NECESSARY CONDITIONS FOR EMBEDDING RINGS INTO FIELDS([1])

BY

ABRAHAM A. KLEIN

1. **Introduction.** If a (noncommutative) integral domain R is embeddable in a (skew-) field D, then the multiplicative semigroup $R^* = R - \{0\}$ is embeddable in the group $D^* = D - \{0\}$. Mal'cev [6] has constructed an integral domain R whose multiplicative semigroup R^* cannot be embedded in a group and hence the ring R cannot be embedded in a field. In [4], we have constructed integral domains that cannot be embedded in fields, but their multiplicative semigroups are embeddable in groups. Our construction was based on the following property of matrix rings over fields:

$\mathfrak{N}_l$: *If a matrix C of order l is nilpotent, then $C^l = 0$.*

Clearly, this condition holds also in subrings of fields, and it is therefore necessary for embedding a ring in a field. We have constructed integral domains R which do not satisfy $\mathfrak{N}_l$, for some $l \geqq 3$, but the semigroups R^* satisfy a certain sufficient condition for embeddability in a group.

The problem arises: What is the relation between the necessary conditions $\mathfrak{N}_l$, $l = 1, 2, \ldots$ and the trivial necessary condition

$\mathfrak{M}$: *R^* is embeddable in a group.*

The aim of this paper is to prove that if an integral domain satisfies all the conditions $\mathfrak{N}_l$, then it also satisfies $\mathfrak{M}$. Thus, if we denote the condition $\bigcap_{l=1}^{\infty} \mathfrak{N}_l$ by $\mathfrak{N}$ and we restrict ourselves to integral domains, we have: $\mathfrak{N} \subseteq \mathfrak{M}$. Moreover, since by [4] there exist integral domains which satisfy $\mathfrak{M}$, but do not satisfy $\mathfrak{N}_l \supseteq \mathfrak{N}$ (for some $l \geqq 3$) we obtain: $\mathfrak{N} \subsetneqq \mathfrak{M}$. The problem whether the condition $\mathfrak{N}$ is sufficient for an integral domain to be embeddable in a field remains open.

Our main result is achieved by showing that if R is an integral domain and satisfies $\mathfrak{N}$, then the semigroup R^* satisfies all the necessary and sufficient conditions of Mal'cev [7] for embeddability in a group. Our formulation of Mal'cev's conditions is a slight modification of that given by Cohn [1]. We show that for cancellative semigroups, a restricted set of conditions, which we call "cancelled conditions" is sufficient for embeddability. We prove that the cancelled conditions have a certain structure and using it we obtain that $\mathfrak{N}_l$ implies all the cancelled conditions of order

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l in R^*. Hence $\mathfrak{N} = \bigcap_{i=1}^{\infty} \mathfrak{N}_l$ implies that all the cancelled conditions hold in R^* and therefore R^* is embeddable in a group.

In the last section of the paper we study the relation between $\mathfrak{N}_l$ and the condition

$\mathfrak{M}_l$: R^* *satisfies all the conditions of Mal'cev of order l.*

The result is $\mathfrak{N}_1 = \mathfrak{M}_1$, $\mathfrak{N}_2 = \mathfrak{M}_2$ and $\mathfrak{N}_l \subsetneqq \mathfrak{M}_l$ for $l \geq 3$.

At the end of the paper we give an application to our main result and some remarks about the condition $\mathfrak{N}$ which have been pointed out by the referee.

2. Mal'cev's conditions. As in [1] we obtain Mal'cev's conditions by the aid of finite chains of brackets of two kinds, round (,) and square [,], which are labelled by suffixes. We need only "normal" chains in the sense of

DEFINITION 1. A chain of brackets will be called *normal* if the round brackets occur in the chain in pairs of the form

$$\cdots (_i \cdots)_i \cdots$$

$i = 1, \ldots, r$, and if $1 \leq i < i' \leq r$, then $(_{i'}$ appears in the chain after $(_i$, and if it appears before $)_i$ then $)_{i'}$ appears also before $)_i$. Similar conditions are imposed on the square brackets.

We shall say that a normal chain is of *order l* if it contains r pairs of round brackets and $s = l - r$ pairs of square brackets.

