.2,

$$D^p T_k = x_0^{m-p} E_p T_{k-1} \quad \text{and} \quad D^q T_k = x_0^{m-q} E_q T_{k-1}.$$

Since $E_p T_{k-1}$ and $E_q T_{k-1}$ are elements of $C(x_1, x_2, \ldots, x_k)$ and $m-q < m-p$, it follows from the definition of $<$ that $D^q T_k < D^p T_k$.

NOTATION. For $f \in C(x_0, x_1, \ldots, x_k)$, by $\partial_j f$ we will mean the formal partial derivative of f with respect to x_j.

LEMMA 5.4. *For any pair of integers* (p, q) *such that* $0 \leqq p < q$, $\partial_n D^q T_k < \partial_n D^p T_k$ *if* $\partial_n D^p T_k \neq 0$ *and* $0 \leqq n \leqq k$.

Proof. Similar to the proof of Lemma 5.3.

6. **The differential field** $(C\langle T_p \rangle, D)$. In this section we will show that the transcendence degree of $(C\langle T_p \rangle, D)$ is $p+1$ over C.

NOTATION. In the sequel the minor of $\partial_p D^i T_{p-1}$ in the Jacobian determinant

$$\frac{\partial(T_{p-1}, DT_{p-1}, \ldots, D^p T_{p-1})}{\partial(x_0, x_1, \ldots, x_p)}$$

will be denoted by A_i.

LEMMA 6.1. *Suppose the Jacobian determinant*

$$J_{p-1} = \frac{\partial(T_{p-1}, DT_{p-1}, \ldots, D^{p-1} T_{p-1})}{\partial(x_0, x_1, \ldots, x_{p-1})} \neq 0,$$

then $A_i < J_{p-1}$ *for* $i = 0, 1, 2, \ldots, p-1$.

Proof. Write $D^i T_{p-1} = x_0^{m-i} E_i T_{p-2}$, by Lemma 5.2, and write J_{p-1} and A_i in the determinant form. By direct calculation

(3) $$J_{p-1} \sim d x_0^{p[m-(p-1)/2]-1} \cdot x_1^{k_1} \cdot x_2^{k_2} \cdot \cdots \cdot x_{p-1}^{k_{p-1}} \quad \text{with } d \in C - \{0\},$$

and either $A_i = 0$ or

(4) $$A_i \sim c x_0^{p[m-(p-1)/2]-1-(p-i)} \cdot x_1^{j_1} x_2^{j_2} \cdot \cdots \cdot x_{p-1}^{j_{p-1}}$$

where $k_1, k_2, \ldots, k_{p-1}, j_1, j_2, \ldots, j_{p-1}$ are integers. $A_i < J_{p-1}$ for $i = 0, 1, \ldots, p-1$ by comparing the right sides of (3) and (4).

COROLLARY. *If* $J_{p-1} \neq 0$, $\sum_{i=1}^{p} mE_0 A_i \sim mE_0 J_{p-1}$.

Proof. Follows from Lemma 6.1 and the identity $A_p = J_{p-1}$.

LEMMA 6.2. *If* $J_{p-1} \neq 0$, *then*

$$A = \sum_{i=0}^{p} (-1)^i D^i T_{p-1} A_i < T_{p-1} \cdot J_{p-1}.$$

Proof. By the properties (iii) and (iv) of §3, and Lemmas 5.3 and 6.1,

$$(-1)^i D^i T_{p-1} A_i < T_{p-1} J_{p-1} \quad \text{for } i = 0, \ldots, p-1$$

and

$$(-1)^p D^p T_{p-1} \cdot A_p = (-1)^p D^p T_{p-1} \cdot J_{p-1} < T_{p-1} \cdot J_{p-1}$$

by Lemma 5.3 and the property (iv) of §3. Therefore by (v) of §3, $A < T_{p-1} \cdot J_{p-1}$.

COROLLARY. $E_0 A < E_0 T_{p-1} \cdot E_0 J_{p-1}$.

Proof. By Lemma 3.1.

