GEOMETRIC DIMENSION OF VECTOR BUNDLES OVER LENS SPACES

BY

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1. **Introduction.** Throughout p will denote an odd prime. Let $q_1, \ldots, q_{n+1}$ be relatively prime to p and let Z_p denote the cyclic group with generator t and relation $t^p = 1$. If $\theta = \exp(2\pi\sqrt{(-1)}/p)$, then we can define a free smooth action of Z_p on the sphere S^{2n+1} by $t^k(z_1, \ldots, z_{n+1}) = (\theta^{kq_1}z_1, \ldots, \theta^{kq_{n+1}}z_{n+1})$, where $(z_1, \ldots, z_{n+1})$ is a complex $n+1$ tuple representing a point of S^{2n+1}. The orbit space of this action is called a (generalized) lens space and is denoted by L^{2n+1}. It is a compact, connected, orientable manifold of dimension $2n+1$ and the quotient map $\pi: S^{2n+1} \to L^{2n+1}$ is a p-fold covering. If all the q_k are equal then L^{2n+1} is known as a standard lens space.

Suppose ξ is a stable vector bundle over a finite CW complex X. Then ξ may be interpreted as the homotopy class of a map $X \to BO$, where BO is the classifying space for the infinite orthogonal group, and the geometric dimension of ξ is the least nonnegative integer k such that the following lifting problem may be solved:

$$\begin{tikzcd} & BO(k) \arrow[d] \\ X \arrow[ur, dashed] \arrow[r] & BO \end{tikzcd}$$

We write g. dim ξ for the geometric dimension of ξ.

If we let P_i denote either the ith Pontrjagin class or its mod p reduction and if c is any integer satisfying $4c+1 \equiv 0 \pmod{p}$, then our main theorems are:

THEOREM A. *If* $\xi \in (KO)^{\sim}(L^{2n+1}) \cap \ker \pi^*$, *then* g. dim $\xi \leqq 2[n/2]+1$.

THEOREM B. *Suppose* $n \geqq 2, p \geqq [n/2]+3$, *and* $\xi \in (KO)^{\sim}(L^{2n+1}) \cap \ker \pi^*$. *Then* g. dim $\xi \leqq 2[n/2]$ *if, and only if, there exists a cohomology class* $\bar{u} \in H^{2[n/2]}(L^{2n+1}; Z_p)$ *such that* $cP_{[n/2]}(\xi) + \bar{u}^2 = 0$.

THEOREM C. (i) L^{2n+1} *immerses in* $R^{2n+2[n/2]+2}$ *for all* n *and all odd primes* p;

(ii) *if all the* q_k *are equal,* $n \geqq 2$, *and* $p \geqq [n/2]+3$, *then* L^{2n+1} *immerses in* $R^{2n+2[n/2]+1}$ *if, and only if,*

$$(-1)^{[n/2]}\binom{n+[n/2]}{n}$$

is a quadratic residue mod p;

Received by the editors July 31, 1967.

(iii) *if all the* q_k *are equal and*

$$\binom{n+[n/2]}{n} \not\equiv 0 \pmod{p},$$

then L^{2n+1} *does not immerse in* $R^{2n+2[n/2]}$.

Therefore Theorem C solves the immersion problem for lens spaces with no twisting (i.e., all the q_k equal) if the prime p is sufficiently large.

Recently there have been published papers on the immersion problem for standard lens spaces. In particular see [2], [3], [4], and [7]. In [4] the case $p=3$ is treated and in [3] the following is proved: if $p \geqq 5$ and $n=\alpha p^k+\beta p^1$, where α, β, k, 1 must satisfy certain restrictions, then L^{2n+1} does not immerse in $R^{2n+2[n/2]+1}$. This result does not overlap with Theorem C.

Uchida's result [7] also applies to the standard lens spaces and says that L^{2n+1} immerses in $R^{2n+2[n/2]+4}$. The technique used in proving Theorems A and B is similar to that used by Uchida. The difference lies in the treatment of the top dimensional obstruction in the Postnikov resolution of the lifting problem:

(1.1)
$$\begin{tikzcd} & BSO(k) \arrow[d] \\ L^{2n+1} \arrow[ur, dashed] \arrow[r, "\xi"'] & BSO(d) \end{tikzcd}$$

where $k=2[n/2]$ or $2[n/2]+1$ and d is a sufficiently large integer. The fiber is the Stiefel manifold $V_{d,d-k}$ and therefore we must know something about their homotopy groups. What we need is stated in §2. To the best of my knowledge this result does not appear in the literature, even though I am sure it is well known. Therefore, the proof is omitted.

Finally I would like to express my thanks to Professors E. Spanier and P. E. Thomas of Berkeley for their help.

