NORM DECREASING HOMOMORPHISMS OF MEASURE ALGEBRAS

BY
ROGER RIGELHOF

1. **Introduction.** Let G be a locally compact group (=locally compact Hausdorff topological group). By the measure algebra of G we mean the Banach $*$-algebra $M(G)$ of bounded regular Borel measures on G. It is well known that $M(G)$ is the dual of $C_0(G)$ the Banach space of continuous complex-valued functions on G which "vanish at infinity". The group algebra $L^1(G)$ is the $*$-subalgebra of $M(G)$ consisting of all measures that are absolutely continuous with respect to the Haar measure on G. Alternatively $L^1(G)$ can be defined as the Banach $*$-algebra of (equivalence classes of) Haar summable functions on G.

Let F and G be locally compact groups, α a bicontinuous isomorphism of F onto G, and γ a character on F. For μ in $M(F)$ and f in $C_0(G)$ let $T\mu(f)=\mu(\gamma(f\circ\alpha))$. Then the mapping $\mu\mapsto T\mu$ is an isometric $*$-isomorphism of $M(F)$ onto $M(G)$ (Lemma 2). In §3 we show that every norm decreasing isomorphism of $M(F)$ onto $M(G)$ is of the above form and consequently is an isometric $*$-isomorphism (Theorem 1). A number of other results follow from this, in particular Wendel's theorem on norm decreasing isomorphisms of group algebras.

Theorem 1 generalizes a result of B. E. Johnson [7] on isometric isomorphisms of measure algebras. In [7] Johnson used norm properties of measures in $L^1(F)$ to show that each isometric isomorphism maps $L^1(F)$ onto $L^1(G)$ and consequently by applying Wendel's result it follows that each isometric isomorphism is of the form described above. Our generalization of Johnson's result has the advantage that Wendel's theorem is a consequence.

In §4 we use the results of §3 to prove a similar structure theorem for norm decreasing homomorphisms T of $M(F)$ onto $M(G)$ such that $T(\mu * L^1(F))=\{0\}$ implies $T\mu=0$. (In this case the bicontinuous isomorphism of F onto G mentioned above becomes a continuous and open homomorphism of F onto G.)

The final section of the paper is concerned with bipositive homomorphisms of $M(F)$ onto $M(G)$. Here we show that the hypothesis of norm decreasing may be dropped provided that the homomorphism maps the positive cone onto the positive cone.

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2. **Preliminaries.** Let (X, τ) be a topological space, we write X_τ in place of (X, τ) and if $Y \subseteq X$, Y_τ is Y with the relative τ-topology. If $(x_j : j \in J)$ is a net in X which converges to x we shall write this as $x_j \to x$. The τ-closure of a subset $Y \subseteq X$ is written $\mathrm{Cl}_\tau\, Y$.

Let G be a locally compact group, $C(G)$ is the Banach space of bounded continuous complex valued functions on G, $K(G)$ the subspace of $C(G)$ consisting of functions whose support is compact. $C_0(G)$ may be defined as the closure of $K(G)$ in $C(G)$.

As remarked in the introduction, $M(G)$ is the dual of $C_0(G)$. The weak topology on $M(G)$ is the $\sigma(M(G), C_0(G))$-topology, i.e. the coarsest topology on $M(G)$ such that for each f in $C_0(G)$ the mapping $\mu \mapsto \mu(f)$ of $M(G)$ into the complex field is continuous. We shall frequently write σ in place of $\sigma(M(G), C_0(G))$.

For x in G, ε_x is the Dirac measure at x, i.e. the element of $M(G)$ defined by $\varepsilon_x(f) = f(x)$ for $f \in C_0(G)$. G^ε is the collection of all Dirac measures.

If μ is any measure on G, then we shall use μ for the corresponding integral, writing $\mu(f) = \int f\, d\mu$ for any μ-integrable function f.

The support of a measure μ, written Supp (μ) is the set of all x in G such that for each neighborhood U of x, there is an f in $K(G)$ which vanishes outside U with $\mu(f) \neq 0$.

PROPOSITION 1. (i) *G is isomorphic and homeomorphic to G^ε_σ.*

(ii) *The mappings $\mu \mapsto \mu * \lambda$, $\mu \mapsto \lambda * \mu$ and $\mu \mapsto \mu^*$ of $M(G)$ into itself are $\sigma(M(G), C_0(G))$-continuous.*

(iii) *Let V be the linear span of G^ε. Then for any $\mu \in M(G)$,*

$$\mu \in \mathrm{Cl}_\sigma \{\lambda \in V : \|\lambda\| \leqq \|\mu\| \text{ and } \mathrm{Supp}\,(\lambda) \subseteq \mathrm{Supp}\,(\mu)\}.$$

Proof. (i) It is straightforward to see that the mapping $x \mapsto \varepsilon_x$ of G onto G^ε is a bicontinuous isomorphism.