LEMMA 6.3. *Let* $\bar{B}_p = \sum_{i=1}^p mE_0 A_i$ *and* $\bar{B} = \sum_{i=1}^p (-1)^i \cdot i \cdot S_{i,i}(m) B_i$ *where* B_i *is the minor of* $\partial_0 E_i T_{p-1}$ *in the Jacobian determinant*

$$\frac{\partial(E_0 T_{p-1}, E_1 T_{p-1}, \ldots, E_p T_{p-1})}{\partial(x_0, x_1, \ldots, x_p)}$$

then $\bar{B}_p = \bar{B}$.

Proof. $\bar{B}_p$ and $\bar{B}$ may be viewed as the expansions of two $(p+1)\times(p+1)$ determinants by the minors of the first columns. These first columns are respectively $(0, -m, +m, \ldots, (-1)^p m)^t$ and $(0, -S_{1,1}(m), -2S_{2,2}(m), \ldots, -pS_{p,p}(m))^t$. We introduce the $(p+1)\times(p+1)$ determinants $\bar{C}_0, \bar{C}_1, \ldots, \bar{C}_p$ recursively as follows. Define $\bar{C}_0 = \bar{B}$ and define $\bar{C}_{k+1}$ as the determinant obtained from $\bar{C}_k$ by adding $-S_{k+1,k+2-j}(m)$ times the jth row of $\bar{C}_k$, $j = 1, 2, \ldots, k+1$, to the $(k+2)$th row of $\bar{C}_k$. Obviously $\bar{B} = \bar{C}_0 = \bar{C}_1 = \cdots = \bar{C}_p$. Evidently $\bar{C}_0$ has the form of $\bar{B}_p$ in the first row, and the form of B in the remaining rows. Suppose now that for some k in $\{0, 1, \ldots, p-1\}$, $\bar{C}_k$ has the form of $\bar{B}_p$ in the first $k+1$ rows and the form of $\bar{B}$ in the remaining rows. By the above construction $\bar{C}_{k+1}$ has the form of $\bar{B}_p$ in the first $k+2$ rows. In fact, the $(k+2)$th row of $\bar{C}_{k+1}$ is

$$\left(\left\{-(k+1)S_{k+1,k+1}(m) - m\sum_{i=1}^{k} (-1)^i S_{k+1,k+1-i}(m)\right\}, \partial_1 E_0 D^{k+1} T_{p-1}, \ldots, \partial_p E_0 D^{k+1} T_{p-1}\right).$$

Hence the first entry in this row is

$$(-1)^{k+1}\left\{(-1)^{k+2}\cdot(k+1)S_{k+1,k+1}(m) - (-1)^{k+1} m \sum_{i=1}^{k} (-1)^i S_{k+1,k+1-i}(m)\right\} = (-1)^{k+1} m \quad \text{by Lemma 4.1.}$$

The remaining entries of this row have the asserted form by the Corollary of Lemma 5.1.

COROLLARY. *Suppose* $J_{p-1} \neq 0$, *then* $\bar{B} \sim mE_0 J_{p-1}$.

Proof. Follows from the Corollary of Lemma 6.1 and Lemma 6.3.

LEMMA 6.4. *If* $J_{p-1} \neq 0$, *then*

$$J_p = \frac{\partial(T_p, DT_p, \dots, D^pT_p)}{\partial(x_0, x_1, \dots, x_p)} \sim mx_0^{(p+1)[m-p/2]-1} \cdot E_0T_{p-1} \cdot E_0J_{p-1}.$$

Proof. Write the Jacobian J_p in the determinant form, then

$$J_p = x_0^{(p+1)[m-p/2)]-1} \cdot \{m\tilde{B} + \tilde{C}\}$$

where $\tilde{B}$ and $\tilde{C}$ are $(p+1) \times (p+1)$ determinants whose expansions in the minors of the first columns are respectively $\sum_{i=0}^{p} (-1)^i E_i T_{p-1} B_i$ and $\sum_{i=0}^{p} (-1)^{i+1} \cdot i \cdot E_i T_{p-1} B_i$. We assert that

$$\tilde{B} = E_0 \sum_{i=0}^{p} (-1)^i D^i T_{p-1} A_i. \tag{5}$$

This can be verified by applying to $\tilde{B}$ the same row operations as were applied to $\bar{B}$ in Lemma 6.3. Each column of $\tilde{B}$, except the first, is thereby transformed into the corresponding column of a determinant E_0A whose expansion in the minors of the first column is the right side of (5). The verification that the first column of $\tilde{B}$ is transformed into the first column of E_0A is done inductively as in Lemma 6.3 and turns upon the identity