Using a normal chain of order l we obtain a *condition of Mal'cev of order l* in the following way:

For each i, $i = 1, \ldots, r$ we adjoin 4 elements (indeterminates) a_i, p_i, b_i, b_i' to the ith pair of round brackets, and in the chain we insert the words $a_i b_i$ before $(_i$, $p_i b_i$ after it; $p_i b_i'$ before $)_i$, $a_i b_i'$ after it and obtain

$$\cdots a_i b_i (_i p_i b_i \cdots p_i b_i')_i a_i b_i' \cdots.$$

Similarly for each j, $j = 1, \ldots, s$, we adjoin 4 elements d_j, q_j, c_j, c_j' to the jth pair of square brackets, and in the chain we insert $c_j d_j$ before $[_j$, $c_j q_j$ after it; $c_j' q_j$ before $]_j$, $c_j' d_j$ after it and obtain

$$\cdots c_j d_j [_j c_j q_j \cdots c_j' q_j]_j c_j' d_j \cdots.$$

Now, we have two words between any two brackets of the chain and we equate them. Since the chain contains $2l$ brackets we obtain $2l - 1$ equations which will be called the *internal equations* of the chain. Equating the left word of the first bracket with the right word of the last bracket, we obtain an equation which will be called the *external equation* of the chain.

The condition of Mal'cev corresponding to the above chain is: If for any $4l$ elements of the semigroup $a_i, p_i, b_i, b_i', d_j, q_j, c_j, c_j'$, $i = 1, \ldots, r$, $j = 1, \ldots, s$, the $2l - 1$ internal equations are satisfied, then the external equation is also satisfied.

Mal'cev's theorem is: A semigroup with 1 is embeddable in a group if and

only if it satisfies the conditions of Mal'cev corresponding to all normal chains [1, p. 268].

In the $4l$ words of two letters, which we have inserted in a normal chain of order l, the elements a_i, p_i, c_j, c_j' occur as left terms and will be called *left elements*; the elements d_j, q_j, b_i, b_i' occur as right terms and will be called *right elements*.

From now on, we shall deal with cancellative semigroups since we are interested in the multiplicative semigroups R^* of an integral domain which clearly satisfies the cancellation laws. For such semigroups we shall show that a restricted set of conditions which we call "cancelled conditions" is sufficient for embeddability in a group.

DEFINITION 2. A chain will be called *cancelled* if it is normal and for each $i, i = 1, \ldots, r$, the bracket $(_i$ is not followed by $)_i$ and if $(_1$ is the first bracket of the chain then $)_1$ is *not* the last; similar conditions are imposed on the square brackets. A condition of Mal'cev is called *cancelled* if it corresponds to a cancelled normal chain.

LEMMA 1. *If a cancellative semigroup S satisfies all the cancelled conditions, then it is embeddable in a group.*

Proof. The conditions corresponding to the chains of order 1 are implied by the cancellation laws. Again using the cancellation laws it is easily shown that each noncancelled condition of order l follows from a condition of order $l-1$, and by induction we obtain that S satisfies all the conditions of Mal'cev. Hence by Mal'cev's theorem if $1 \in S$ then S is embeddable in a group. If $1 \notin S$, embed S in $S_1 = S \cup \{1\}$ and observe that any chain of equations in S_1 becomes a chain in S on multiplying by an element $t \in S$ (which can again be cancelled) from left and right.

3. **The structure of the cancelled conditions.** Let us fix a cancelled condition of order l and suppose it contains r pairs of round brackets and $s = l - r$ pairs of square brackets. By Definition 2, it follows that $r \geq 1$ and $s \geq 1$. Consider the corresponding cancelled condition which is obtained, as in the previous section, by the aid of $4l$ elements $a_i, p_i, b_i, b_i', d_j, q_j, c_j, c_j', i = 1, \ldots, r, j = 1, \ldots, s$. Again by Definition 2, it follows that the two left (right) elements of the words of an equation of the condition are distinct. Hence each element appears in exactly two equations.