(ii) For $f \in C_0(G)$, define $\lambda^-(f)$ by $\lambda^-(f)(x) = \lambda({}_x f)$ where ${}_x f(y) = f(xy)$. Then $\lambda^-(f) \in C_0(G)$ [6, p. 264], and $\mu * \lambda(f) = \mu(\lambda^-(f))$. It follows that $\mu \mapsto \mu * \lambda$ is $\sigma(M(G), C_0(G))$-continuous. Similarly the definition of μ^* yields the continuity of the mapping $\mu \mapsto \mu^*$ [6, p. 299]. To show that $\mu \mapsto \lambda * \mu$ is continuous observe that this mapping can be written $\mu \mapsto \mu^* \mapsto \mu^* * \lambda^* = (\lambda * \mu)^* \mapsto (\lambda * \mu)^{**} = \lambda * \mu$.

The proof of (iii) is the same as the proof of Corollary 1 on p. 71 of [2].

COROLLARY. *Given any $\mu \in M(G)$ there is a net $(\mu_j : j \in J)$ in*

$$A = \{\lambda \in V : \mathrm{Supp}\,(\lambda) \subseteq \mathrm{Supp}\,(\mu);\ \|\lambda\| \leqq \|\mu\|\}$$

such that $\mu_j \xrightarrow{\sigma} \mu$ and $\|\mu\| = \lim \|\mu_j\|$.

Proof. $\mu_j \xrightarrow{\sigma} \mu$ implies $\liminf \|\mu_j\| \geqq \|\mu\|$ since the mapping $\mu \mapsto \|\mu\|$ is lower semicontinuous in $M(G)_\sigma$.

Let $L^1(G)$ be the subset of $M(G)$ consisting of measures that are absolutely continuous with respect to the Haar measure on G. It is well known that $L^1(G)$ is a two-sided norm closed ideal in $M(G)$. Let $B(L^1(G))$ be the algebra of all bounded linear operators which map $L^1(G)$ into itself, taken with the topology of simple convergence (i.e. the coarsest topology such that for each $\lambda \in L^1(G)$, the mapping $T \mapsto T\lambda$ is continuous. $L^1(G)$ is taken with the norm topology). For $\mu \in M(G)$ we define $T_\mu \in B(L^1(G))$ by $T_\mu \lambda = \mu * \lambda$. The mapping $\mu \mapsto T_\mu$ is one-one. The *so*-topology on $M(G)$ is the coarsest topology such that this map is continuous. A theorem of Wendel [8] states that the image of $M(G)$ under this mapping is closed in $B(L^1(G))$.

PROPOSITION 2. (i) *On norm bounded sets* $so \supset \sigma(M(G), C_0(G))$.

(ii) $G^\varepsilon_\sigma = G^\varepsilon_{so}$.

(iii) *The mappings* $\mu \mapsto \mu * \lambda$, $\mu \mapsto \lambda * \mu$ *are so-continuous.*

(iv) *Let V be the linear span of the Dirac measures. For any* $\mu \in M(G)$ *we have* $\mu \in \mathrm{Cl}_{so}\{\lambda \in V : \mathrm{Supp}(\lambda) \subseteq \mathrm{Supp}(\mu)$ *and* $\|\lambda\| \leqq \|\mu\|\}$.

(v) *Let* $\mu \in M(G)$ *then* $\mu \in \mathrm{Cl}_{so}\{\lambda \in L^1(G) : \|\lambda\| \leqq \|\mu\|\}$.

Proof. (i) follows from Lemma 1.1.1 of [4] and the fact that any $f \in C_0(G)$ is left (and right) uniformly continuous [6, p. 185]. (ii) follows from (i) together with Proposition 1(i) and the fact that the mapping $x \mapsto \varepsilon_x * \lambda$ is continuous for λ in $L^1(G)$. The proof of (iii) is an easy calculation. To prove (iv) use Lemmata 1.1.2 and 1.1.3 of [4]. The proof of (v) is an easy consequence of the facts that $L^1(G)$ is an ideal in $M(G)$ and $L^1(G)$ contains an approximate unit of norm 1.

COROLLARY. *The mapping* $\lambda \mapsto \|\lambda\|$ *is lower semicontinuous in the so-topology.*

The next proposition is the well-known relationship between the Dirac measures on G and the extreme points of the unit ball in $M(G)$.

PROPOSITION 3. *The extreme points of the unit ball of $M(G)$ are precisely the measures* $\gamma\varepsilon_x$, $|\gamma| = 1$, $x \in G$.