$$E_{k+1}T_{p-1} - \sum_{j=1}^{k+1} S_{k+1,k+2-j}(m) E_0 D^{j-1} T_{p-1} = E_0 D^{k+1} T_{p-1}$$

which follows from the Corollary of Lemma 5.1. As for $\tilde{C}$, $\tilde{C}$ can be written

$$\tilde{C} = (E_0T_{p-1})\bar{B} + \sum_{i=0}^{p} (-1)^{i+1} \cdot i \cdot \sum_{j=0}^{i} S_{i,j}(m)(E_0 D^{i-j} T_{p-1}) B_i. \tag{6}$$

It can be shown by the same reduction as employed in Lemma 6.3 that the second term in the right side of (6) is $< E_0T_{p-1}E_0J_{p-1}$. It now follows from the Corollary of Lemma 6.3 that $\tilde{C} \sim mE_0T_{p-1}E_0J_{p-1}$. Thus, by the Corollary of Lemma 6.2, $m\tilde{B} + \tilde{C} \sim mE_0T_{p-1} \cdot E_0J_{p-1}$. This establishes the lemma.

COROLLARY. *If* $J_{p-1} \neq 0$, *then* $J_p \neq 0$.

THEOREM 6.1. *The differential field* $(C\langle T_p \rangle, D)$ *is of transcendence degree* $p+1$ *over* C.

Proof. Since $(C\langle T_p \rangle, D)$ as a field is contained in the field $C(x_0, x_1, \dots, x_p)$, the transcendence degree of $C\langle T_p \rangle$ over C is at most $p+1$. It is enough, therefore, to show that $T_p, DT_p, \dots, D^pT_p$ are algebraically independent over C. By induction on p and the Corollary of Lemma 6.4, it is enough to show that the Jacobian $J_0 \neq 0$. This by direct calculation is $mx_0^{m-1} \neq 0$.

THEOREM 6.2. (i) $(C(x_0, x_1, \dots, x_p), D) = (C\langle x_0 \cdot x_1 \cdot \cdots \cdot x_p \rangle, D)$,
(ii) $(C(x_0, x_1, \dots, x_p), D) = (C\langle x_p \rangle, D)$.

Proof of (i). By Theorem 6.1, $(C\langle x_0 \cdot x_1 \cdot \cdots \cdot x_p\rangle, D)$ is of transcendence degree $p+1$ over C. Thus $(C(x_0, x_1, \ldots, x_p), D)$ is an algebraic extension of

$$(C\langle x_0 \cdot x_1 \cdot \cdots \cdot x_p\rangle, D).$$

By Lemma 1.1, it is enough to show $Dx_0, Dx_1, \ldots, Dx_p$ are in

$$(C\langle x_0, x_1, \ldots, x_p\rangle, D).$$

$Dx_0 = 1$; therefore $x_0 \in (C\langle x_0 \cdot x_1 \cdot \cdots \cdot x_p\rangle, D)$. Suppose

$$x_0, x_1, \ldots, x_{k-1} \in (C\langle x_0 \cdot x_1 \cdot \cdots \cdot x_p\rangle, D).$$

Since $Dx_k = (x_0 \cdot x_1 \cdot \cdots \cdot x_{k-1})^{-1}$, then $x_k \in (C\langle x_0 \cdot x_1 \cdot \cdots \cdot x_p\rangle, D)$.

Proof of (ii). Since $Dx_p = (x_0 \cdot x_1 \cdot \cdots \cdot x_{p-1})^{-1}$, $x_0 \cdot x_1 \cdot \cdots \cdot x_{p-1} \in (C\langle x_p\rangle, D)$. Thus $x_0 \cdot x_1 \cdot \cdots \cdot x_p \in (C\langle x_p\rangle, D)$. Therefore

$$(C\langle x_0 \cdot x_1 \cdot \cdots \cdot x_p\rangle, D) \subset (C\langle x_p\rangle, D) \subset (C(x_0, x_1, \ldots, x_p), D).$$