2. **Preliminary lemmas.** Let J denote any finite abelian group whose order is relatively prime to p and let $\rho: Z$ or $Z \oplus J \to Z_p$ denote mod p reduction. The following lemma is well known:

LEMMA (2.1). (i) $H^*(L^{2n+1}; Z_p)$ *is a truncated polynomial algebra in the even dimensions on a two dimensional generator*;

(ii) $H^i(L^{2n+1}; Z) \cong Z$ *if* $i=0, 2n+1$,
$\cong Z_p$ *if i even and* $2 \leqq i \leqq 2n$,
$\cong 0$ *otherwise.*

Moreover there is a two dimensional generator x such that in the even dimensions $H^*(L^{2n+1}; Z)$ *is a polynomial algebra truncated by* $x^{n+1}=0$;

(iii) $H^i(L^{2n+1}; J)=0$ *for* $0<i<2n+1$;

(iv) $\pi^*: H^{2n+1}(L^{2n+1}; J) \to H^{2n+1}(S^{2n+1}; J)$ *is an isomorphism*;

(v) $\rho_*: H^{2i}(L^{2n+1}; Z)$ *or* $H^{2i}(L^{2n+1}; Z \oplus J) \to H^{2i}(L^{2n+1}; Z_p)$ *is an isomorphism if* $0<2i<2n+1$.

This lemma could be proved by using the cell decomposition given in [5]. For the next lemma let $V_{n,n-q}$ be the Stiefel manifold of $n-q$ frames in n-space. Denote by $\bar{n}$ (resp. $\bar{q}$) the greatest (resp. least) odd integer $\leqq n$ (resp. $\geqq q$) and denote by S the product $S^{2\bar{q}+1}\times S^{2\bar{q}+5}\times\cdots\times S^{2\bar{n}-3}$, where we must include the factor S^{n-1} if n is even and the factor S^q if q is even. If $\bar{n}=\bar{q}$ replace $S^{2\bar{q}+1}\times\cdots\times S^{2\bar{n}-3}$ by a point. Putting n_1 equal to the least dimension of all the spheres in S we have:

LEMMA (2.2). (i) *For all* i, $\pi_i(V_{n,n-q})$ *and* $\pi_i(S)$ *are isomorphic modulo finite groups*;

(ii) *if* $n_1>2$, *then the p-primary part of* $\pi_i(V_{n,n-q})$ *is zero for* $i<n_1+2p-3$.

3. **Proof of Theorem C.** Let $\nu: L^{2n+1}\to BO$ be the stable normal bundle of L^{2n+1}. Because $\pi: S^{2n+1}\to L^{2n+1}$ is a p-fold covering we see that $\pi^*(\nu)=0$. Therefore Theorem A applies and g. dim $\nu \leqq 2[n/2]+1$. By a well-known theorem of Hirsch (see [1]) this proves (i).

According to [5] the total Pontrjagin class of the tangent bundle τ of L^{2n+1} is given by

$$P(\tau) = (1+q_1^2x^2)\cdots(1+q_{n+1}^2x^2)$$

where x is as in (2.1). If all the $q_k=q$ we get $P(\tau)=(1+q^2x^2)^{n+1}$. Since there is no 2-torsion in $H^*(L^{2n+1}; Z)$ this implies that

$$P(\nu) = \sum_{i\geqq 0}(-1)^i\binom{n+i}{i}q^{2i}x^{2i}.$$

In particular we have

$$P_{[n/2]}(\nu) = (-1)^{[n/2]}\binom{n+[n/2]}{[n/2]}q^{2[n/2]}x^{2[n/2]}.$$

If L^{2n+1} immerses in $R^{2n+2[n/2]}$ we must have $P_i(\nu)=0$ for $i\geqq[n/2]$. This proves (iii).

Now assume all the $q_k=q$, $n\geqq 2$, and $p\geqq[n/2]+3$. Theorem B applies so that g. dim $\nu\leqq 2[n/2]$ if, and only if, there exists $\bar{u}\in H^{2[n/2]}(L^{2n+1}; Z_p)$ so that

$$c(-1)^{[n/2]}\binom{n+[n/2]}{[n/2]}q^{2[n/2]}\bar{x}^{2[n/2]}+\bar{u}^2 = 0,$$

where $\bar{x}$ is the mod p reduction of x. But $\bar{x}$ generates the even dimensions of the truncated polynomial algebra $H^*(L^{2n+1}; Z_p)$ and therefore we can write $\bar{u}=a\bar{x}^{[n/2]}$. Thus g. dim $\nu\leqq 2[n/2]$ if, and only if, there exists an integer a satisfying

$$c(-1)^{[n/2]}\binom{n+[n/2]}{n}q^{2[n/2]}+a^2 \equiv 0 \pmod{p}.$$

By simple number theory this is equivalent to

$$(-1)^{[n/2]}\binom{n+[n/2]}{n}$$

being a quadratic residue mod p. This proves (ii).

4. **Proof of Theorem A.** Consider the lifting problem in (1.1) for $k=2[n/2]+1$. If V denotes the fiber then (2.2) implies that $\pi_i(V)$ is finite with no "p-part" for $i\leqq 2n$. Therefore, according to (2.1), in the Postnikov resolution we encounter only one possible nonzero obstruction:

$$\begin{tikzcd}
 & & BSO(2[n/2]+1) \arrow[d] & \\
 & & E \arrow[r, "k"] \arrow[d] & K(\pi_{2n}(V), 2n+1) \\
S^{2n+1} \arrow[r, "\pi"'] & L^{2n+1} \arrow[r, "\xi"'] \arrow[ur, "\eta"] & BSO(d) &
\end{tikzcd}$$

η lifts ξ up to E and the last obstruction to lifting ξ is $\eta^*(k)\in H^{2n+1}(L^{2n+1};\pi_{2n}(V))$. Since $\xi\circ\pi$ is trivial it must lift all the way to $BSO(2[n/2]+1)$, and in particular, up to E. Moreover, its lifting up to E must be unique and is therefore $\eta\circ\pi$. Since $\xi\circ\pi$ lifts past E so does $\eta\circ\pi$ and this implies $\pi^*\eta^*(k)=0$. But (2.1) now gives $\eta^*(k)=0$. This proves Theorem A.