PROPOSITION 4. *Let T be a norm bounded homomorphism of $M(F)$ onto $M(G)$ and suppose that for μ in $M(F)$, $T(\mu * L^1(F)) = \{0\}$ implies $T\mu = 0$. Then T restricted to norm bounded sets is continuous as a mapping of $M(F)_{so}$ onto $M(G)_\sigma$.*

Proof. Let $(e_j : j \in J)$ be an approximate unit in $L^1(F)$. By Lemma 4.1.2 of Greenleaf [4], $Te_j \xrightarrow{\sigma} \varepsilon$ where ε is an idempotent such that $\varepsilon * \mu = \mu * \varepsilon = \mu$ for all μ in $\mathrm{Cl}_\sigma T(L^1(F))$. Since T is onto and $T(\mu * L^1(F)) = \{0\}$ implies $T\mu = 0$, it follows that ε is the unit of $M(G)$. Let $(\mu_j : j \in J)$ be a norm bounded net in $M(F)$ and suppose that $\mu_j \xrightarrow{so} \mu \in M(F)$. For any λ in $L^1(F)$ we have $\mu_j * \lambda \to \mu * \lambda$ so that $T\mu_j * T\lambda \to T\mu * T\lambda$. Since $(T\mu_j : j \in J)$ is norm bounded and the unit ball of $M(G)$ is $\sigma(M(G), C_0(G))$-compact it follows that there is a $\mu' \in M(G)$ and a subnet $(T\mu_{j(i)})$ such that $T\mu_{j(i)} \xrightarrow{\sigma} \mu'$. Consequently for any λ in $L^1(F)$ we have $T\mu * T\lambda = \mu' * T\lambda$, since multiplication is separately continuous in $M(G)_\sigma$. Thus $T\mu * Te_j = \mu' * Te_j$, and since $Te_j \xrightarrow{\sigma} \varepsilon$, we have $T\mu = \varepsilon * T\mu = \mu'$. This completes the proof.

LEMMA 1. *Let F, G be locally compact groups and let T be a norm decreasing homomorphism of $M(F)$ onto $M(G)$. Then for each $x \in F$ there is a $y \in G$, and a complex number γ, $|\gamma|=1$ such that $T\varepsilon_x=\gamma\varepsilon_y$.*

Proof. Let L_x be the operator on $M(G)$ defined by $L_x\mu=T\varepsilon_x * \mu$. Since T is a norm decreasing homomorphism of $M(F)$ onto $M(G)$ it follows that L_x^{-1} exists; $L_x^{-1}=L_{x^{-1}}$ and $\|L_x\| \leqq 1$, $\|L_x^{-1}\| \leqq 1$. Let μ, $\lambda \in M(G)$, $\|\mu\| \leqq 1$, $\|\lambda\| \leqq 1$, and suppose that $T\varepsilon_x=a\mu+(1-a)\lambda$ for $0<a<1$. Then $\varepsilon_e=aL_x^{-1}\mu+(1-a)L_x^{-1}\lambda$, and since ε_e is an extreme point of the unit ball, we have $\mu=\lambda$. Thus $T\varepsilon_x$ is an extreme point of the unit ball of $M(G)$ so now the lemma follows from Proposition 3.

3. Norm decreasing isomorphisms of measure algebras.

LEMMA 2. *Let F and G be locally compact groups, α a bicontinuous isomorphism of F onto G and γ a character on F. Let T be the mapping of $M(F)$ into $M(G)$ defined by*

$$T\mu(f) = \mu(\gamma(f \circ \alpha)), \qquad \mu \in M(F),\ f \in C_0(G).$$

Then T is an isometric $*$-isomorphism of $M(F)$ onto $M(G)$; and T is a bicontinuous mapping of $M(F)_\sigma$ onto $M(G)_\sigma$.

Proof. Consider the mapping $f \mapsto \gamma(f \circ \alpha)$ of $C_0(G)$ into $C_0(F)$. It is easily seen that this mapping is an isometry of $C_0(G)$ onto $C_0(F)$ and that T is its adjoint. Consequently T is an isometry of $M(F)$ onto $M(G)$ and is a bicontinuous mapping of $M(F)_\sigma$ onto $M(G)_\sigma$. That $T(\mu * \lambda)=T\mu * T\lambda$ and $(T\mu)^*=T\mu^*$ hold may be shcwn by calculations.

THEOREM 1. *Let F and G be locally compact groups and let T be a norm decreasing isomorphism of $M(F)$ onto $M(G)$. Then there is a bicontinuous isomorphism α of F onto G, and a character γ on F such that*

$$T\mu(f) = \mu(\gamma(f \circ \alpha)), \qquad \mu \in M(F),\ f \in C_0(G).$$

Proof. For $x \in F$ we have by Lemma 1, that there is a complex number $\gamma(x)$ with $|\gamma(x)|=1$, and an element $\alpha(x)$ of G such that $T\varepsilon_x=\gamma(x)\varepsilon_{\alpha(x)}$. Consider the mappings

$$\gamma : x \mapsto \gamma(x) \quad \text{and} \quad \alpha : x \mapsto \alpha(x).$$

We first show that α is a homomorphism of F into G, and γ is a homomorphism of F into the complex numbers of absolute value 1. Clearly $T\varepsilon_{xy}=\gamma(xy)\varepsilon_{\alpha(xy)}$, and

$$T\varepsilon_x * T\varepsilon_y = \gamma(x)\gamma(y)\varepsilon_{\alpha(x)} * \varepsilon_{\alpha(y)} = \gamma(x)\gamma(y)\varepsilon_{\alpha(x)\alpha(y)}.$$

Since $T\varepsilon_x * T\varepsilon_y=T(\varepsilon_x * \varepsilon_y)=T\varepsilon_{xy}$, we have

$$\gamma(xy)\varepsilon_{\alpha(xy)} = \gamma(x)\gamma(y)\varepsilon_{\alpha(x)\alpha(y)};$$

and since the Dirac measures are pairwise linearly independent, we have $\gamma(xy)=\gamma(x)\gamma(y)$, and $\alpha(xy)=\alpha(x)\alpha(y)$. Thus α is a homomorphism of F into G, and γ is a homomorphism of F into the complex numbers.