For two left elements u, v define $u \sim v$ if there exist left elements $u = u_0, u_1, \ldots, u_k$, $u_{k+1} = v$ and $k+1$ equations of the condition as follows

(1) $\qquad u_0 x_0 = u_1 y_1, \qquad u_1 x_1 = u_2 y_2, \ldots, \qquad u_k x_k = u_{k+1} y_{k+1}.$

It is easily verified that " $\sim$ " is an equivalence relation in the set of left elements and we obtain a partition of this set into disjoint classes. A similar process leads to a partition of the set of right elements into disjoint classes. The class which is determined by an element z will be denoted by $\bar{z}$.

LEMMA 2. *If in (1) $u_{k+1} = u_0$ then $\bar{u}_0 = \{u_0, u_1, \ldots, u_k\}$.*

Proof. Clearly $\bar{u}_0 \supseteq \{u_0, u_1, \ldots, u_k\}$. Let $w \in \bar{u}_0$ then $u_0 \sim w$ and since each element is contained in exactly two equations we have either

$$u_0 x_0 = u_1 y_1, \qquad u_1 x_1 = u_2 y_2, \ldots, u_i x_i = wy,$$

or

$$u_0 y_{k+1} = u_k x_k, \qquad u_k y_k = u_{k-1} x_{k-1}, \ldots, u_j y_j = wx.$$

In both cases w belongs to $\{u_0, u_1, \ldots, u_k\}$ and the result follows.

LEMMA 3. *For $i = 1, \ldots, r$ all the elements of the class $\bar{p}_i$ are contained in the equations which appear in the interval $(_i \cdots)_i$ of the chain.*

Proof. For each square bracket the words before and after it have the same left-hand factor. For any round bracket occuring in the interval $(_i \ldots)_i$ its mate also occurs; this shows that all left-hand factors in the interval $(_i \ldots)_i$ occur in pairs and the result follows from this.

Going through the proof of Lemma 3 we obtain

LEMMA 4. (a) *If $a_{i'} \in \bar{p}_i$, then $i' > i$, and in particular $a_1 \notin \bar{p}_i$ for all $i = 1, \ldots, r$.*
(b) *If $(_{i+1} \cdots)_{i+1}$ is contained in $(_i \cdots)_i$, then $a_{i+1} \in \bar{p}_i$.*
(c) *If $i' \neq i$, then $\bar{p}_{i'} \cap \bar{p}_i = \varnothing$.*

The analog of Lemma 4 for the right elements is

LEMMA 4'. (a) *If $d_{j'} \in \bar{q}_j$, then $j' > j$, and in particular $d_1 \notin \bar{q}_j$ for all $j = 1, \ldots, s$.*
(b) *If $[_{j+1} \cdots]_{j+1}$ is contained in $[_j \cdots]_j$, then $d_{j+1} \in \bar{q}_j$.*
(c) *If $j' \neq j$, then $\bar{q}_{j'} \cap \bar{q}_j = \varnothing$.*

The structure of our cancelled condition is given by

THEOREM 5. *The set of left elements consist of $r + 1$ classes which are $\bar{a}_1, \bar{p}_1, \ldots, \bar{p}_r$, and the set of right elements consists of $s + 1$ classes which are $\bar{d}_1, \bar{q}_1, \ldots, \bar{q}_s$.*

Proof. We prove the theorem for the left elements and in a similar way it can be proved for the right elements.

By (a) and (c) of Lemma 4, the classes $\bar{a}_1, \bar{p}_1, \ldots, \bar{p}_r$ are disjoint. It remains to show that all the left elements $a_i, p_i, c_j, c'_j, i = 1, \ldots, r, j = 1, \ldots, s$ belong to one of these classes.

The assertion is clear for $p_1, \ldots, p_r$ and let us prove it for $a_i, i = 1, \ldots, r$. For $i = 1$ we have $a_1 \in \bar{a}_1$. Let $i \geq 2$ and assume that the assertion is true for $i', i' < i$. We consider two cases:

(a) $(_i \cdots)_i$ is contained in $(_{i-1} \cdots)_{i-1}$. Then by Lemma 4(b) with $i - 1$ replacing i we obtain $a_i \in \bar{p}_{i-1}$.