5. **Proof of Theorem B.** For convenience put $m=[n/2]$. Then we are assuming ξ is a real stable vector bundle over L^{2n+1} and that $\pi^*(\xi)$ is trivial and

(5.1) $$p\geqq m+3, \qquad n\geqq 2.$$

Put $B=BSO(2n+3)$ and $B'=BSO(2m)$ and let $p\colon B\to B'$ be the standard fibration with fiber the Stiefel manifold $V=V_{2n+3,2n+3-2m}$. As mentioned in the introduction to prove Theorem B we study the Postnikov resolution of the lifting problem:

$$\begin{tikzcd}
 & B' \arrow[d, "p"] \\
L^{2n+1} \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

The following facts are well known and will be used implicitly in what follows:

(1) $H^*(BSO(2k+1); Z) \cong Z[P_1, \ldots, P_k] \oplus T$;

(2) $H^*(BSO(2k); Z) \cong Z[P_1, \ldots, P_k, \chi_{2k}] \oplus T$ with the relation $P_k = \chi_{2k}^2$;

(3) $H^*(BSO(2k+1); Z_p) \cong Z_p[P_1, \ldots, P_k]$;

(4) $H^*(BSO(2k); Z_p) \cong Z_p[P_1, \ldots, P_k, \chi_{2k}]$ with the relation $P_k = \chi_{2k}^2$; where χ_{2k} is the Euler class and T is a subring consisting of elements of order 2.

The fiber V of $p: B \to B'$ is $2m-1$ connected and $\pi_{2m}(V) \cong Z$. Thus there is a fundamental class $v \in H^{2m}(V; Z)$ and the first Postnikov invariant is $k_0 = \tau(v) \in H^{2m+1}(B; Z)$, where τ is the transgression in the fibration $V \to B' \xrightarrow{p} B$. Inducing a principal fibration over B with k_0 as classifying map we get the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{i} & B' \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle q_1} \\ K(Z, 2m) & \xrightarrow{j} & E_1 \\ & & \downarrow{\scriptstyle p_1} \\ & & B \xrightarrow[k_0]{} K(Z, 2m+1) \end{array}$$

LEMMA (5.2). *$k_0 = \delta w_{2m}$, where $w_{2m} \in H^{2m}(B; Z_2)$ is a Stiefel-Whitney class and δ is the Bockstein coboundary associated to the coefficient sequence $0 \to Z \to Z \to Z_2 \to 0$.*

Proof. Let $B'' = BSO(2m+1)$. Then we have a commutative diagram of fibrations

$$\begin{array}{ccc} S^{2m} & \longrightarrow & V \\ \downarrow & & \downarrow \\ B' & = & B' \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B'' & \xrightarrow[p'']{} & B \end{array}$$

where p', p'' are the natural fibrations. If $s^* \in H^{2m}(S^{2m}; Z)$ is the fundamental class and τ' the transgression for p', then

$$p''^*(k_0) = p''^*\tau(v) = \tau'(s^*) = \chi_{2m+1} \in H^{2m+1}(B''; Z),$$

the Euler class (see [6]). The fiber of p'' is $2m$ connected and therefore

$$p''^*: H^{2m+1}(B; Z) \to H^{2m+1}(B''; Z)$$

is a monomorphism. Now $\delta w_{2m} = \chi_{2m+1}$ is a relation in $H^*(B''; Z)$ and thus $p''^*(\delta w_{2m}) = p''^*(k_0)$, i.e., $k_0 = \delta w_{2m}$. Q.E.D.

Now consider the Serre exact sequence of $p\colon B' \to B$:

$$\cdots \longrightarrow H^{2m}(B; Z) \longrightarrow H^{2m}(B'; Z) \longrightarrow H^{2m}(V; Z) \xrightarrow{\tau} H^{2m+1}(B; Z) \longrightarrow \cdots.$$

This reduces to

$$\cdots \longrightarrow H^{2m}(B; Z) \xrightarrow{p^*} H^{2m}(B'; Z) \xrightarrow{i^*} Z \longrightarrow Z_2 \longrightarrow 0$$

because $H^{2m}(V; Z) \cong Z$ is generated by v, $\tau(v) = k_0 \neq 0$, and $2k_0 = 0$. Since $\chi_{2m} \in H^{2m}(B'; Z)$ generates $\operatorname{coker} p^*$ in dimension $2m$ we see that $i^*(\chi_{2m}) = \pm 2v$. If need be we change the sign of v so that

$$i^*(\chi_{2m}) = 2v. \tag{5.3}$$

This will not change the Postnikov resolution since $k_0 = -k_0$.