We now show that γ is continuous. For any nonnegative f in $C_0(G)$ we have $|T\varepsilon_x(f)| = \varepsilon_{\alpha(x)}(f)$ consequently $\gamma(x)|T\varepsilon_x(f)| = T\varepsilon_x(f)$. Thus to show the continuity of γ it suffices to show that the mapping $x \mapsto T\varepsilon_x(f)$ is continuous at e in F where f is a nonnegative function in $C_0(G)$ such that $T\varepsilon_e(f) \neq 0$. By the definition of the weak topology the mapping $T\varepsilon_x \mapsto T\varepsilon_x(f)$ is continuous from $M(G)_\sigma$ into the complex numbers. The mapping $x \mapsto \varepsilon_x$ is continuous from F to F^ε_{so} by Propositions 1 and 2 and the mapping $\varepsilon_x \mapsto T\varepsilon_x$ is continuous from F^ε_{so} to $M(G)_\sigma$ by Proposition 4. It follows that γ is continuous and thus γ is a character.

The continuity of α follows by considering the mappings

$$x \mapsto \varepsilon_x \mapsto T\varepsilon_x = \gamma(x)\varepsilon_{\alpha(x)} \mapsto \varepsilon_{\alpha(x)} \mapsto \alpha(x).$$

The only mapping we have to check is the mapping $\gamma(x)\varepsilon_{\alpha(x)} \mapsto \varepsilon_{\alpha(x)}$, of a subset of $M(G)_\sigma$ into $M(G)_\sigma$. But since this mapping is multiplication by the continuous character $x \mapsto (\gamma(x))^-$, it is continuous. Thus α is continuous since it may be written as a composite of continuous mappings. Moreover since each of the above mappings is one to one we have that α is a continuous one to one homomorphism. Now consider T^{-1}. By the open mapping theorem T^{-1} is a bounded isomorphism of $M(G)$ onto $M(F)$. Thus Proposition 4 applies and we have that $\varepsilon_{\alpha(x)} \mapsto T^{-1}\varepsilon_{\alpha(x)} = (\gamma(x))^-\varepsilon_x$ is continuous from G^ε_{so} into $M(F)_\sigma$. The continuity of α^{-1} restricted to $\alpha(F)$ now follows by considering the mappings

$$\alpha(x) \mapsto \varepsilon_{\alpha(x)} \mapsto T^{-1}\varepsilon_{\alpha(x)} = (\gamma(x))^-\varepsilon_x \mapsto \varepsilon_x \mapsto x.$$

Thus F is homeomorphic to $\alpha(F)$ and since a locally compact group is complete $\alpha(F)$ is complete and therefore closed.

Now suppose α is not onto; then there is a y in $G \backslash \alpha(F)$ and a compact neighborhood V of y such that $V \cap \alpha(F) = \varnothing$ because $\alpha(F)$ is closed. Since $T^{-1}\varepsilon_y$ is in $M(F)$, by Proposition 2 there is a net $(\mu_j : j \in J)$ such that

$$\mu_j = \sum_{i=1}^{n_j} b_{i,j}\varepsilon_{x_{i,j}}, \qquad x_{i,j} \in F,\ b_{i,j} \text{ complex},$$
$$\|\mu_j\| \leqq \|T^{-1}\varepsilon_y\|$$

and $\mu_j \xrightarrow{so} T^{-1}\varepsilon_y$. Thus by Proposition 4, $T\mu_j \xrightarrow{\sigma} \varepsilon_y$. Note that $T\varepsilon_x = \gamma(x)\varepsilon_{\alpha(x)}$, thus

$$T\mu_j = \sum_{i=1}^{n_j} b_{i,j}\gamma(x_{i,j})\varepsilon_{\alpha(x_{i,j})}.$$

Since G is locally compact there is a function f in $C_0(G)$ such that $f(y) = 1$ and $f(G \backslash V) = 0$ and $0 \leqq f(x) \leqq 1$ for all $x \in G$. Since $T\mu_j \xrightarrow{\sigma} \varepsilon_y$ we have

$$\sum_{i=1}^{n_j} b_{i,j}\gamma(x_{i,j})\varepsilon_{\alpha(x_{i,j})}(f) \to f(y).$$

But since $\alpha(x_{i,j}) \in \alpha(F) \cap (G \backslash V)$, $\varepsilon_{\alpha(x_{i,j})}(f) = f(\alpha(x_{i,j})) = 0$, so that $f(y) = 0$, a contradiction. Thus α is onto.