(b) $(_i$ appears after $)_{i-1}$. Then the interval $)_{i-1} \cdots (_i$ does not contain round brackets of the form $(.$ Let $)_{i'}$ be the last round bracket of this interval and $i' = i - 1$ if this interval does not contain round brackets. We have $i' < i$ and considering the interval $)_{i'} \cdots (_i$ we obtain that $a_{i'}$ and a_i belong to the same class. By the induction

hypothesis $a_{i'}$ belongs to one of the classes $\bar{a}_1, \bar{p}_1, \ldots, \bar{p}_r$ and hence a_i also belongs to one of these classes.

It remains to prove that each of the elements $c_j, c_j', j = 1, \ldots, s$ belongs to one of the classes $\bar{a}_1, \bar{p}_1, \ldots, \bar{p}_r$.

For c_j consider the nearest round bracket to $[_j$ and let it belong to the ith pair. We obtain either $c_j \in \bar{p}_i$ or $c_j \in \bar{a}_i$ and we have already proved that $\bar{a}_i$ is one of the classes $\bar{a}_1, \bar{p}_1, \ldots, \bar{p}_r$. In a similar way we obtain the result for c_j' which completes the proof of the theorem.

The following result will be used in the next section.

LEMMA 6. *If $(_1$ is the first bracket of the chain, then $b_1 \in \bar{d}_1$.*

Proof. The first square bracket of the chain is $[_1$ and the interval $(_1 \cdots [_1$ is either empty or contains round brackets of the form $($. If it is empty then $p_1 b_1 = c_1 d_1$ and hence $b_1 \in \bar{d}_1$. In the other case we have for some $i > 1$:

$$p_1 b_1 = a_2 b_2, \ldots, p_{i-1} b_{i-1} = a_i b_i, \ p_i b_i = c_1 d_1.$$

Hence again we obtain $b_1 \in \bar{d}_1$.

4. The main result. If R is a ring which satisfies the condition $\mathfrak{N}_l$ and if $A, B \in R_l$ and $(AB)^l = 0$ then $(BA)^l = 0$. Indeed, BA is nilpotent since $(BA)^{l+1} = B(AB)^l A = 0$, and therefore $(BA)^l = 0$ since R satisfies $\mathfrak{N}_l$. Using this remark and the results of the previous sections we shall obtain the main result of this paper which is:

THEOREM 7. *If R is an integral domain which satisfies the condition $\mathfrak{N} = \bigcap_{l=1}^{\infty} \mathfrak{N}_l$, then R^* is embeddable in a group, i.e., $\mathfrak{N} \subseteq \mathfrak{M}$.*

Proof. By Lemma 1 it suffices to prove that R^* satisfies all the cancelled conditions.

Let us consider the cancelled condition of the previous section and assume that for $4l$ elements of R^*

$$(2) \qquad a_i, p_i, b_i, b_i', d_j, q_j, c_j, c_j', \qquad i = 1, \ldots, r, \quad j = 1, \ldots, s,$$

the internal equations hold and we have to prove that the external equation holds. Assume that $(_1$ is the first bracket of the chain and a similar proof can be given in the other case. Denote by ef the right word of the last bracket of the chain and let $w = a_1 b_1 - ef$. The external equation is $a_1 b_1 = ef$ and hence the result will follow if we prove that $w = 0$.

Using the $4l$ elements of (2) and 0 we first construct two matrices A_1, B_1 of orders $(r+1) \times l$ and $l \times (s+1)$ respectively. Then by the aid of 0 and an element $t \neq 0$ we shall enlarge A_1, B_1 to $A, B \in R_l$ which will serve to obtain the result.

The two words appearing near each bracket of the given chain have one common letter: b_i, b_i', c_j, c_j' for $(_i$, $)_i$, $[_j$, $]_j$ respectively. We put a minus sign before the elements which are common to the words of the $2k$th bracket of the chain, $k = 1, 2, \ldots, l$. Note that since $(_1$ is the first bracket of the chain " $-$ " is not adjoined to b_1.

A_1 is obtained by the aid of the $r+1$ classes of left elements (Theorem 5) and zeros. We put the elements of $\bar{a}_1$ (with the adjoined signs) in the first row, the elements of $\bar{p}_1$ in the second row, ..., the elements of $\bar{p}_r$ in the $(r+1)$th row. The place of each element in the row of its class will be determined by the table:

1	...	r	r+1	...	l
a_1	...	a_r	$\pm c_s$	...	$\pm c_1$
p_1	...	p_r	$\pm c'_s$	...	$\pm c'_1$

We put zeros in the remaining places.