Now consider the higher Postnikov invariants:

(5.4)

$$\begin{tikzcd}
V_1 \arrow[r] & V \arrow[r, "i"] \arrow[dddddd, "v"'] & B' \arrow[d, "q_{2m-1}"] & & \\
K(\pi_{4m-2}, 4m-2) \arrow[rr, "j_{2m-1}"] & & E_{2m-1} \arrow[r, "k_{2m-1}"] \arrow[d] \arrow[rr, bend right, "k''"'] \arrow[ddddd, bend right=60, "f"'] & K(\pi_{4m-1}, 4m) \arrow[r, "\rho"] & K(Z_p, 4m) \\
& & \vdots \arrow[d] & & \\
K(\pi_{i+2m-1}, i+2m-1) \arrow[rr, "j_i"] & & E_i \arrow[r, "k_i"] \arrow[d, "p_i"] & K(\pi_{i+2m}, i+2m+1) & \\
& & E_{i-1} \arrow[d] & & \\
& & \vdots \arrow[d] & & \\
& K(Z, 2m) \arrow[r, "j"] & E_1 \arrow[d, "p_1"] & & \\
& L^{2n+1} \arrow[r, "\xi"'] & E_0 = B \arrow[r, "k_0"'] & K(\pi_{2m}, 2m+1) &
\end{tikzcd}$$

where $\pi_i = \pi_i(V)$. Let f be the composite $E_{2m-1} \to \cdots \to E_1$. Now from (2.2) it follows that

(5.5) (i) π_i has no "p-part" for $i < 4m+3$;

(ii) π_i is finite for $2m < i < 4m-1$ and $4m-1 < i < 4m+3$;

(iii) π_{4m-1} and π_{4m+3} are isomorphic to Z modulo finite groups.

Recall that $\rho: \pi_{4m-1} \to Z_p$ is mod p reduction. Then define k'' to be $\rho_*(k_{2m-1})$.

In the remainder of this section, unless stated otherwise, all cohomology groups will have Z_p coefficients.

LEMMA (5.6). *There is a unique class $k \in H^{4m}(E_1)$ such that $f^*(k) = k''$. Moreover $q_1^*(k) = 0$.*

Proof. From (5.4) we see that f is a composition of fibrations whose fibers are Eilenberg-MacLane spaces $K(\pi_r, r)$, $2m+1 \leqq r \leqq 4m-2$. By (5.5) it follows that the mod p cohomology of these fibers is trivial and therefore $f^*: H^*(E_1) \cong H^*(E_{2m-1})$ by some Serre exact sequences. Finally $q_1^*(k) = q_{2m-1}^*(k'') = \rho_* q_{2m-1}^*(k_{2m-1}) = 0$. Q.E.D.

The reason for introducing k is given by:

LEMMA (5.7). *ξ lifts past E_{2m-1} in the Postnikov resolution* (5.4) *if, and only if, there exists a lifting $\eta: L^{2n+1} \to E_1$ so that $\eta^*(k) = 0$.*

(5.8)
$$\begin{tikzcd}
V_1 \arrow[r] & V \arrow[r, "i"] \arrow[dd, "v"'] & B' \arrow[d, "q_{2m-1}"] & \\
 & & E_{2m-1} \arrow[r, "k''"] \arrow[d, "f"] & K(Z_p, 4m) \\
 & K(Z, 2m) \arrow[r, "j"] \arrow[rr, bend right, "\xi_1"' pos=0.1, "\theta" pos=0.7] & E_1 \arrow[r, "k"] \arrow[d, "p_1"] & K(Z_p, 4m) \\
 & L^{2n+1} \arrow[r, "\xi"'] \arrow[ur, "\xi_2"] & B &
\end{tikzcd}$$

Proof. First note that ξ lifts to E_1 since $\xi^*(k_0) = 0$ by (2.1). The obstructions to lifting ξ up to E_{2m-1} lie in the groups $H^i(L^{2n+1}; \pi_{i-1})$ for $2m+2 \leqq i \leqq 4m-1$. But all these groups are zero according to (2.1) and (5.5) so that ξ lifts to $\bar{\eta}: L^{2n+1} \to E_{2m-1}$. Let $\eta = f \circ \bar{\eta}: L^{2n+1} \to E_1$. Then ξ lifts past E_{2m-1} if, and only if, we can choose $\bar{\eta}$ so that $\bar{\eta}^*(k_{2m-1})=0$. But $\rho_*: H^{4m}(L^{2n+1}; \pi_{4m-1}) \cong H^{4m}(L^{2n+1}; Z_p)$ and therefore $\bar{\eta}^*(k_{2m-1})=0$ if, and only if, $\bar{\eta}^*(k'')=0$. Now f^* is an isomorphism for Z_p coefficients and so $\bar{\eta}^*(k'')=0$ if, and only if, $\eta^*(k)=0$. This shows that if ξ lifts past E_{2m-1} then such a map η exists. Conversely, given η we can lift it to E_{2m-1} and proceed as above. Q.E.D.

The situation now is shown in the preceding equation (5.8).