All that remains now is to establish the formula $(T\mu)(f)=\mu(\gamma(f\circ\alpha))$.

Let T_1 be the mapping defined by

$$(T_1\mu)(f) = \mu(\gamma(f\circ\alpha)), \qquad \mu\in M(F),\ f\in C_0(G).$$

By Lemma 2 we have that T_1 is an isomorphism and isometry from $M(F)$ onto $M(G)$. Hence in view of Proposition 4, T_1 is continuous on norm bounded sets from $M(F)_{so}$ onto $M(G)_\sigma$. Now observe that

$$T_1\varepsilon_x(f) = \varepsilon_x(\gamma(f\circ\alpha)) = \gamma(x)f(\alpha(x)) = \gamma(x)\varepsilon_{\alpha(x)}(f) = T\varepsilon_x(f)$$

Thus T and T_1 coincide on F^ε, and by Proposition 2 each $\mu\in M(F)$ is a *so*-adherence point of a norm bounded set of linear combinations of Dirac measures so we have $T=T_1$. This completes the proof.

COROLLARY 1. *Every norm decreasing isomorphism of $M(F)$ onto $M(G)$ is an isometric $*$-isomorphism.*

Proof. This follows from Lemma 2 and the above theorem.

COROLLARY 2. *Let T be a norm decreasing isomorphism of $M(F)$ onto $M(G)$; then T is a bicontinuous mapping of $M(F)_\sigma$ onto $M(G)_\sigma$.*

Proof. This follows from Lemma 2 and the above theorem.

COROLLARY 3. *Each norm decreasing isomorphism of $M(F)$ onto $M(G)$ maps $L^1(F)$ onto $L^1(G)$.*

Proof. This follows from the formula.

COROLLARY 4. *$L^1(G)$ is invariant under norm decreasing automorphisms of $M(G)$.*

The following example shows that a $*$-isomorphism of $M(F)$ onto $M(G)$ need not be norm decreasing. Let F and G be finite abelian groups of order n, and suppose that F and G are not isomorphic. Let $\gamma_1,\gamma_2,\ldots,\gamma_n$ be the n characters of F and define functions f_i, $i=1,2,\ldots,n$ on F by

$$f_i(x) = (1/n)\gamma_i(x), \qquad x\in F,\ i = 1,2,\ldots,n.$$

Using the orthogonality relations for characters on abelian groups it follows that $f_1,f_2,\ldots,f_n$ are orthogonal idempotents which span $M(F)$. Clearly $f_i^*=f_i$. Let $g_1,g_2,\ldots,g_n$ be a basis for $M(G)$ defined in the analogous manner. Define T by $Tf_i=g_i$, $i=1,2,\ldots,n$ and extend T to a linear mapping of $M(F)$ into $M(G)$. It is easily seen that T is a $*$-isomorphism of $M(F)$ onto $M(G)$. T cannot be norm decreasing if F and G are not isomorphic since this would violate Theorem 1.

As a consequence of Theorem 1 we shall derive a theorem due to Wendel [8] on isomorphisms of $L^1(F)$ onto $L^1(G)$. First we shall need the following lemma.

LEMMA 3. *Let T be a bounded isomorphism of $L^1(F)$ onto $L^1(G)$. Then there is a unique bounded isomorphism $T^\#$ of $M(F)$ onto $M(G)$ which extends T. Moreover, $\|T^\#\|=\|T\|$.*

Proof. Clearly T is continuous as a mapping of $L^1(F)_{so}$ onto $L^1(G)_{so}$. Since $M(G)_{so}$ is quasi-complete (see the remark preceding Proposition 2) and since each μ in $M(F)$ is a *so*-adherence point of a bounded set in $L^1(F)$ (Proposition 2), T has a unique extension $T^\#$ to a continuous linear mapping of $M(F)_{so}$ into $M(G)_{so}$ [1, Chapitre III, §2, No. 5].

To show that $T^\#(\mu * \lambda) = T^\#\mu * T^\#\lambda$, μ, λ in $M(F)$, let $(\mu_j : j \in J)$ and $(\lambda_k : k \in K)$ be nets in $L^1(F)$ such that $\mu_j \xrightarrow{so} \mu$ and $\lambda_j \xrightarrow{so} \lambda$. Then since multiplication is separately continuous in $M(F)_{so}$ (Proposition 2) we have

$$\mu * \lambda = \lim_j \left(\lim_k \mu_j * \lambda_k\right).$$

Therefore the *so*-continuity of $T^\#$ implies that

$$T^\#(\mu * \lambda) = \lim_j \left(\lim_k T(\mu_j * \lambda_k)\right).$$

Since $T^\# = T$ on $L^1(F)$ we have $T^\#(\mu_j * \lambda_k) = T\mu_j * T\lambda_k$. Now using the fact that multiplication is separately continuous in $M(G)_{so}$ we have

$$\lim_j \left(\lim_k T\mu_j * T\lambda_k\right) = T^\#\mu * T^\#\lambda.$$

Combining the above we have $T^\#(\mu * \lambda) = T^\#\mu * T^\#\lambda$.