B_1 is obtained by the aid of the $s+1$ classes of right elements and zeros. We put the elements of $\bar{q}_s$ in the first column, ..., the elements of $\bar{q}_1$ in the sth column and those of $\bar{d}_1$ in the $(s+1)$th column. The place of each element in the column of its class will be determined by the table:

1	b_1	$\pm b'_1$
⋮	⋮	⋮
r	$\pm b_r$	$\pm b'_r$
r+1	d_s	q_s
⋮	⋮	⋮
l	d_1	q_1

As in A_1 the remaining places are filled in by zeros.

By Lemma 4(a) if $a_{i'} \in \bar{p}_i$ then $i' > i$. Whence $a_{i'}$ appears in A_1 on the right side of p_i and this implies that on the left side of p_i appear only zeros. Similarly, by Lemma 4'(a) we obtain that in B_1 appear only zeros on the left side of q_j. By Lemma 6 $b_1 \in \bar{d}_1$ and hence b_1 appears in the last column of B_1.

The general element of A_1B_1 is a sum whose ith term, $1 \leq i \leq r$, is 0 or one of the words $\pm a_i b_i$, $\pm p_i b_i$, $\pm p_i b'_i$, $\pm a_i b'_i$ and whose $(r+s+1-j)$th term, $1 \leq j \leq s$, is 0 or one of the words $\pm c_j d_j$, $\pm c_j q_j$, $\pm c'_j q_j$, $\pm c'_j d_j$.

Each of the words

(3) $a_i b_i,\ p_i b_i,\ p_i b'_i,\ a_i b'_i,\ c_j d_j,\ c_j q_j,\ c'_j q_j,\ c'_j d_j,\qquad i = 1,\ldots,r,\quad j = 1,\ldots,s$

perhaps with " $-$ " is obtained as a summand of one and only one element of A_1B_1.

If xy and uv are words of (3) and $xy = uv$ is one of the $2l$ equations of the condition, then x, u belong to the same left class, and y, v belong to the same right class. Hence $\pm x$, $\pm u$ are in the same row of A_1, say the mth; $\pm y$, $\pm v$ are in the same column of B_1, say the nth. Considering the tables which have determined the positions of the elements in A_1 and B_1, we obtain that $\pm xy$ and $\pm uv$ are summands of the (m, n)-entry of $A_1 B_1$. But it is readily seen that " $-$ " was adjoined to only one element of each equation. Hence if $xy = uv$ is an internal equation the contribution of xy and uv to the (m, n)-entry of $A_1 B_1$ is $\pm (xy - uv) = 0$. If $xy = uv$ is the external equation, one of the words xy and uv is $a_1 b_1$ and the other is ef, and their contribution to the $(1, s+1)$-entry of $A_1 B_1$ is $a_1 b_1 - ef = w$. Thus, the $(1, s+1)$-entry of $A_1 B_1$ is w and the other entries are 0 and we have:

$$(4) \qquad A_1 B_1 = \begin{pmatrix} 0 & w \\ 0_{rs} & 0 \end{pmatrix}$$

where 0_{rs} is the $r \times s$ zero matrix.

Now, let A' be defined as the $(s-1) \times (r+s)$ matrix

$$t(E_{1, r+2} + E_{2, r+3} + \cdots + E_{s-1, r+s})$$

and B' as the $(r+s) \times (r-1)$ matrix $t(E_{11} + E_{22} + \cdots + E_{r-1, r-1})$. Note that A' is defined for $s > 1$ and B' is defined for $r > 1$. We enlarge the matrices A_1, B_1 to matrices A, $B \in R_l$ by the aid of A' and B' as follows:

$$A = \begin{pmatrix} A' \\ A_1 \end{pmatrix}, \qquad B = (B_1 | B').$$

If $s = 1$ we take $A = A_1$ and if $r = 1$ we take $B = B_1$. We obtain

$$(5) \qquad AB = \left(\begin{array}{c|c} A'B_1 & A'B' \\ \hline A_1 B_1 & A_1 B' \end{array} \right) \begin{array}{l} \}s-1 \\ \}r+1 \end{array}$$
$$\underbrace{}_{s+1} \underbrace{}_{r-1}$$