Define a cohomology operation θ by $\theta = j^*(k) \in H^{4m}(Z, 2m; Z_p)$. $p_1: E_1 \to B$ is a principal fibration and therefore there is an action of the fiber $K(Z, 2m)$ on the total space E_1, $\mu_1: K(Z, 2m) \times E_1 \to E_1$ (see [6]). This induces an action on the homotopy level $\mu_1: [\circ, K(Z, 2m)] \times [\circ, E_1] \to [\circ, E_1]$. If ξ_1, ξ_2 are liftings of ξ to E_1 then they differ by this action, i.e., there exists a class $u \in H^{2m}(L^{2n+1}; Z)$ such that ξ_2 is the composite

$$L^{2n+1} \xrightarrow{d} L^{2n+1} \times L^{2n+1} \xrightarrow{u \times \xi_1} K(Z, 2m) \times E_1 \xrightarrow{\mu_1} E_1$$

where d is diagonal. Let $\iota \in H^{2m}(Z, 2m; Z_p)$ be the fundamental class. In §6 it is proved that $\theta = \iota^2$. If we use the notation $\bar{u}$ for u reduced mod p we have:

Lemma (5.9). (i) $\mu_1^*(k) = 1 \otimes k + \iota^2 \otimes 1 + \iota \otimes x$ *for some* $x \in H^{2m}(E_1)$;
(ii) $\xi_2^*(k) - \xi_1^*(k) = \bar{u}^2 + u \cup \xi_1^*(x)$.

Proof. Let $s_1: K(Z, 2m) \to K(Z, 2m) \times E_1$ and $s_2: E_1 \to K(Z, 2m) \times E_1$ be the canonical inclusions with respect to chosen base points. Then $\mu_1 \circ s_1 \simeq j$ and $\mu_1 \circ s_2 \simeq 1_{E_1}$ (see [6]). By the Künneth formula $\mu_1^*(k)$ must contain $1 \otimes k + \iota^2 \otimes 1$. Because of (5.1) it follows that $H^i(Z, 2m; Z_p) = 0$ for $2m < i < 4m$ and $H^{4m}(Z, 2m; Z_p) \cong Z_p$ generated by ι^2. Thus there is at most one more term in $\mu_1^*(k)$ and that is a cross term $\iota \otimes x$. Finally

$$\begin{aligned}\xi_2^*(k) &= (u \cdot \xi_1)^*(k) = d^*(u \times \xi_1)^* \mu_1^*(k) \\ &= d^*(u \times \xi_1)^*(1 \otimes k + \iota^2 \otimes 1 + \iota \otimes x) = \xi_1^*(k) + \bar{u}^2 + u \cup \xi_1^*(x). \quad \text{Q.E.D.}\end{aligned}$$

Defining a cohomology operation φ by $\varphi = j^*(x) \in H^{2m}(Z, 2m; Z_p)$ we get by a similar proof:

Lemma (5.10). (i) $\mu_1^*(x) = 1 \otimes x + \varphi \otimes 1$;
(ii) $\xi_2^*(x) - \xi_1^*(x) = \varphi(u)$.

If ξ_1, ξ_2 are liftings of ξ to E_1 then $\xi_2 = u \cdot \xi_1$ and $\xi_1 = -u \cdot \xi_2$. Thus (5.9) gives

$$\bar{u}^2 - u \cup \xi_2^*(x) = \xi_1^*(k) - \xi_2^*(k) = -(\xi_2^*(k) - \xi_1^*(k)) = -\bar{u}^2 - u \cup \xi_1^*(x),$$

i.e., $2\bar{u}^2 = u \cup (\xi_2^*(x) - \xi_1^*(x)) = u \cup \varphi(u)$. Since this is true for all $u \in H^{2m}(L^{2n+1}; Z)$ we get

$$\varphi = 2\iota. \tag{5.11}$$

(5.10) now becomes

LEMMA (5.12). (i) $\mu_1^*(x)=1\otimes x+2\iota\otimes 1$;
(ii) $\xi_2^*(x)-\xi_1^*(x)=2\bar{u}$.

The importance of x is now clear since two liftings are homotopic if, and only if, $\xi_1^*(x)=\xi_2^*(x)$. Notice also that there is a unique ξ_1 satisfying $\xi_1^*(x)=0$. In all that follows ξ_1 will denote this particular lifting. Summarizing we have proved:

LEMMA (5.13). (i) $\mu_1^*(k)=1\otimes k+\iota^2\otimes 1+\iota\otimes x$ *for some* $x\in H^{2m}(E_1)$;
(ii) $\varphi=j^*(x)=2\iota$;
(iii) $\xi_2^*(k)-\xi_1^*(k)=\bar{u}^2$;
(iv) $\xi_2^*(x)=2\bar{u}$.

In §7 it is proved that $\xi_1^*(k)=cP_m(\xi)$, where c is as in §1. From (5.13) and (5.7) we immediately get

THEOREM (5.14). *ξ lifts past E_{2m-1} in the Postnikov resolution* (5.4) *if, and only if, there exists $u\in H^{2m}(L^{2n+1};Z)$ so that $cP_m(\xi)+\bar{u}^2=0$. If such a u exists then $\xi_2=\pm u\cdot\xi_1$ are the liftings of ξ that lift past E_{2m-1}.*

Now it is simple to finish the proof of Theorem B. Assume there exists a cohomology class u satisfying (5.14). Then ξ at least lifts past E_{2m-1}. The proof is completed exactly as Theorem A in §4.