To show that $T^\#$ is one-one, let $\lambda, \mu \in M(F)$ and suppose $T^\#\mu = T^\#\lambda$. If $\lambda \neq \mu$ then there is a ν in $L^1(F)$ such that $\lambda * \nu \neq \mu * \nu$. Then $T(\lambda * \nu) = T^\#\lambda * T\nu = T^\#\mu * T\nu = T(\mu * \nu)$ which contradicts the assertion that T is an isomorphism because $\lambda * \nu$ and $\mu * \nu$ are in $L^1(F)$. Therefore $T^\#$ is one-one.

We now show that $T^\#$ is onto. Let $\mu' \in M(G)$, by Proposition 2 there is a net $(\mu_j' : j \in J)$ in $L^1(G)$ such that $\mu_j' \xrightarrow{so} \mu'$ and $\|\mu_j'\| \leqq \|\mu'\|$. By the open mapping theorem T^{-1} is bounded, so that $(T^{-1}\mu_j' : j \in J)$ is a bounded Cauchy net in $L^1(F)_{so}$. Since $M(F)_{so}$ is quasi-complete there is a μ in $M(F)$ such that $T^{-1}\mu_j' \xrightarrow{so} \mu$. Then $T(T^{-1}\mu_j') \xrightarrow{so} T\mu$ so that $T^\#\mu = \mu'$ and $T^\#$ is onto.

We next show $\|T^\#\| = \|T\|$. Clearly $\|T\| \leqq \|T^\#\|$. To show the reverse inequality let $\mu \in M(F)$ be given. By Proposition 2 there is a net $(\mu_j : j \in J)$ in $L^1(F)$ such that $\mu_j \xrightarrow{so} \mu$ and $\|\mu_j\| \leqq \|\mu\|$ so that $T\mu_j \xrightarrow{so} T^\# u$. Since the mapping $\lambda \mapsto \|\lambda\|$ is lower semicontinuous in the *so*-topology (Corollary 1 to Proposition 2), we have $\|T^\#\mu\| \leqq \liminf \|T\mu_j\| \leqq \liminf \|T\| \, \|\mu_j\| \leqq \|T\| \, \|\mu\|$. Therefore $\|T^\#\| \leqq \|T\|$ and hence $\|T^\#\| = \|T\|$.

The uniqueness of $T^\#$ is an easy consequence of Propositions 2 and 4.

THEOREM 2 (WENDEL). *Let T be a norm decreasing isomorphism of $L^1(F)$ onto $L^1(G)$. Then there is a bicontinuous isomorphism α of F onto T, a character γ on G such that*

$$T\lambda(f) = \lambda(\gamma(f \circ \alpha)) \qquad \lambda \in L^1(F), f \in C_0(G).$$

Proof. This follows easily from Theorem 1 in virtue of Lemma 3.

COROLLARY. *Each norm decreasing isomorphism of $L^1(F)$ onto $L^1(G)$ is an isometric $*$-isomorphism.*

4. **Norm decreasing homomorphisms of measure algebras.** In this section we give the structure of all norm decreasing homomorphisms T of $M(F)$ onto $M(G)$ such that for $\mu \in M(F)$, $T(\mu * L^1(F))=\{0\}$ implies $T\mu=0$.

LEMMA 4. *Let F and G be locally compact groups, α a continuous and open homomorphism of F onto G, and γ a character on F. Let T be the mapping of $M(F)$ into $M(G)$ defined by*

$$T\mu(f) = \mu(\gamma(f \circ \alpha)) \qquad \mu \in M(F), f \in C_0(G).$$

Then T is a norm decreasing $*$-homomorphism of $M(F)$ onto $M(G)$; T is continuous as a mapping of $M(F)_{so}$ onto $M(G)_{so}$ and $T(L^1(F))=L^1(G)$.

Proof. Let e be the unit of G and let $F_0=\{x : \alpha(x)=e\}$. Then F_0 is a closed normal subgroup of F. Let $\pi: F \to F/F_0$ be the canonical mapping then there is a bicontinuous isomorphism $\beta: F/F_0 \to G$ such that $\alpha=\beta \circ \pi$. Consider the following mappings:

(1) $\gamma: M(F) \to M(F)$ defined by $\gamma\mu(f)=\mu(\gamma f)$ $\mu \in M(F), f \in C_0(F)$,

(2) $\pi^{**}: M(F) \to M(F/F_0)$ defined by $\pi^{**}\mu(f)=\mu(f \circ \pi)$ $\mu \in M(F), f \in C_0(F/F_0)$,

(3) $\beta^{**}: M(F/F_0) \to M(G)$ defined by $\beta^{**}\mu(f)=\mu(f \circ \beta)$ $\mu \in M(F/F_0), f \in C_0(G)$.