7. **Imbedding of** $(C(x_0, x_1, \ldots, x_p), D)$ **in a graduated logarithmic field**(7). Let K, C, D and the logarithmic sequence $x_0, x_1, \ldots$ be as in §2. Recall that K is a differential field with derivative D, C is the subfield of constants: $C = \{c : c \in K$ such that $Dc = 0\}$, and $x_0, x_1, \ldots$ is a logarithmic sequence in K. Suppose further that K contains a multiplicative subgroup U' such that for every $f \in U'$ and every integer $r \geqq 1$ there is a unique $g \in U'$ such that $g^r = f$; we will denote g by $f^{1/r}$. Furthermore, suppose U' contains the set $\{x_0, x_1, \ldots\}$. Let U_p be the subgroup of U' generated by the elements of the form $x_i^m \in U'$, where m is rational and $0 \leqq i \leqq p$. Let G_p be the differential subfield of K generated by U_p over C. We observe that $(C(x_0, x_1, \ldots, x_p), D)$ is a differential subfield of (G_p, D). Moreover, (G_p, D) is an algebraic extension of $(C(x_0, x_1, \ldots, x_p), D)$. Since the transcendence degree of $(C(x_0, x_1, \ldots, x_p), D)$ over C is $p+1$ by Lemma 2.2, the transcendence degree of (G_p, D) is $p+1$ over C. Let $(G, U, D) = \lim_{p\to\infty} (G_p, U_p, D)$. It is clear that (G, D) is a differential field.

We will introduce for the differential field (G_p, D) a partial order $<$, whose restriction to the differential subfield $(C(x_0, x_1, \ldots, x_p), D)$ coincides with the partial order $<$ defined on $(C(x_0, x_1, \ldots, x_p), D)$ in §3, and which is such that the quadruple $(G_p, <, U_p, C)$ is a graduated field as defined in [7]. Define the order relation $<$ in U_p as follows. Let $\{M, N\} \subset U_p$, then $M = x_0^{m_0} \cdot x_1^{m_1} \cdot \cdots \cdot x_p^{m_p}$ and $N = x_0^{n_0} \cdot x_1^{n_1} \cdot \cdots \cdot x_p^{n_p}$ where the exponents are rational numbers. Set $M < N$ if $m_0 < n_0$, or if for some natural number q, $0 < q \leqq p$, $m_k = n_k$ for $k = 0, 1, \ldots, q-1$ and $m_q < n_q$. Let $\mathscr{M} = C^* \cdot U_p$ where $C^* = C - \{0\}$. Define the order relation in $\mathscr{M}$ as follows. If $\{g^*, h^*\} \subset \mathscr{M}$ then $g^* = cM$ and $h^* = dN$ for some M and N in U_p such that $c, d \in C^*$. Set $g^* < h^*$ if $M < N$ and $g^* \approx h^*$ if $M = N$. It is clear that the order

(7) The graduated differential field (X, D) (see Lemma 7.3) is a graduated logarithmic field if U contains a logarithmic sequence (see also [8, p. 14]).

relation $<$ in $\mathscr{M}$ is compatible with the order relation $<$ in U_p. We now define an order relation $<$ in $G_p-\{0\}$. Suppose first that $f\in G_p-\{0\}$ and $f=\sum_{i=1}^n c_iN_i$ with $c_i\in C^*$ and $N_i\in U_p$ (with $i\neq j \Rightarrow N_i\neq N_j$), then for some r, $c_iN_i \lesssim c_rN_r$ for $i=1, 2,\ldots,n$, with strict inequality if $i\neq r$. The representation $f=\sum_{i=1}^n c_iN_i$ is unique. In fact for an arbitrary nonzero $p+1$-tuple $(r_0, r_1,\ldots,r_p)$ of rational numbers, $x_0^{r_0}, x_1^{r_1},\ldots,x_p^{r_p}$ are algebraically independent over C. In this case we write $f\sim c_rN_r$, and we say c_rN_r is the dominating monomial of f. If now f is any element of $G_p-\{0\}$, then $f=\sum c_iM_i/\sum d_jN_j$. Let g^* and h^* be the dominating monomials of the numerator and the denominator respectively. We write $g^*\cdot h^{*-1}\sim f$, and call $g^*\cdot h^{*-1}$ the dominating monomial of f. Let (c, u) be the unique element of $C\times U_p$ such that $g^*\cdot h^{*-1}=cu$. Then u is called the gauge of f and is denoted by $]f[$(8). If f_1 and f_2 belong to $G_p-\{0\}$, we say $f_1<f_2$ if and only if $]f_1[<]f_2[$. If f_1^* and f_2^* are the dominating monomials of f_1 and f_2 respectively and $f_1^*=f_2^*$, then we write $f_1\sim f_2$. It is clear that this order relation in $G_p-\{0\}$ is compatible with the order relation in $\mathscr{M}$. We extend this definition of order by setting $0<f$ for every $f\in G_p-\{0\}$, and $]0[=0$. Thus $f_1\sim f_2$ if and only if $f_1^*=f_2^*$, and hence if and only if $f_1-f_2<f_2$. It is now clear that the partial order $<$ defined here restricted to the subfield $C(x_0, x_1,\ldots,x_p)$ of G_p coincides with the partial order $<$ defined in §3. We observe that this partial order can be extended to the differential field (G, D).