6. **Proof of $\theta=\iota^2$.** Define ν to be the composite

$$K(Z,2m)\times B'\xrightarrow{1\times q_1}K(Z,2m)\times E_1\xrightarrow{\mu_1}E_1.$$

According to [6] there is a homotopy commutative diagram of fibrations

$$\begin{array}{ccc}
K(Z,2m) & = & K(Z,2m)\\
\downarrow & & \downarrow j\\
K(Z,2m)\times B' & \xrightarrow{\nu} & E_1\\
\downarrow \tilde{p} & & \downarrow p_1\\
B' & \xrightarrow{p} & B
\end{array}$$

where $\tilde{p}$ is the product fibration. Finally there is the relative Serre exact sequence

$$\cdots\longrightarrow H^r(E_1)\xrightarrow{\nu^*}H^r(K(Z,2m)\times B')\xrightarrow{\tau_0}H^{r+1}(B,B')$$
$$\xrightarrow{1^*}H^{r+1}(E_1)\longrightarrow\cdots\xrightarrow{\nu^*}H^{4m}(K(Z,2m)\times B'),$$

where τ_0 is the relative transgression and 1 is the composite $E_1\xrightarrow{p_1}B\subset(B,B')$. The next lemma is technical and will be needed later.

LEMMA (6.1). *In dimensions $\leqq 4m+3$, $H^*(B,B')$ is a free $H^*(B)$ module on one generator $\delta(\chi_{2m})\in H^{2m+1}(B,B')$, where δ is the coboundary for the pair (B,B').*

Proof. Note that $p^*: H^*(B) \to H^*(B')$ is 1-1 in these dimensions and therefore the following sequence is exact for $q \leqq 4m+2$:

$$0 \longrightarrow H^q(B) \xrightarrow{p^*} H^q(B') \xrightarrow{\delta} H^{q+1}(B, B') \longrightarrow 0.$$

Now coker p^* is a free $H^*(B)$ module on one generator $\chi_{2m} \in H^{2m}(B')$ in dimensions $\leqq 4m+3$. Since δ is an $H^*(B)$ morphism the result follows. Q.E.D.

COROLLARY (6.2). *$\nu^*: H^i(E_1) \to H^i(K(Z, 2m) \times B')$ is 1-1 for $i=2m, 4m$.*

Since $H^{4m}(Z, 2m; Z_p) \cong Z_p$ is generated by ι^2 we certainly have $\theta = a\iota^2$ for some integer a. Just as we proved (5.11) we can show that $\varphi = 2a\iota$.

THEOREM (6.3). *$\theta \neq 0$, i.e., $a \not\equiv 0 \pmod p$.*

Proof. Assume the contrary, i.e., $\theta=0$ and $\varphi=0$. By the Serre exact sequence of the fibration $p_1: E_1 \to B$ and the definition of φ there is $y \in H^{2m}(B)$ such that $x = p_1^*(y)$. Now the transgression $\tau_0: H^q(K(Z, 2m) \times B') \to H^{q+1}(B, B')$ is an $H^*(B)$ morphism where $H^*(K(Z, 2m) \times B')$ is made into an $H^*(B)$ module by the map

$$K(Z, 2m) \times B' \xrightarrow{\tilde{p}} B' \xrightarrow{p} B$$

(see [6]). Also τ_0 can be defined on a subset of $H^{4m}(K(Z, 2m) \times B')$ containing image ν^*. Moreover τ_0 is zero on image ν^*. Thus

$$\nu^*(k) = (1 \times q_1)^* \mu_1^*(k) = (1 \times q_1)^*(1 \otimes k + \iota \otimes x) = \iota \otimes p^*(y)$$

since $\mu_1^*(k) = 1 \otimes k + \theta \otimes 1 + \iota \otimes x$ (see the proof of (5.9)) and $q_1^*(k)=0$. This means that $\tau_0(\iota \otimes p^*(y))=0$. But $\tau_0(\iota \otimes p^*(y)) = \tau_0(y \cdot \iota \otimes 1)$ (by the definition of the $H^*(B)$ module structure) $= y \cdot \tau_0(\iota \otimes 1)$. Therefore it follows that either $y=0$ or $\tau_0(\iota \otimes 1)=0$.

Assume $y=0$. Then $x = p_1^*(y) = 0$ and $\nu^*(k) = \iota \otimes p^*(y) = 0$, i.e., $k=0$ by (6.2). This implies that the mod p reduction of the Postnikov invariant k_{2m-1} is zero. Let V_{2m-1} be the fiber of q_{2m-1} in (5.4). Then V_{2m-1} is $4m-2$ connected and $\pi_{4m-1}(V_{4m-1}) \cong \pi_{4m-1}$. If $v \in H^{4m-1}(V_{2m-1}; \pi_{4m-1})$ is the fundamental class, then k_{2m-1} is the transgression of v in the fibration q_{2m-1}. Since $\rho_*(k_{2m-1})=0$ we see that $\rho_*(v)$ transgresses to zero, which implies by the Serre exact sequence of q_{2m-1}, that $\rho_*(v)$ is the image of a class in $H^{4m-1}(B')$. But $H^{4m-1}(B')=0$ and thus $\rho_*(v)=0$. This is a contradiction since $\rho_*(v)$ can be identified with mod p reduction in Hom $(\pi_{4m-1}; Z_p)$.