By Lemma 2, (1) and (3) are isometric isomorphisms. By Lemma 5.1.8 of Greenleaf [4], π^{**} is a norm decreasing homomorphism of $M(F/F_0)$ onto $M(G)$. Observe that $T=\beta^{**}\pi^{**}\gamma$ so that T is a norm decreasing homomorphism of $M(F)$ onto $M(G)$. That T is a $*$-homomorphism follows by a calculation. To complete the proof it suffices to show that $T(L^1(F))=L^1(G)$. In the proof of Theorem 5.2.1 of [4] Greenleaf shows that $\pi^{**}(L^1(F))=L^1(F/F_0)$ consequently by Corollary 3 to Theorem 1, $T(L^1(F))=L^1(G)$.

The next lemma is an observation of Greenleaf's [4] and [5].

LEMMA 5. *Let F and G be locally compact groups; α a continuous homomorphism of F into G and γ a character of F. Then the mapping T defined by $T\mu(f)=\mu(\gamma(f \circ \alpha))$ is a norm decreasing homomorphism of $M(F)$ into $M(G)$ which is continuous on norm bounded sets as a mapping of $M(F)_{so}$ into $M(G)_{\sigma}$. If α is a monomorphism then so is T.*

Proof. It is straightforward to verify that T is a norm decreasing homomorphism of $M(F)$ into $M(G)$. The stated continuity property of T follows from Lemma 1.1.1 of Greenleaf [4] and the fact that for $f \in C_0(G)$, $f \circ \alpha$ is left (and right) uniformly continuous on F. If α is a monomorphism then to show that T is a monomorphism it suffices to show that the mapping α^{**} defined by $\alpha^{**}\mu(f)=\mu(f \circ \alpha)$, $\mu \in M(F)$, $f \in C_0(G)$ is a monomorphism. To show this let $K \subseteq F$ be any compact subset, then $\mu(K)=\mu(\alpha^{-1}\alpha(K))=\alpha^{**}\mu(\alpha(K))$ since α is a monomorphism. It follows that $\alpha^{**}\mu=0$ implies $\mu=0$ [6, p. 175]. This completes the proof.

Theorem 3. *Let T be a norm decreasing homomorphism of $M(F)$ onto $M(G)$. If $T(\mu * L^1(F))=\{0\}$ implies $T\mu=0$, then there is a character γ on F and a continuous and open homomorphism α of F onto G such that*

$$T\mu(f) = \mu(\gamma(f\circ\alpha)) \quad \text{for } \mu\in M(F),\ f\in C_0(G).$$

Proof. For $x\in F$ we have by Lemma 1, a complex number $\gamma(x)$, $|\gamma(x)|=1$ and an $\alpha(x)\in G$ such that $T\varepsilon_x=\gamma(x)\varepsilon_{\alpha(x)}$. It follows as in the proof of Theorem 1 that α is a continuous homomorphism of F into G; γ is a character of F and that $T\mu(f)=\mu(\gamma(f\circ\alpha))$ (in virtue of Lemma 5). To complete the proof it remains to show that α is open and onto. Let e be the unit of G and let $F_0=\{x : \alpha(x)=e\}$. Let $\pi: F\to F/F_0$ be the canonical homomorphism and define $T_1: M(F)\to M(F/F_0)$ by $T_1\mu(f)=\mu(\gamma(f\circ\pi))$. By Lemma 4, T_1 is a norm decreasing homomorphism which is onto. Let $\mu\in\ker T_1$, then $\mu(\gamma(f\circ\pi))=0$ for all f in $C_0(F/F_0)$ and it follows from Lemma 5.2.2 of Greenleaf [4], that $\mu(\gamma(f\circ\pi))=0$ for all $f\in C(F/F_0)$. Let $\beta: F/F_0\to G$ be such that $\alpha=\beta\circ\pi$. For $f\in C_0(G), f\circ\beta\in C(F/F_0)$ so that $\mu(\gamma(f\circ\beta\circ\pi))=0$. Thus we have that $\ker T_1\subseteq\ker T$ so we may define a homomorphism $T_2: M(F/F_0)\to M(G)$ by $T_2(T_1\mu)=T\mu$. To show that T_2 is an isomorphism we need only show that T_2 is a monomorphism since T_2 is clearly onto. Define $T': M(F/F_0)\to M(G)$ by $T'\mu(f)=\mu(f\circ\beta)$, then $T'T_1\mu(f)=T_1\mu(f\circ\beta)=\mu(\gamma(f\circ\beta\circ\pi))=T\mu(f)=T_2T_1\mu(f)$ for all μ in $M(F)$. Since T and T_1 are onto, $T_2=T'$. Now since β is a continuous monomorphism, T_2 is a norm decreasing monomorphism (Lemma 5).