LEMMA 7.1. *If $M\in U_p-\{1\}$ and $e\in U_p$, such that $e<1$, then $MDe<DM$.*

Proof. If $M=x_n^m\cdot x_{n+1}^{m_1}\cdot\cdots\cdot x_p^{m_{p-n}}$ with $m\neq 0$, then by routine calculations $m(x_0\cdot x_1\cdot\cdots\cdot x_n)^{-1}\sim M^{-1}DM$. On the other hand $e=x_m^b\cdot x_{m+1}^{b_1}\cdot\cdots\cdot x_p^{b_{p-m}}$ with $b<0$. Then $De\sim b(x_0\cdot x_1\cdot\cdots\cdot x_m)^{-1}\cdot e<M^{-1}DM$ by lexicographic comparison of exponents. Thus $MDe<DM$.

LEMMA 7.2. *If $\{M, N\}\subset U_p$, $N\neq 1$, such that $M<N$, then $DM<DN$.*

Proof. $M<N$ implies $M=Ne$ for some $e\in U_p$ with $e<1$. Hence $DM=eDN+NDe$. Now $eDN<DN$ since $e<1$ and $NDe<DN$ by Lemma 7.1. Thus $DM<DN$.

LEMMA 7.3. *Let G_p, G, D, C, U_p, U, $<$ and the logarithmic sequence $\{x_0, x_1,\ldots\}$ be as defined above, then*

(i) *the ordered pairs $(G_p, <)$, $(G, <)$ are fields with asymptotic order (for the definition of asymptotic order* (see [7, p. 231])).

(ii) *the ordered quadruples $X_p=(G_p, <, U_p, C)$ and $X=(G, <, U, C)$ are graduated fields* (see [7, p. 231]).

(iii) *(X_p, D) is a graduated differential field. This means $DC=\{0\}$ and D is stable at $U_p-\{1\}$*(9). (See also [8, Definition 20].)

(iv) *(X, D) is a graduated logarithmic field.*

(8) See Definition 17 of [7].

(9) D is stable at $U_p-\{1\}$ if $M<N$ implies $DM<DN$ whenever $M\in G_p$ and $N\in U_p-\{1\}$ (see also [8, Definition 18]).

Proof. (i) and (ii) follow from the order relation defined in G_p. To show (iii) it is enough to show D is stable on $U_p-\{1\}$(9). This follows from Lemma 7.2. (iv) is obvious.

LEMMA 7.4. *Let $y \in (C(x_0, x_1, \ldots, x_p), D)$. Let m be a nonzero rational number. Then $(C\langle y\rangle, D)$ and $(C\langle y^m\rangle, D)$ have the same transcendence degree over C.*

Proof. Suppose $m=a/b$ with a and b integers, $b>0$. Then $(C\langle y\rangle, D)$ is an algebraic extension of $(C\langle y^a\rangle, D)$. Hence the transcendence degree of $(C\langle y^a\rangle, D)$ is same as the transcendence degree of $(C\langle y\rangle, D)$. Similarly $(C\langle y^{a/b}\rangle, D)$ is an algebraic extension of $(C\langle y^a\rangle, D)$. Thus $(C\langle y^m\rangle, D)$, $(C\langle y^a\rangle, D)$ and $(C\langle y\rangle, D)$ have the same transcendence degree over C.