Therefore $y \neq 0$ and $\tau_0(\iota \otimes 1)=0$. Then there exists $z \in H^{2m}(E_1)$ such that $\nu^*(z) = \iota \otimes 1$. But

$$\nu^*(z) = (1 \times q_1)^* \mu_1^*(z) = (1 \otimes q_1)^*(1 \otimes z + j^*(z) \otimes 1) = 1 \otimes q_1^*(z) + j^*(z) \otimes 1.$$

Thus $q_1^*(z)=0$ and $j^*(z)=\iota$. But q_1^* is an isomorphism in dimension $2m$ and again we have a contradiction. Q.E.D.

We have proved that $\theta = a\iota^2$, where $a \not\equiv 0 \pmod p$. Let a' be such that $aa' \equiv 1 \pmod p$. Then multiplication by a' in Z_p induces a homotopy equivalence of $K(Z_p, 4m)$ with itself. It is clear that as far as the obstructions to lifting ξ are concerned we may replace k by $a'^*(k)$ as the Postnikov invariant, i.e., without loss of generality we may assume $\theta = \iota^2$.

7. **Proof of $\xi_1^*(k) = cP_m(\xi)$.** mod p reduction of the first Postnikov invariant k_0 is zero since $H^{2m+1}(B; Z_p) = 0$. Thus there is a commutative diagram of fibrations

$$(7.1)\qquad \begin{array}{ccc} K(Z, 2m) & \xrightarrow{\rho} & K(Z_p, 2m) \\ \downarrow{\scriptstyle j} & & \downarrow \\ E_1 & \xrightarrow{g} & K(Z_p, 2m) \times B \\ \downarrow{\scriptstyle p_1} & & \downarrow \\ B & = & B \\ \downarrow{\scriptstyle k_0} & & \downarrow{\scriptstyle \rho k_0} \\ K(Z, 2m+1) & \xrightarrow{\rho} & K(Z_p, 2m+1) \end{array}$$

(7.2) Let $h: B' \to K(Z_p, 2m) \times B$ be the composite $g \circ q_1$ and let F denote the fiber of h.

Then we have the commutative diagram

$$(7.3)\qquad \begin{array}{ccccccc} V_1 & \longrightarrow & B' & \xrightarrow{q_1} & E_1 & \xrightarrow{p_1} & B \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1} \\ F & \longrightarrow & B' & \xrightarrow{h} & K(Z_p, 2m) \times B & \longrightarrow & B \end{array}$$

where $V_1 = \text{fiber } q_1$. $K(Z_p, 2m)$ is an H-space via loop multiplication $\mu_2: K(Z_p, 2m) \times K(Z_p, 2m) \to K(Z_p, 2m)$. The action of the fiber on the total space of the trivial fibration induced by ρk_0 is simply $\mu_2 \times 1: K(Z_p, 2m) \times K(Z_p, 2m) \times B \to K(Z_p, 2m) \times B$ and we have the commutative diagram

$$(7.4)\qquad \begin{array}{ccc} K(Z, 2m) \times E_1 & \xrightarrow{\mu_1} & E_1 \\ \downarrow{\scriptstyle \rho \times g} & & \downarrow{\scriptstyle g} \\ K(Z_p, 2m) \times K(Z_p, 2m) \times B & \xrightarrow{\mu_2 \times 1} & K(Z_p, 2m) \times B \end{array}$$

Let $\iota_1 \in H^{2m}(Z_p, 2m; Z_p)$ be the fundamental class.

LEMMA (7.5). (i) *$h^*(\iota_1 \otimes 1) = (1/2)\chi_{2m} + p^*(y)$ for some $y \in H^{2m}(B)$;*
(ii) *$h^*: H^r(K(Z_p, 2m) \times B) \to H^r(B')$ is an isomorphism if $r < 4m$;*

(iii) $1 \otimes b+\iota_1 \otimes b'+\lambda\iota_1^2 \otimes 1 \in H^{4m}(K(Z_p, 2m)\times B) \cap \ker h^*$, *where* $b \in H^{4m}(B)$ *and* $b' \in H^{2m}(B)$, *if and only if*, $b'+2\lambda y=0$ *and* $2yb'+4b+\lambda P_m=0$.

Proof. $h^*(\iota_1 \otimes 1)=q_1^*g^*(\iota_1 \otimes 1)$ and so $i^*h^*(\iota_1 \otimes 1)=v^*j^*g^*(\iota_1 \otimes 1)=v^*\rho^*(\iota_1)$ $=v^*(\iota)=\bar{v}=\rho_*(v)$. But we also have $i^*(\frac{1}{2}\chi_{2m})=\bar{v}$ according to (5.3). From the Serre exact sequence of $p: B' \to B$ we get $h^*(\iota_1 \otimes 1)=\frac{1}{2}\chi_{2m}+p^*(y)$ for some $y \in H^{2m}(B)$. This proves (i). (ii) and (iii) are short calculations. Q.E.D.