and we have

$$(6)\ A'B_1 = \begin{pmatrix} 0 & tq_{s-1} & & & \\ & \cdot & \cdot & & \text{\Large *} \\ & & \cdot & \cdot & \\ \bigcirc & & & \cdot & \cdot \\ & & 0 & tq_1 & td_1 \end{pmatrix}, \ A_1 B' = \begin{pmatrix} a_1 t & & & \\ p_1 t & & \text{\Large *} & \\ 0 & \cdot & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ \bigcirc & & \cdot & P_{r-1} t \\ & & & 0 \end{pmatrix}, \ A'B' = 0.$$

By (4), (5) and (6) we obtain

$$AB = \begin{pmatrix} 0 & tq_{s-1} & & & & * & & & & \\ & \cdot & \cdot & & & & & & \bigcirc & \\ & & \cdot & \cdot & & & & & & \\ & \bigcirc & & \cdot & \cdot & & & & & \\ & & & & 0 & tq_1 & td_1 & & & \\ 0 & \cdots & 0 & & & w & a_1 t & & & \\ & & & & & 0 & p_1 t & & * & \\ & & & & & \cdot & 0 & \cdot & & \\ & \bigcirc & & & & \cdot & & \cdot & & \\ & & & & & \cdot & & & \cdot & \\ & & & & & & \bigcirc & \cdot & p_{r-1}t \\ & & & & 0 & & & & 0 \end{pmatrix}$$

Now, we complete the proof of Theorem 7. We have $(AB)^l = 0$ since AB is triangular. Hence by the remark at the beginning of this section it follows that $(BA)^l = 0$. We also see that

$$(AB)^{l-1} = (tq_{s-1})\cdots(tq_1)w(p_1 t)\cdots(p_{r-1}t)E_{1l}.$$

The $(r+1, 1)$-entry of B is q_s and the (l, r)-entry of A is p_r. But $E_{r+1,1}E_{1l}E_{lr} = E_{r+1,r}$ and therefore we obtain for the $(r+1, r)$-entry of $B(AB)^{l-1}A = (BA)^l = 0$ the equation

$$q_s tq_{s-1} \cdots tq_1 wp_1 t \cdots p_{r-1}tp_r = 0.$$

Since R is an integral domain and $p_1, \ldots, p_r, q_1, \ldots, q_s, t$ are nonzero elements it follows that $w = 0$ and this completes the proof of the theorem.

In [4] we have constructed integral domains R which do not satisfy $\mathfrak{N}$, but R^* are embeddable in groups and therefore R satisfy $\mathfrak{M}$. Hence by Theorem 7 we obtain

THEOREM 8. $\mathfrak{N} \subsetneq \mathfrak{M}$.

5. **The relation between $\mathfrak{N}_l$ and $\mathfrak{M}_l$.** In this section we obtain the relation between the conditions $\mathfrak{N}_l$ and $\mathfrak{M}_l$ (defined in the introduction) for integral domains.

The conditions of Mal'cev of order 1 are satisfied by a cancellative semigroup and in particular they are satisfied by R^* if R is an integral domain. Thus, each integral domain satisfies $\mathfrak{M}_1$. Since an integral domain has no nonzero nilpotent elements it satisfies $\mathfrak{N}_1$ and this implies $\mathfrak{N}_1 = \mathfrak{M}_1$.

To obtain the relation between $\mathfrak{N}_l$ and $\mathfrak{M}_l$ for $l \geq 2$ we first prove

LEMMA 9. $\mathfrak{N}_l \supseteq \mathfrak{N}_{l+1}$.