If we now apply Theorem 1 to T_2 we have that there is a character γ' on F/F_0 and a bicontinuous isomorphism α' of F/F_0 onto G such that

$$T_2(T_1\mu)(f) = T_1\mu(\gamma'(f\circ\alpha')), \qquad \mu\in M(F) \text{ and } f\in C_0(G).$$

Thus using the definition of T_1 we have

$$\mu(\gamma(\gamma'\circ\pi)(f\circ\alpha'\circ\pi)) = \mu(\gamma(f\circ\alpha)), \qquad \mu\in M(F) \text{ and } f\in C_0(G).$$

It follows that $\alpha'\circ\pi=\alpha$ and consequently α is open and onto.

Corollary 1. *Let T be a norm decreasing homomorphism of $M(F)$ onto $M(G)$. If $T(\mu * L^1(F))=\{0\}$ implies $T\mu=0$, then $T(L^1(F))=L^1(G)$ and consequently T is continuous as a mapping of $M(F)_{so}$ onto $M(G)_{so}$.*

Proof. This follows from the theorem and Lemma 4.

Corollary 2. *Each norm decreasing homomorphism of $M(F)$ onto $M(G)$ satisfying the hypothesis of Theorem* 3 *is a $*$-homomorphism.*

Remark. There are homomorphisms T which are onto but which do not satisfy the hypothesis of the theorem as the following example shows.

Let F be a nondiscrete locally compact group and let $G=\{e\}$ be the group consisting of only one element. Define $T: M(F)\to M(G)$ by $T\mu=\sum_{x\in F}\mu(\{x\})\varepsilon_e$. Then T

is a norm decreasing homomorphism which is onto and T' annihilates $L^1(F)$. Corollary 1 to the theorem shows that the conclusion of Theorem 2 cannot hold for T.

5. Bipositive homomorphisms of measure algebras.

DEFINITION. A homomorphism T of $M(F)$ onto $M(G)$ is said to be bipositive if T maps the set of positive measures onto the set of positive measures.

THEOREM 4. *Let T be a bipositive homomorphism of $M(F)$ onto $M(G)$. If $T(\mu * L^1(F))=\{0\}$ implies $T\mu=0$ then there is a continuous and open homomorphism α of F onto G such that*

$$T\mu(f) = \mu(f \circ \alpha), \qquad \mu \in M(F),\ f \in C_0(G).$$

If T is an isomorphism then so is α.

Proof. Note that a positive measure μ on G is a real multiple of a Dirac measure if and only if each $\lambda \in M(G)$ for which $\mu-\lambda$ is a positive measure is a real multiple of μ. Since this property is clearly preserved under bipositive maps we have that for each $x \in F$, there is a real number $\gamma(x)$ and an $\alpha(x) \in G$ such that $T\varepsilon_x=\gamma(x)\varepsilon_{\alpha(x)}$. It is readily verified that the mapping $x \mapsto \gamma(x)$ is a homomorphism of G into the positive reals. Since T is positive, T is bounded, thus there is a constant M such that $\|T\mu\| \leqq M\|\mu\|$, for all $\mu \in M(F)$. It follows that $\{\gamma(x) : x \in G\}$ is a bounded subgroup of the positive reals. Thus we must have $\gamma(x)=1$ for all $x \in F$. Applying Proposition 4 we see that $T\mu(f)=\mu(f \circ \alpha)$ for $\mu \in M(F), f \in C_0(G)$. By Lemma 5, T is a norm decreasing homomorphism and Theorems 1 and 3 now apply.

COROLLARY. *Let T be a bipositive homomorphism of $M(F)$ onto $M(G)$. If $T(\mu * L^1(F))=\{0\}$ implies $T\mu=0$ then T is a norm decreasing $*$-homomorphism.*

REMARK. The example given after Corollary 2 to Theorem 3 shows that there are bipositive homomorphisms which do not satisfy the hypothesis of the theorem.

The author has recently learned that G. Gaudry has proven Theorem 4 for the case when T is a bipositive isomorphism (to appear, Canad. J. Math.).

REFERENCES

1. N. Bourbaki, *Éléments de mathématique*. Livre V. *Espaces vectoriels topologiques*, Chapitres I–V, Hermann, Paris, 1953–1955.

2. ———, *Éléments de mathématique*. Livre VI. *Intégration*, Chapitres I–IV, Hermann, Paris, 1965.

3. N. Dunford and J. Schwartz, *Linear operators*. Part I: *General theory*, Interscience, New York, 1958.

4. F. Greenleaf, *Norm decreasing homomorphisms of group algebras*, Pacific J. Math. **15** (1965), 1187–1220.

5. ———, "Closed subalgebras of group algebras which are group algebras" in *Function algebras*, Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965, Scott-Foresman, Chicago, Ill., 1966, pp. 276–281.

6. E. Hewitt and K. Ross, *Abstract harmonic analysis*. I, Springer-Verlag, Berlin, 1963.

7. B. E. Johnson, *Isometric isomorphisms of measure algebras*, Proc. Amer. Math. Soc. **15** (1964), 186–188.

8. J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. **2** (1952), 251–261.

McMaster University,
Hamilton, Canada
McGill University,
Montreal, Canada