From the Serre exact sequence of h we easily get:

COROLLARY (7.6).
$$H^r(F) \cong Z_p \quad \text{if } r = 0, 4m-1,$$
$$\cong 0 \quad \text{if } 0 < r < 4m-1.$$

According to (7.1) and (7.5) there exists uniquely $k' \in H^{4m}(K(Z_p, 2m)\times B)$ such that

$$h^*(k') = 0 \quad \text{and} \quad j^*g^*(k') = \iota^2, \tag{7.7}$$

namely $k'=1 \otimes b+\iota_1 \otimes b'+\iota_1^2 \otimes 1$, where $b'=-2y$ and $b=-(\frac{1}{4})P_m+y^2$. Thus $g^*(k')$ satisfies $q_1^*(g^*(k'))=0$ and $j^*(g^*(k'))=\iota^2$. Both of these conditions are satisfied by k and we would like to show that $g^*(k')=k$.

Let V_i be the fiber of $q_i: B' \to E_i$. Then we have the fibrations $V_{i+1} \to V_i \xrightarrow{v_i} K(\pi_{2m+i}, 2m+i)$, $i \geqq 0$, where $V_0=V$, $v_0=v$, and $v_i \in H^{2m+i}(V_i, \pi_{2m+i})$ is the fundamental class. The maps $V_{i+1} \to V_i$ compose to give a map $V_{2m-1} \to V_1$. From (5.5) it follows that this composition induces an isomorphism between $H^*(V_1)$ and $H^*(V_{2m-1})$. Therefore

$$\begin{aligned} H^r(V_1) &\cong Z_p \quad \text{if } r = 0 \text{ or } 4m-1,\\ &\cong 0 \quad \text{if } 0 < r < 4m-1, \end{aligned} \tag{7.8}$$

since V_{2m-1} is $4m-2$ connected and $H^{4m-1}(V_{2m-1})\cong \operatorname{Hom}(\pi_{4m-1}, Z_p)\cong Z_p$.

From (7.3) we get the commutative diagram

$$\begin{tikzcd} H^{4m-1}(F) \arrow[r, "j'^*"] \arrow[d, "\tau'"] & H^{4m-1}(V_1) \arrow[d, "\tau_1"] \\ H^{4m}(K(Z_p, 2m)\times B) \arrow[r, "g^*"] & H^{4m}(E_1) \end{tikzcd} \tag{7.9}$$

where τ_1, τ' are the respective transgressions. Since $h^*(k')=0$ the Serre exact sequence of h says that there is $v' \in H^{4m-1}(F)$ such that $k'=\tau'(v')$. Thus $g^*(k')$ $=\tau_1 j'^*(v')$. Since $q_1^*(k)=0$ we see that k is in the image of τ_1 and therefore if we could show that $j'^*: H^{4m-1}(F)\cong H^{4m-1}(V_1)$ it would follow that k is in the image of g^*. To do this we need only show that $g^*(k')\neq 0$. After a straightforward calculation based on (7.4) we get

$$\nu^*g^*(k') = \iota \otimes \chi_{2m}+\iota^2 \otimes 1. \tag{7.10}$$

Because k' is the only class satisfying (7.7) we conclude $k=g^*(k')$.

Now $\nu^*(k)=(1\times q_1)^*\mu_1^*(k)=(1\times q_1)^*(1\otimes k+\iota^2\otimes 1+\iota\otimes x)=\iota^2\otimes 1+\iota\otimes q_1^*(x)$ since $q_1^*(k)=0$. Therefore we get

$$q_1^*(x) = \chi_{2m}. \tag{7.11}$$

$h^*(\iota_1\otimes 1)=q_1^*g^*(\iota_1\otimes 1)=\frac{1}{2}\chi_{2m}+p^*(y)$ by (7.5). Since q_1^* is an isomorphism in dimension $2m$ it follows that

$$g^*(\iota_1\otimes 1) = \tfrac{1}{2}x+p_1^*(y). \tag{7.12}$$

Finally

$$\begin{aligned}\xi_1^*(k) &= \xi_1^*g^*(k') = \xi_1^*g^*(1\otimes b+\iota_1\otimes b'+\iota_1^2\otimes 1)\\ &= \xi^*(b)+\xi_1^*g^*(\iota_1\otimes 1)\xi^*(b')+\xi_1^*g^*(\iota_1\otimes 1)^2\\ &= \xi^*(b)+\xi^*(y)\xi^*(b')+\xi^*(y)^2 = \xi^*(b+yb'+y^2)\\ &= \xi^*(-\tfrac{1}{4}P_m) = cP_m(\xi).\end{aligned}$$

A FINAL REMARK. Since $(KO)^\sim(S^{2n+1})=0$ if $n\not\equiv 0 \pmod 4$ and Z_2 otherwise and since it can be shown that $(KO)^\sim(L^{2n+1})$ has order twice a power of p if $n\equiv 0 \pmod 4$ it follows that the restriction $\xi\in\ker\pi^*$ in Theorems A and B is not so restrictive.

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