Proof. Let R be an integral domain which does not satisfy $\mathfrak{N}_l$ and we shall prove that it does not satisfy $\mathfrak{N}_{l+1}$. Let $A \in R_l$ be nilpotent of order $n > l$. Thus, $A^n = 0$, $A^{n-1} \neq 0$ and assume that the (p, q)-entry of A^{n-1} is $a \neq 0$. Let $B_1 \in R_{l+1}$ be the matrix obtained from A by adjoining a row and a column of zeros and let $B = B_1 + tE_{q,l+1}$, where $0 \neq t \in R$. Computing the powers of B we obtain $B^{n+1} = 0$ and $B^n \neq 0$ since its $(q, l+1)$-entry is $at \neq 0$. Thus, $B \in R_{l+1}$ is nilpotent of order $n+1 > l+1$, hence R does not satisfy $\mathfrak{N}_{l+1}$.

Using the previous lemma and the result proved in Theorem 7, we obtain

LEMMA 10. $\mathfrak{N}_l \subseteq \mathfrak{M}_l$.

Proof. By the proof of Theorem 7 it follows that if R satisfies $\mathfrak{N}_l$, then R^* satisfies all the cancelled conditions of order l. But, by the previous lemma, R satisfies $\mathfrak{N}_k$ for $k \leq l$, hence R^* satisfies all the cancelled conditions of order $\leq l$. Now, each condition of order l which is not cancelled can be proved using the cancellation laws and perhaps also a cancelled condition of lower order. This proves that R^* satisfies all the conditions of order l and hence R satisfies $\mathfrak{M}_l$.

In [4] we have constructed for each $l \geq 3$ an integral domain R which does not satisfy $\mathfrak{N}_l$, and R^* is embeddable in a group. Hence R satisfies $\mathfrak{M}$ and in particular it satisfies $\mathfrak{M}_l$. This result together with the result of Lemma 10 implies

THEOREM 11. $\mathfrak{N}_l \subsetneqq \mathfrak{M}_l$ *for each* $l \geq 3$.

Now, we deal with the case $l = 2$. We first prove

LEMMA 12. *If R is an integral domain and $A \in R_2$ is nilpotent of order $k+1$, $k \geq 2$, then all the entries of A and of A^k are $\neq 0$.*

Proof. Let $A = (a_{ij})$ and $A^k = (b_{ij})$. Since $A^k \neq 0$ and $k \geq 2$ we have $A^2 \neq 0$.

If $a_{12} = 0$ then the $(1, 1)$ and $(2, 2)$ entries of A^{k+1} are a_{11}^{k+1} and a_{22}^{k+1} respectively. But $A^{k+1} = 0$ and since R is an integral domain it follows $a_{11} = a_{22} = 0$. Hence we obtain $A^2 = 0$, a contradiction. Thus, $a_{12} \neq 0$ and similarly $a_{21} \neq 0$.

If $a_{11} = 0$ then since $AA^k = A^{k+1} = 0$ we obtain $a_{12}b_{2j} = a_{11}b_{1j} + a_{12}b_{2j} = 0$ for $j = 1, 2$. Hence $b_{21} = b_{22} = 0$ and since $A^{2k} = 0$ its $(1, 1)$-entry is $b_{11}^2 = 0$ which implies that $b_{11} = 0$. By the relation $A^k A = A^{k+1} = 0$ we obtain $b_{12}a_{21} = b_{11}a_{11} + b_{12}a_{21} = 0$ and therefore $b_{12} = 0$. Thus, we have proved $A^k = 0$, a contradiction. Hence $a_{11} \neq 0$ and similarly $a_{22} \neq 0$ and this completes the proof that all the entries of A are $\neq 0$.

Now, we prove that all $b_{ij} \neq 0$. We have $a_{11}b_{1j} + a_{12}b_{2j} = 0$ and since $a_{11}, a_{12} \neq 0$ it follows that $b_{1j} = 0$ if and only if $b_{2j} = 0$. Similarly the equation $b_{i1}a_{11} + b_{i2}a_{21} = 0$ implies $b_{i1} = 0$ if and only if $b_{i2} = 0$. Hence if $b_{ij} = 0$, then the second entry of the jth column of A^k is 0, and therefore the second column of A^k is also 0, a contradiction. This proves that each $b_{ij} \neq 0$.

THEOREM 13. $\mathfrak{N}_2 = \mathfrak{M}_2$.

Proof. By Lemma 10 we have $\mathfrak{N}_2 \subseteq \mathfrak{M}_2$. To obtain equality it suffices to prove that if R is an integral domain which does not satisfy $\mathfrak{N}_2$, then it does not satisfy $\mathfrak{M}_2$.