THEOREM 7.1. *Let m be a nonzero rational number, then*

(i) *the transcendence degree of $(C\langle(x_0\cdot x_1\cdot\cdots\cdot x_p)^m\rangle, D)$ is $p+1$ over C.*

(ii) *the transcendence degree of $(C\langle x_p^m\rangle, D)$ is $p+1$ over C.*

(iii) *let $V_p=(x_0\cdot x_1\cdot\cdots\cdot x_p)^m$, then $V_p, DV_p, \ldots, D^pV_p$ are algebraically independent over C.*

(iv) *$x_p^m, Dx_p^m, \ldots, D^px_p^m$ are algebraically independent over C.*

(v) *V_p and x_p^m satisfy no algebraic differential equation of order less than $p+1$.*

Proof. (i) and (ii) follow from Theorems 6.1 and 6.2 and Lemma 7.4 above. (iii) and (iv) follow from (i) and (ii). (v) follows from (iii) and (iv).

8. Applications.

DEFINITION 8.1(10). *Let P be the algebraic differential operator defined by*

$$P(y) = \sum a_i y^{i_0}(Dy)^{i_1}\cdot\cdots\cdot(D^py)^{i_p}, \quad a_i \in C. \tag{I}$$

We say P is homogeneous of degree d if $i_0+i_1+\cdots+i_p=d$, and is isobaric of weight W if $i_1+2i_2+\cdots+pi_p=W$ for every monomial effectively present in the right side of formula (I).

LEMMA 8.1. *Let P be the algebraic differential operator given by formula* (I) *above. Let d and W be positive integers. Let P be homogeneous of degree d and isobaric of weight W. Let m be a rational number and $q=dm-W$. Then under the substitution $y=x_0^m\cdot z$, the expression $x_0^{-q}\cdot P(y)$ is transformed into*

$$Q(z) = \sum b_j z^{j_0}(x_0Dz)^{j_1}\cdot\cdots\cdot((x_0D)^pz)^{j_p}$$

where $b_j \in C$.

Proof. By formula 66(c) of [8], or induction on k

$$D^k[x_0^m z] = x_0^{m-k}\cdot\sum_{i=0}^{k} S_{k,i}(m)(x_0D)^{k-i}\cdot z \tag{II}$$

(10) See §§2–4 of [8].

for any positive integer k. Substitution of $y=x_0^m z$ in $P(y)$ with the aid of formula (II) establishes the Lemma.

DEFINITION 8.2. *The operator Q of Lemma* 8.1 *will be called the m-image of the operator P given by the formula* (I).

THEOREM 8.1 (STRODT). *Under the hypotheses of Lemma* 8.1, *if m is nonzero* (*rational*), *then Q effectively involves terms of weight less than W.*

NOTE. This theorem was proven by W. Strodt in the general context of a graduated logarithmic field (cf. [8, §69]). The proof in [8] depends upon the analytic proof of a special case of this theorem [6, §61]. We eliminate here this dependence upon the analytic proof.

Proof of Theorem 8.1. $P(Y)=\sum a_i Y^{i_0}(DY)^{i_1}\cdot\cdots\cdot(D^pY)^{i_p}$ and

$$Q(Y) = \sum a_i Y^{i_0}\cdot(x_0DY+S_{11}(m)Y)^{i_1}\cdot\cdots\cdot\left([x_0D]^pY+\sum_{i=1}^{p} s_{p,i}(m)[x_0D]^{p-i}Y\right)^{i_p}$$

$$= P(Y, x_0DY, \ldots, [x_0D]^pY)+H(Y, x_0DY, \ldots, [x_0D]^pY)$$

where

$$P(Y, x_0DY, \ldots, [x_0D]^pY) = \sum a_i Y^{i_0}(x_0DY)^{i_1}\cdot\cdots\cdot([x_0D]^pY)^{i_p}.$$