Let R be an integral domain and let $A \in R_2$ be nilpotent of order $k+1$, $k \geq 2$. We shall show that R does not satisfy $\mathfrak{M}_2$. Let us consider the following condition of Mal'cev of order 2:

$$(7) \qquad a_1 b_1 (_1 p_1 b_1 = c_1 d_1 [_1 c_1 q_1 = p_1 b_1')_1 a_1 b_1' = c_1' q_1]_1 c_1' d_1.$$

As in the previous lemma let $A = (a_{ij})$, $A^k = (b_{ij})$. Since $A^k \neq 0$ and $k \geq 2$ we have $A^2 \neq 0$ and suppose the (i, j)-entry of A^2 is not 0. We take:

$$a_1 = a_{i1}, \qquad p_1 = b_{i1}, \qquad b_1 = a_{1j}, \qquad b_1' = b_{1j}$$
$$d_1 = a_{2j}, \qquad q_1 = b_{2j}, \qquad c_1 = -b_{i2}, \qquad c_1' = -a_{i2}.$$

By Lemma 12 all these elements are $\neq 0$ (belong to R^*) and since $A^k A = A A^k = A^{k+1} = 0$ and $A^k A^k = 0$ we obtain

$$p_1 b_1 = b_{i1} a_{1j} = -b_{i2} a_{2j} = c_1 d_1,$$
$$c_1 q_1 = -b_{i2} b_{2j} = b_{i1} a_{1j} = p_1 b_1',$$
$$a_1 b_1' = a_{i1} b_{1j} = -a_{i2} b_{2j} = c_1' q_1.$$

Thus, for 8 elements of R^* the internal equations of (7) hold, but since the (i, j)-entry of A^2 is not 0 we have $a_1 b_1 = a_{i1} a_{1j} \neq -a_{i2} a_{2j} = c_1' d_1$ which means that the external equation of (7) does not hold. Hence R does not satisfy $\mathfrak{M}_2$ and this completes the proof of the theorem.

The following is an application to our main result which has been communicated to us by the referee[2].

A semifir ($=$ local fir in [2]) is an integral domain in which every finitely generated left (or right) ideal is free of unique rank.

THEOREM 14. *A semifir satisfies $\mathfrak{N}$ and hence its multiplicative semigroup is embeddable in a group.*

Proof. Let A be an $l \times l$ matrix over a semifir R and assume $A^n = 0$ for some n, but $A^{n-1} \neq 0$.

Write $V = R^l$ considered as left R-module; by [2, Proposition 2.1] all finitely generated submodules of V are free. Moreover multiplication (on the right) by A defines an R-homomorphism. Thus writing $V_i = V A^i$ we have the chain of submodules

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{n-1} \supseteq V_n = 0.$$

All these submodules are free of diminishing ranks, hence if $n > l$ we have equality at some stage, say rank $(V_{i-1}) = $ rank (V_i). But this means that A acting on V_{i-1} is injective, whence so is $A^n = 0$, i.e. $V_{i-1} = 0$, a contradiction unless $n \leq l$.

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(2) The author wishes to thank the referee for this interesting application.

We conclude with the following remark.

A ring R satisfying $\mathfrak{N}$ has the following property for every integer n: If A, $B \in R_n$ and $AB = 1$ then $BA = 1$. Indeed, if $AB = 1$ and $BA \neq 1$ then as in [5, p. 35] define $e_{ij} = B^{i-1}A^{j-1} - B^i A^j$ for $i, j = 1, 2, \ldots$, and observe that for each $m \geq 2$, $e_{12} + e_{23} + \cdots + e_{m-1,m}$ is nilpotent of order m, a contradiction.

In [3] the property "A, $B \in R_n$ and $AB = 1$ then $BA = 1$" is expressed in the following equivalent form:

III. In a free R-module of any rank n, any generating set of n elements is free. Hence a ring satisfying $\mathfrak{N}$ has the property III and also satisfies II and I of [3] and in particular it has the invariant basis property.

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TEL-AVIV UNIVERSITY,
    TEL-AVIV, ISRAEL