It is clear that all the terms of $H(Y, x_0DY, \ldots, [x_0D]^pY)$ have coefficients in C and are of weight less than W. Suppose $H(Y, x_0DY, \ldots, [x_0D]^pY)\equiv 0$, then

$$\text{(III)} \qquad Q(Y) \equiv P(Y, x_0DY, \ldots, [x_0D]^pY).$$

If $y\in C(x_0, x_1, \ldots, x_k)$, $k<p$, then

$$\text{(IV)} \qquad \begin{aligned} P(Ey, x_0DEy, \ldots, [x_0D]^pEy) &= P(Ey, EDy, \ldots, ED^py) \\ &= EP(y) \quad \text{by Lemma 5.1,} \end{aligned}$$

where $Ey(x_0, x_1, \ldots, x_k)=y(x_1, x_2, \ldots, x_{k+1})$, (Definition 3.1). Since $W>0$, $P(1)=0$ hence

$$\begin{aligned} Q(1) = x_0^{-dm+W}P(x_0^m) &= P(1, x_0D1, \ldots, [x_0D]^p1) \quad \text{by (III)} \\ &= 0. \end{aligned}$$

Thus $P(x_0^m)=0$. Now suppose that for $k>0$, $P((x_0\cdot x_1\cdot\cdots\cdot x_{k-1})^m)=0$. Then

$$\begin{aligned} x_0^{-dm+W}P((x_0\cdot x_1\cdot\cdots\cdot x_k)^m) &= Q((x_1\cdot x_2\cdot\cdots\cdot x_k)^m) \\ &= P((x_1\cdot x_2\cdot\cdots\cdot x_k)^m, x_0D(x_1\cdot x_2\cdot\cdots\cdot x_k)^m, \ldots, \\ &\qquad [x_0D^p](x_1\cdot x_2\cdot\cdots\cdot x_k)^m) \quad \text{by (III)} \\ &= EP((x_0\cdot x_1\cdot\cdots\cdot x_{k-1})^m) \quad \text{by (IV)} \\ &= 0. \end{aligned}$$

It thus follows by induction that $P((x_0\cdot x_1\cdot\cdots\cdot x_p)^m)=0$. This contradicts Theorem 7.1(v) and completes the proof.

THEOREM 8.2 (S. BANK). *Under the hypothesis of Lemma* 8.1 *if* $m=0$ *and* $P(Y)\neq cY^{d-W}\cdot(DY)^W$, $c\in C-\{0\}$, *then* Q *effectively involves terms of weight less than* W.

NOTE. In the case where C is the field of complex numbers and $x_0=x$, $x_1=\log x,\ldots,x_p=\log x_{p-1}$, this theorem has been proven by S. Bank [1, Lemma 13].

Proof of Theorem 8.2. As in the proof of the previous theorem

$$Q(Y) = P(Y, x_0DY,\ldots,[x_0D]^pY)+H(Y,x_0DY,\ldots,[x_0D]^pY).$$

If we suppose $P(Y)=cY^{d-W}\cdot(DY)^W+T(Y)$ where $T(Y)\not\equiv 0$ and has its coefficients in C and is homogeneous of degree d, isobaric of weight W and is of order $\geqq 2$, then $Q(Y)=cY^{d-W}\cdot(x_0DY)^W+R(Y,x_0DY,\ldots,[x_0D]^pY)$ where R is the 0-image of $T(Y)$ (see Definition 8.2). If the conclusion of the theorem is assumed false, then $H(Y,x_0DY,\ldots,[x_0D]^pY)\equiv 0$. Thus

$$\text{(V)}\qquad T(Y,x_0DY,\ldots,[x_0D]^pY)\equiv R(Y,x_0DY,\ldots,[x_0D]^pY).$$

Since every term of $T(Y)$ is of order $\geqq 2$, $T(x_0)=0$. Now suppose that for $k>0$, $T(x_{k-1})=0$. Then

$$\begin{aligned} x_0^WT(x_k) &= R(x_k,x_0Dx_k,\ldots,[x_0D]^px_k)\\ &= T(x_k,x_0Dx_k,\ldots,[x_0D]^px_k)\quad \text{by (V)}\\ &= ET(x_{k-1})\quad \text{by (IV)}\\ &= 0. \end{aligned}$$

It follows by induction that $T(x_p)=0$. This contradicts Theorem 7.1(v) and establishes the theorem.

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UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS