ON THE DIMENSION THEORY OF OVERRINGS OF AN INTEGRAL DOMAIN(1)

BY

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Let D be an integral domain with identity which has quotient field L. If there exists a chain $P \subset P_1 \subset \cdots \subset P_n$ of $n+1$ prime ideals of D, where $P_n \subset D$, but no such chain of $n+2$ prime ideals, then we say that D has dimension n and we write $\dim D = n$ [6]. In [6] and [7] Seidenberg has shown that if $\dim D = n$, and if D is a Noetherian domain or a Prüfer domain, then $\dim D[X_1, \ldots, X_m] = n+m$, where $X_1, \ldots, X_m$ are indeterminates over D. In the special case in which $\dim D = 1$ he has proved that the following statements are equivalent.

(1) $\dim D[X_1] = 2$.

(2) $\dim D[X_1, \ldots, X_m] = m+1$ for any m.

More recently Gilmer has established the equivalence of the following properties for an n-dimensional domain D [1].

(3) Every domain between D and L has dimension less than or equal to n.

(4) $\dim D[t_1, \ldots, t_n] \leqq n$ for $\{t_1, \ldots, t_n\} \subseteq L$.

For $n=1$ he further showed that (3) and (4) are equivalent to (1).

In this paper we consider domains D having finite dimension n and having the property that each domain between D and its quotient field has dimension less than or equal to ω for some positive integer $\omega \geqq n$. For such a domain we obtain equivalent statements analogous to statements (1)–(4). The main results of this paper are contained in Theorems 2 and 5.

Throughout this paper D will denote an integral domain with identity having quotient field L, and $X, X_1, \ldots, X_m$ will denote indeterminates over D. By an *overring* of D we mean an integral domain D' such that $D \subseteq D' \subseteq L$. By a *valuation overring* of D we mean an overring of D which is a valuation ring. Our notation will be that of Zariski-Samuel [8] with the one exception: $\subset$ denotes proper containment and $\subseteq$ denotes containment.

I. If $\dim D = n$ and $\omega \geqq n$, we wish to find necessary and sufficient conditions in order that each overring of D have dimension less than or equal to ω. One such set of conditions is given by the following theorem.

THEOREM 1. *Suppose that* $\dim D = n$. *Then the following statements are equivalent.*

(1) *Each overring of D has dimension less than or equal to ω.*

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(2) *Each valuation overring of D has dimension less than or equal to ω.*
(3) *For* $\{t_1, \ldots, t_\omega\} \subseteq L$, $\dim D[t_1, \ldots, t_\omega] \leqq \omega$.

Proof. Clearly (2) and (3) follow from (1). To show that (2) implies (1) we suppose that D' is an overring of D such that $\dim D' > \omega$. Then there exists a chain $(0) \subset P_1 \subset \cdots \subset P_s \subset D'$ of prime ideals of D' such that $s > \omega$. By Theorem 11.9 of [5, p. 37] there exists a valuation overring V of D' such that V has prime ideals $Q_1, \ldots, Q_s$ which lie over $P_1, \ldots, P_s$ respectively. It follows that V is a valuation overring of D such that $\dim V > \omega$.

The proof that (3) implies (1) is given by Gilmer in [1] for the case $\omega = n$. The same method of proof is used here.

Suppose that there exists an overring D' of D such that $\dim D' \geqq \omega + 1$. Let $(0) \subset P_{\omega+1} \subset \cdots \subset P_1 \subset D'$ be a chain of prime ideals of D' and let $t_i \in P_i - P_{i+1}$, $1 \leqq i \leqq \omega$. If $P_i' = P_i \cap D[t_1, \ldots, t_\omega]$, $1 \leqq i \leqq \omega + 1$, then P_i' is a prime ideal of $D[t_1, \ldots, t_\omega]$ and $t_i \in P_i' - P_{i+1}'$, $1 \leqq i \leqq \omega$. Now let $r/s \in P_{\omega+1}$, where $r, s \in D - \{0\}$. Then $r = s(r/s)$ is an element of $P_{\omega+1} \cap D$ so that $r \in P_{\omega+1}'$. Further, $1 \notin P_1'$ since $1 \notin P_1$. Thus, $(0) \subset P_{\omega+1}' \subset \cdots \subset P_1' \subset D[t_1, \ldots, t_\omega]$ and $\dim D[t_1, \ldots, t_\omega] \geqq \omega + 1$.

Theorem 1 leads us to the consideration of domains D such that $\dim D[t_1, \ldots, t_\omega] \leqq \omega$ for any subset $\{t_1, \ldots, t_\omega\}$ of L. More generally, for a fixed positive integer m, we wish to find necessary and sufficient conditions in order that $\dim D[t_1, \ldots, t_m] \leqq \omega$, where $\{t_1, \ldots, t_m\}$ is any subset of L, and where $\omega \geqq \dim D$. Sufficient conditions are given by the following theorem.

THEOREM 2. *Suppose that* $\dim D[X_1, \ldots, X_m] = \omega + m$. *Then* $\omega \geqq \dim D$, *and given* $\{t_1, \ldots, t_m\} \subseteq L$, *we have* $\dim D[t_1, \ldots, t_m] \leqq \omega$.

If P is a prime ideal of an integral domain R, we shall denote by $h(P)(d(P))$ the height (depth) of P in R. Before proving Theorem 2, we require Lemma 1.

LEMMA 1. *Let $t_1, \ldots, t_m$ be elements of L and let ϕ be the canonical D-homomorphism from $D[X_1, \ldots, X_m]$ onto $D[t_1, \ldots, t_m]$ such that $\phi(X_i) = t_i$, $1 \leqq i \leqq m$. If Q is the kernel of ϕ, then Q has height m in $D[X_1, \ldots, X_m]$.*

Proof. We have $Q \cap D = (0)$, for if $d \in D$, then $\phi(d) = d$. Thus, if $N = D - \{0\}$, Q extends to a proper prime ideal of

$$(D[X_1, \ldots, X_m])_N = D_N[X_1, \ldots, X_m] = L[X_1, \ldots, X_m].$$

Further, if $h(Q) = s$ in $D[X_1, \ldots, X_m]$, the extension of Q has height s in

$$L[X_1, \ldots, X_m].$$

However, $\dim L[X_1, \ldots, X_m] = m$ so that $s \leqq m$.

Let $Q_i = Q \cap D[X_1, \ldots, X_i]$ for $1 \leqq i \leqq m$. If $t_i = a_i/b_i$, $a_i, b_i \in D$, then $b_iX_i - a_i \in Q_i$. Consequently, $Q_i \neq (0)$ and for $1 \leqq i \leqq m-1$, $Q_i[X_{i+1}] \subset Q_{i+1}$, since $b_{i+1}X_{i+1} - a_{i+1} \notin Q_i[X_{i+1}]$. It now follows that $(0) \subset Q_1[X_2, \ldots, X_m] \subset \cdots \subset Q_{m-1}[X_m] \subset Q_m = Q$ so that $h(Q) \geqq m$. Thus equality holds and the lemma is proved.

Proof of Theorem 2. Suppose that $\{t_1, \ldots, t_m\} \subseteq L$ and let Q be the kernel of the D-homomorphism ϕ of $D[X_1, \ldots, X_m]$ onto $D[t_1, \ldots, t_m]$, where ϕ is such that $\phi(X_i) = t_i$ for each i. Further, let $(0) \subset Q_1 \subset \cdots \subset Q_k \subset D[t_1, \ldots, t_m]$ be a chain of prime ideals of $D[t_1, \ldots, t_m]$. Then there exists a chain

$$Q \subset P_1 \subset \cdots \subset P_k \subset D[X_1, \ldots, X_m]$$

of prime ideals of $D[X_1, \ldots, X_m]$ such that $\phi(P_i) = Q_i$, $1 \leqq i \leqq k$. Since Q has height m, $m+k \leqq \dim D[X_1, \ldots, X_m] = m+\omega$. Therefore, $k \leqq \omega$ as we wished to show.

Theorem 5 shows that the conditions given in Theorem 2 are also necessary in order that $\dim D[t_1, \ldots, t_m] \leqq \omega$ for $\{t_1, \ldots, t_m\} \subseteq L$. However, before proving Theorem 5 we need several other results.

THEOREM 3. *Suppose that* $\dim D[t_1, \ldots, t_m] \leqq \omega$ *for* $\{t_1, \ldots, t_m\} \subseteq L$. *If J is an integral domain containing D such that J is integral over D, and if F is the quotient field of J, then* $\dim J[s_1, \ldots, s_m] \leqq \omega$ *for* $\{s_1, \ldots, s_m\} \subseteq F$.

In order to prove Theorem 3, we use the following lemma.

LEMMA 2. *Let* $f(X) = f_n X^n + \cdots + f_1 X + f_0 \in D[X]$, $f_n \neq 0$, *and let s be a root of $f(X)$ in an extension field of L. Then s is integral over $D[1/f_n]$ and $f_n s$ is integral over D.*

Proof. Since $f(X)/f_n \in D[1/f_n][X]$ and $f(s)/f_n = 0$, it follows that s is integral over $D[1/f_n]$. Also

$$0 = f_n^{n-1} f(s) = (f_n s)^n + f_{n-1}(f_n s)^{n-1} + \cdots + f_1 f_n^{n-2}(f_n s) + f_0 f_n^{n-1}$$

so that $f_n s$ is integral over D.

Proof of Theorem 3. F is algebraic over L since J is integral over D. Therefore, if $s_1, \ldots, s_m$ are elements of F, there exists $f_i(X) \in D[X] - \{0\}$ such that $f_i(s_i) = 0$, $1 \leqq i \leqq m$. It follows from Lemma 2 that if d_i is the leading coefficient of $f_i(X)$, then s_i is integral over $D[1/d_1, \ldots, 1/d_m]$ for each i, $1 \leqq i \leqq m$. Hence, $J[s_1, \ldots, s_m]$ is integral over $D[1/d_1, \ldots, 1/d_m]$. Therefore, $\dim J[s_1, \ldots, s_m] = \dim D[1/d_1, \ldots, 1/d_m]$ by [6, Theorem 5]. But $\dim D[1/d_1, \ldots, 1/d_m] \leqq \omega$ since $\{1/d_1, \ldots, 1/d_m\} \subseteq L$.

COROLLARY 1. *Suppose that* $\dim D[t_1, \ldots, t_m] \leqq \omega$ *for* $\{t_1, \ldots, t_m\} \subseteq L$. *Then if* $\{s_1, \ldots, s_m\}$ *is a set of elements algebraic over D, we have* $\dim D[s_1, \ldots, s_m] \leqq \omega$.

Proof. Suppose $f_i(X) \in D[X] - \{0\}$ is such that $f_i(s_i) = 0$, $1 \leqq i \leqq m$. From Lemma 2 it follows that if d_i is the leading coefficient of $f_i(X)$, then $J = D[d_1 s_1, \ldots, d_m s_m]$ is integral over D. Moreover, $\{s_1, \ldots, s_m\}$ is a subset of the quotient field of J. Therefore, since $J[s_1, \ldots, s_m] = D[s_1, \ldots, s_m]$, it follows from Theorem 3 that $\dim D[s_1, \ldots, s_m] \leqq \omega$.

THEOREM 4. *If each overring of D has dimension less than or equal to ω, then* $\dim D[X_1, \ldots, X_m] \leqq \omega + m$.

Proof. Suppose that dim $D[X_1, \ldots, X_m]=m+k$, $k \geqq 0$. It follows from Theorem 2 of [6] that $k \geqq \dim D$ so we have $D=L$ if $k=0$. Therefore, since the theorem is true for $D=L$, we assume that $k>0$. By Theorem 11.9 of [5, p. 37], there exists a valuation overring W of $D[X_1, \ldots, X_m]$ such that dim $W \geqq m+k$. If $V=W \cap L$, then V is a valuation overring of D.

Suppose now that dim $V=\mu$. Then by assumption $\mu \leqq \omega$, and by Theorem 4 of [7], if $Z_1, Z_2, \ldots, Z_r$ is any set of indeterminates over V, then dim $V[Z_1, \ldots, Z_r] =\mu+r$. In particular, if $Y_1, \ldots, Y_{\mu+m}$ are indeterminates over $V[X_1, \ldots, X_m]$, then dim $V[X_1, \ldots, X_m][Y_1, \ldots, Y_{\mu+m}]=2\mu+2m$. Therefore, by Theorem 2, if $\delta_1, \ldots, \delta_{\mu+m}$ are elements of $L[X_1, \ldots, X_m]$, the quotient field of $V[X_1, \ldots, X_m]$; then dim $V[X_1, \ldots, X_m][\delta_1, \ldots, \delta_{\mu+m}] \leqq \mu+m$. It then follows from Theorem 1 that every overring of $V[X_1, \ldots, X_m]$ has dimension less than or equal to $\mu+m$. But W is an overring of $V[X_1, \ldots, X_m]$ so we have $m+k \leqq \dim W \leqq m+\mu$. Therefore, $k \leqq \mu \leqq \omega$ so that dim $D[X_1, \ldots, X_m] \leqq \omega+m$ as we wished to show.

LEMMA 3. *Suppose that* dim $D[t_1, \ldots, t_m] \leqq \omega$ *for* $\{t_1, \ldots, t_m\} \subseteq L$, *and let P be a prime ideal of D such that $h(P)=k$. If F is the quotient field of D/P, then*

$$\dim (D/P)[s_1, \ldots, s_m] \leqq \omega-k \quad \textit{for } \{s_1, \ldots, s_m\} \subseteq F.$$

Proof. F is isomorphic to D_P/PD_P, since $D_P/PD_P \cong (D/P)_{P/P}$ [8, p. 227], and D/P is isomorphic to $\{d+PD_P \mid d \in D\} \subseteq D_P/PD_P$. Thus suppose that $\{s_1, \ldots, s_m\} \subseteq D_P/PD_P$—say $s_i=t_i+PD_P$, where $t_i \in D_P$, and let $D'=D[t_1, \ldots, t_m]$.

If $(0) \subset P_1 \subset \cdots \subset P_k=P$ is a chain of prime ideals of D, then $(0) \subset P_1 D_P \subset \cdots \subset P_k D_P=PD_P \subset D_P$ is a chain of prime ideals of D_P such that $P_i D_P \cap D=P_i$, $1 \leqq i \leqq k$. Now $D \subseteq D' \subseteq D_P$ so that if $P_i'=P_i D_P \cap D'$, then $P_i=P_i' \cap D$. Therefore, $(0) \subset P_1' \subset \cdots \subset P_k'=P' \subset D'$ is a chain of prime ideals of D' and $h(P') \geqq k$.

It is easily seen that

$$\begin{aligned} D'/P' &\cong \{d'+PD_P \mid d' \in D'\} \\ &= \{f(t_1, \ldots, t_m\}+PD_P \mid f(X_1, \ldots, X_m) \in D[X_1, \ldots, X_m]\} \\ &= \{(d_0+ \textstyle\sum d_{n_1 \cdots n_m} t_1^{n_1} \cdots t_m^{n_m})+PD_P \mid d_i \in D\} \\ &\cong \{d_0+PD_P+ \textstyle\sum (d_{n_1 \cdots n_m}+PD_P)(t_1+PD_P)^{n_1} \cdots (t_m+PD_P)^{n_m} \mid d_i \in D\} \\ &\cong (D/P)[s_1, \ldots, s_m]. \end{aligned}$$

But by assumption dim $D' \leqq \omega$, and we have seen that $h(P') \geqq k$. Therefore, dim $D'/P' \leqq \omega-k$; that is, dim $(D/P)[s_1, \ldots, s_m] \leqq \omega-k$, and the proof of Lemma 3 is complete.

LEMMA 4. *Let P be a prime ideal of D, $P \neq D$, and let $Q_1 \subset Q_2 \subset \cdots \subset Q_s$ be a chain of prime ideals of $D[X_1, \ldots, X_m]$ such that $Q_i \cap D=P$ for each i, $1 \leqq i \leqq s$. Then $s \leqq m+1$ and there exists a chain $P[X_1, \ldots, X_m]=\Gamma_1 \subset \cdots \subset \Gamma_{m+1}$ of prime ideals of $D[X_1, \ldots, X_m]$ such that $\Gamma_i \cap D=P$ for each i and such that $\{Q_1, \ldots, Q_s\} \subseteq \{\Gamma_1, \ldots, \Gamma_{m+1}\}$.*

Proof. If $P^e = P[X_1, \ldots, X_m]$, then $Q_1/P^e \subset \cdots \subset Q_s/P^e$ is a chain of prime ideals $D[X_1, \ldots, X_m]/P^e = (D/P)[X_1, \ldots, X_m]$ meeting D/P in (0). Thus, it suffices to prove Lemma 4 for the case in which $P=(0)$. But if $P=(0)$, then

$$(D[X_1, \ldots, X_m])_{D-P} = L[X_1, \ldots, X_m].$$

Lemma 4 now follows from the results in [9, p. 194].

LEMMA 5. *Let D be a quasi-local domain with maximal ideal M. If* $\dim D = n \leqq \omega \leqq m$ *and if* $\dim D[X_1, \ldots, X_m] \geqq \omega + m + 1$, *then there exists a chain of prime ideals of $D[X_1, \ldots, X_m]$ of the form $M[X_1, \ldots, X_m] \supset Q_\omega \supset \cdots \supset Q_1 \supset (0)$, where either $Q_1 = P[X_1, \ldots, X_m]$ for some prime ideal P of D, or $Q_1 \cap D = (0)$ but $Q_1 \cap D[X_1] \neq (0)$.*

For convenience we number the following remark since it will be used repeatedly in the proof of Lemma 5.

REMARK 1. If $\dim D[X_1, \ldots, X_m] \geqq \omega + m + 1$, then by Theorem 4 some overring of D has dimension greater than or equal to $\omega + 1$. Hence, by Theorem 1, we have $\dim D[t_1, \ldots, t_\omega] \geqq \omega + 1$ for some $\{t_1, \ldots, t_\omega\} \subseteq L$ so that, by Theorem 2, $\dim D[X_1, \ldots, X_\omega] \geqq 2\omega + 1$.

Proof of Lemma 5. If the lemma were true in the special case in which $m = \omega$, there would exist a chain of prime ideals of $D[X_1, \ldots, X_\omega]$ of the form

$$M[X_1, \ldots, X_\omega] \supset Q_\omega \supset \cdots \supset Q_1 \supset (0),$$

where either $Q_1 = P[X_1, \ldots, X_\omega]$ for some prime ideal P of D, or $Q_1 \cap D = (0)$ but $Q_1 \cap D[X_1] \neq (0)$.

Since $\omega \leqq m$,

$$M[X_1, \ldots, X_m] \supset Q_\omega[X_{\omega+1}, \ldots, X_m] \supset \cdots \supset Q_1[X_{\omega+1}, \ldots, X_m] \supset (0)$$

is a chain of prime ideals of $D[X_1, \ldots, X_m]$ having the desired form.

Therefore, it suffices to prove Lemma 5 for the special case in which $\omega = m$. The proof will be by induction on n, where $n = \dim D$.

We first consider the special case in which there exists a chain of prime ideals $D[X_1, \ldots, X_m] \supset Q_{2m+1} \supset \cdots \supset Q_1 \supset (0)$ such that if $Q_i \cap D \neq (0)$, then $Q_i \cap D = M$, $1 \leqq i \leqq 2m+1$. Since this is the case when $n=1$ we will have the first step of an induction argument.

By taking $P=(0)$ in Lemma 4, we see that $Q_{m+1} \cap D \neq (0)$ so that, by hypothesis, $Q_{m+1} \cap D = M$. Then $Q_{m+1} \supseteq M[X_1, \ldots, X_m]$, and it follows from Lemma 4 that $M[X_1, \ldots, X_m]$ has depth m in $D[X_1, \ldots, X_m]$. But $d(Q_{m+1}) \geqq m$, so $Q_{m+1} = M[X_1, \ldots, X_m]$.

Since $Q_{m+1} \supset Q_m$, our assumption implies that $Q_m \cap D = (0)$. However, by Lemma 4, $Q_m \cap D[X_1] \neq (0)$—say $Q_m \cap D[X_1] = Q_1'$. If $D' = D[X_1]_{Q_1'}$, then $D' \supseteq L$ since $Q_1' \cap D = (0)$, and every valuation overring of D' has dimension less than or equal to one [9, p. 50]. Therefore, by Theorem 1, D' is a one-dimensional domain such that every overring has dimension less than or equal to one. It then follows from Theorem 4 that $\dim (D')[X_2, \dots, X_m]=m$. Let $(Q_1')^e=(Q_1')D'$. By Lemma 4, $(Q_1')^e[X_2, \dots, X_m]$ has depth $m-1$ in $(D')[X_2, \dots, X_m]$, so it is minimal. Since $(D')[X_2, \dots, X_m]$ is a quotient ring of $D[X_1, \dots, X_m]$ with respect to the multiplicative system

$$D[X_1]-Q_1',\ Q_1'' = (Q_1')[X_2, \dots, X_m] = (Q_1')^e[X_2, \dots, X_m] \cap D[X_1, \dots, X_m]$$

is minimal in $D[X_1, \dots, X_m]$.

Now $Q_1'' \subseteq Q_m$ and $Q_1'' \cap D[X_1]=Q_1'=Q_m \cap D[X_1]$. If $Q \supset Q_m$, then $h(Q) \geqq m+1$, so by Lemma 4 we have $Q \cap D \neq (0)$. Hence $Q \cap D[X_1] \neq Q_1'$ since $Q_1' \cap D=(0)$. Since Q_1'' is minimal there exists, by Lemma 4, prime ideals $Q_2'', \dots, Q_{m-1}''$ of $D[X_1, \dots, X_m]$ such that $(0) \subset Q_1'' \subset Q_2'' \subset \cdots \subset Q_{m-1}'' \subset Q_m$. Then $M[X_1, \dots, X_m] \supset Q_m \supset Q_{m-1}'' \supset \cdots \supset Q_2'' \supset Q_1'' \supset (0)$ is the desired chain.

We now assume that the result is true for $n<k$ and that $\dim D=k$. If

$$\dim D[X_1, \dots, X_m] \geqq 2m+1$$

and if $D[X_1, \dots, X_m] \supset Q_{2m+1} \supset \cdots \supset Q_1 \supset (0)$ is a chain of prime ideals of

$$D[X_1, \dots, X_m],$$

then, from what we have just shown, we may assume that $(0) \subset Q_i \cap D \subset M$ for some i, $1 \leqq i \leqq 2m+1$. Thus we choose α, $1 \leqq \alpha \leqq 2m+1$, such that $Q_\alpha \cap D \neq (0)$ but $Q_{\alpha-1} \cap D=(0)$. (We take $Q_0=(0)$.) Suppose that $Q_\alpha \cap D=P$ and suppose that $\dim D_P=\mu$. By assumption $P \subset M$ so that $\mu<k$. Let $(0) \subset P_1 \subset \cdots \subset P_{\mu-1} \subset P$ be a chain, having length μ, of prime ideals of D which are contained in P. Let λ be the maximal length of a proper chain of prime ideals of $D[X_1, \dots, X_m]$ which is contained properly between Q_α and $P[X_1, \dots, X_m]$ (let $\lambda=-1$ if $Q_\alpha=P[X_1, \dots, X_m]$), and let $t=(2m+1)-(\alpha-1)=2m-\alpha+2$. Then $P_1[X_1, \dots, X_m]$ has depth greater than or equal to $t+\lambda+\mu-1$ in $D[X_1, \dots, X_m]$.

If $t+\lambda+\mu \geqq 2m+1$, then $\dim (D/P_1)[X_1, \dots, X_m] \geqq 2m=(m-1)+m+1$ and $m-1 \geqq \dim D/P_1$ (since $m \geqq \dim D$ and $P_1 \neq (0)$). Taking $\omega=m-1$, Remark 1 implies that $\dim (D/P_1)[X_1, \dots, X_{m-1}] \geqq 2(m-1)+1$. But $\dim D/P_1<\dim D=k$, so by the induction hypothesis there is a chain of prime ideals of

$$(D/P_1)[X_1, \dots, X_{m-1}]$$

of the form $(M/P_1)[X_1, \dots, X_{m-1}] \supset Q_{m-1}'' \supset \cdots \supset Q_1'' \supset (0)$. If for $1 \leqq i \leqq m-1$, Q_i' is the unique prime ideal of $D[X_1, \dots, X_m]$ such that $Q_i' \supseteq P_1[X_1, \dots, X_{m-1}]$ and $(Q_i')/P_1[X_1, \dots, X_{m-1}] \cong Q_i''$, then

$$M[X_1, \dots, X_{m-1}] \supset Q_{m-1}' \supset \cdots \supset Q_1' \supset P_1[X_1, \dots, X_{m-1}].$$

Clearly, $M[X_1, \dots, X_m] \supset (Q_{m-1}')[X_m] \supset \cdots \supset (Q_1')[X_m] \supset P_1[X_1, \dots, X_m]$ is a chain of the desired form.

We now suppose that $t+\lambda+\mu<2m+1$. We first show the existence of a chain $M[X_1, \dots, X_m] \supset Q_\beta' \supset \cdots \supset Q_1' \supset P[X_1, \dots, X_m]$ of prime ideals of $D[X_1, \dots, X_m]$ such that $\beta+m+1 \geq t+\lambda$.

Thus, if $t+\lambda \leqq m+\dim D/P$, we take $\beta+1=\dim D/P$. Then there exists a chain $M \supset Q_\beta \supset \cdots \supset Q_1 \supset P$ of prime ideals of D and $M[X_1, \ldots, X_m] \supset Q_\beta[X_1, \ldots, X_m] \supset \cdots \supset Q_1[X_1, \ldots, X_m] \supset P[X_1, \ldots, X_m]$ is the desired chain. If, on the other hand, $t+\lambda \geqq \dim D/P+m+1$, we take β to be such that $t+\lambda=\beta+m+1$. Then $\beta \geqq \dim D/P$ and by assumption $t+\lambda+\mu<2m+1$, so that $\beta+m+1=t+\lambda<2m+1$. Therefore we have $\beta<m$; that is, $\dim D/P<m$. But $\dim (D/P)[X_1, \ldots, X_m] \geqq t+\lambda=\beta+m+1$, so it follows from Remark 1 that $\dim (D/P)[X_1, \ldots, X_\beta] \geqq 2\beta+1$. Since $\dim D/P<k$, the induction hypothesis is applicable. Hence, using the same method of proof given above for D/P_1, there exist prime ideals $Q'_\beta, \ldots, Q'_1$ of $D[X_1, \ldots, X_m]$ such that $M[X_1, \ldots, X_m] \supset Q'_\beta \supset \cdots \supset Q'_1 \supset P[X_1, \ldots, X_m]$.

We now consider the domain D_P. Since $Q_i \cap D \subseteq P$ for $1 \leqq i \leqq \alpha$, if we set $Q_i^e=Q_iD_P[X_1, \ldots, X_m]$, $1 \leqq i \leqq \alpha$, $P_i^e=P_iD_P$, $1 \leqq i \leqq \mu-1$, and $P^e=PD_P$, then we have $(0) \subset Q_1^e \subset \cdots \subset Q_\alpha^e$, $(0) \subset (P_1^e)[X_1, \ldots, X_m] \subset \cdots \subset (P_{\mu-1}^e)[X_1, \ldots, X_m] \subset (P^e) \cdot [X_1, \ldots, X_m] \subseteq Q_\alpha^e$, $Q_i^e \cap D_P=(0)$ for $1 \leqq i \leqq \alpha-1$, $Q_\alpha^e \cap D_P=P^e$, and λ is the maximum length of a proper chain of prime ideals of $D_P[X_1, \ldots, X_m]$ contained properly between Q_α^e and $(P^e)[X_1, \ldots, X_m]$ ($\lambda=-1$ if $Q_\alpha^e=P^e[X_1, \ldots, X_m]$). By Lemma 4 there is a chain of prime ideals of $D_P[X_1, \ldots, X_m]$ of the form

$$(P^e)[X_1, \ldots, X_m] = H_{m+1} \subset H_m \subset \cdots \subset H_1$$

such that $H_i \cap D_P=P^e$ for each i, and $Q_\alpha^e=H_s$ for some s, $1 \leqq s \leqq m+1$. Then $(0) \subset Q_1^e \subset \cdots \subset Q_{\alpha-1}^e \subset H_s \subset \cdots \subset H_1 \subset D_P[X_1, \ldots, X_m]$ is a chain of prime ideals of $D_P[X_1, \ldots, X_m]$ so that $\dim D_P[X_1, \ldots, X_m] \geqq \alpha-1+s$. But by assumption $\mu+\lambda+t<2m+1$. Hence, $\mu+\lambda<2m+1-t=\alpha-1$ and we have $\mu+\lambda+s<\alpha-1+s$. By choice of the integer λ and the ideals $H_1, \ldots, H_s=Q_\alpha^e$, it follows from Lemma 4 that $\lambda+s=m$. Consequently, $\mu+m<\alpha-1+s$. By Lemma 4, $\alpha-1 \leqq m$ (since $i \leqq \alpha-1$ implies $Q_i \cap D=(0)$), and $s \leqq m+1$ by choice. Then $\alpha-1+s \leqq 2m+1$, so we may choose $\gamma \leqq m$ such that $\alpha-1+s=\gamma+m+1$. We now have $\mu+m<\alpha-1+s=\gamma+m+1 \leqq \dim D_P[X_1, \ldots, X_m]$, from which it follows that $\mu \leqq \gamma \leqq m$ (we recall that $\mu=\dim D_P$). Remark 1 now implies that $\dim D_P[X_1, \ldots, X_\gamma] \geqq 2\gamma+1$.

Since $P \subset M$, $\dim D_P<k$, so by the induction hypothesis there is a chain of prime ideals of $D_P[X_1, \ldots, X_\gamma]$ of the form $P^e[X_1, \ldots, X_\gamma] \supset \Gamma'_\gamma \supset \cdots \supset \Gamma'_1 \supset (0)$, where either $\Gamma'_1=P'[X_1, \ldots, X_\gamma]$ for some prime ideal P' of D_P, or $\Gamma'_1 \cap D_P=(0)$ but $\Gamma'_1 \cap D_P[X_1] \neq (0)$. If we let $\Gamma_i=\Gamma'_i \cap D[X_1, \ldots, X_\gamma]$, $1 \leqq i \leqq \gamma$, then

$$P[X_1, \ldots, X_\gamma] \supset \Gamma_\gamma \supset \cdots \supset \Gamma_1 \supset (0)$$

is a chain of prime ideals of $D[X_1, \ldots, X_\gamma]$. Further, if $\Gamma'_1=(P')[X_1, \ldots, X_\gamma]$, then $\Gamma_1=(P'')[X_1, \ldots, X_\gamma]$ where $P''=P' \cap D$; or, if $\Gamma'_1 \cap D_P=(0)$ but $\Gamma'_1 \cap D[X_1] \neq (0)$, then $\Gamma_1 \cap D=(0)$ but $\Gamma_1 \cap D[X_1] \neq (0)$. We now show that

$$\begin{aligned} M[X_1, \ldots, X_m] &\supset Q'_\beta \supset \cdots \supset Q'_1 \supset P[X_1, \ldots, X_m] \\ &\supset L_\gamma[X_{\gamma+1}, \ldots, X_m] \supset \cdots \supset \Gamma_1[X_{\gamma+1}, \ldots, X_m] \supset (0) \end{aligned}$$

is the desired chain of prime ideals.

Certainly $\Gamma_1[X_{\gamma+1}, \ldots, X_m]$ has the desired form, so it suffices to show that $\beta+\gamma+1 \geqq m$. But $\gamma+\lambda+s+1=\gamma+m+1=\alpha-1+s=2m+1-t+s$. Hence, $\gamma+\lambda+1=2m+1-t$ so that $\gamma+\lambda+t+1=2m+1$. By choice of β, $\beta+m+1 \geqq t+\lambda$ and it follows that $\gamma+\beta+m+2 \geqq \gamma+t+\lambda+1=2m+1$. Therefore, $\beta+\gamma+1 \geqq m$ as we wished to show.

This completes the proof of Lemma 5.

We are now in position to show that the conditions given in Theorem 2 are necessary in order that $\dim D[t_1, \ldots, t_m] \leqq \omega$ for $\{t_1, \ldots, t_m\} \subseteq L$. Theorem 5 is the main result of this paper.

THEOREM 5. *If* $\dim D=n$ *and if* m, ω *are nonnegative integers such that*

$$\dim D[t_1, \ldots, t_m] \leqq \omega \quad \textit{for } \{t_1, \ldots, t_m\} \subseteq L,$$

then the following conditions hold.

(1) $\dim D[X_1, \ldots, X_m] \leqq \omega+m$.

(2) *If there exist elements* $t_1, \ldots, t_m$ *in* L *such that* $\dim D[t_1, \ldots, t_m]=\omega$, *then* $\dim D[X_1, \ldots, X_m]=\omega+m$.

Proof of (1). The proof of (1) will be by induction on n and m. Thus, we first show that (1) is true when either $n=1$ or $m=1$.

Suppose that $n=1$. By a theorem of Sedenberg [7, p. 608], D one-dimensional implies that for any m, $\dim D[X_1, \ldots, X_m] \leqq 2m+1$. Clearly then (1) holds if $\omega \geqq m+1$, so we assume that $\omega \leqq m$. Since $\dim D[t_1, \ldots, t_m] \leqq \omega$ for $\{t_1, \ldots, t_m\} \subseteq L$, it follows, by taking $t_\omega=t_{\omega+1}=\cdots=t_m$, that $\dim D[t_1, \ldots, t_\omega] \leqq \omega$ for $\{t_1, \ldots, t_\omega\} \subseteq L$. Theorem 1 now implies that each overring of D has dimension less than or equal to ω so that, by Theorem 4 $\dim D[X_1, \ldots, X_m] \leqq m+\omega$.

Now suppose that $m=1$. We have just seen that (1) holds for $n=1$, so we assume that (1) is true for $n<h$, that $\dim D=h$, and that $\dim D[t] \leqq \omega$ for $t \in L$. Let $(0) \subset Q_1 \subset \cdots \subset Q_s \subset D[X]$ be a chain of prime ideals of $D[X]$, where Q_1 is chosen to be minimal. If $Q_1 \cap D=(0)$, then $D[X]/Q_1 \cong D[\bar{X}]$, where $\bar{X}=X+Q_1$, and $f(\bar{X})=0$ for any $f \in Q_1$, so $\bar{X}$ is algebraic over D. It follows from Corollary 1 that $\dim D[\bar{X}] \leqq \omega$ and this implies that Q_1 has depth less than or equal to ω in $D[X]$. Therefore, $s \leqq \omega+1$. On the other hand, if $Q_1 \cap D \neq (0)$, then $Q_1 \cap D \supseteq P$, where P is a minimal prime ideal of D. By choice, Q_1 is minimal, and $Q_1 \supseteq P[X]$. Thus, $Q_1=P[X]$. From Lemma 3 we have $\dim (D/P)[\sigma] \leqq \omega-1$ for each σ in the quotient field of D/P. Further, $\dim (D/P)<h$, and by assumption (1) holds; that is,

$$\dim (D/P)[X] \leqq \omega.$$

But $(D/P)[X] \cong D[X]/P[X]$ so that $P[X]=Q_1$ has depth less than or equal to ω. Consequently $s \leqq \omega+1$ and it follows by induction that (1) is true for $m=1$.

From what we have just shown, we may make the following inductive assumptions.

(A) Suppose that (1) is true for any n when $m<k$.

(B) Suppose that (1) is true for $n<h$ when $m=k$.

Now let $\dim D=h$ and suppose that $\dim D[X_1,\ldots,X_k]\geqq\omega+k+1$, $\omega\geqq h$. We wish to establish the existence of $t_1,\ldots,t_k$ in L such that $\dim D[t_1,\ldots,t_k]\geqq\omega+1$.

If $\dim D[X]\geqq\omega+2$, then by the case in which $m=1$ there exists $t\in L$ such that $\dim D[t]\geqq\omega+1$. If we set $t=t_1=\cdots=t_k$, it follows that $\dim D[t_1,\ldots,t_k]\geqq\omega+1$ and we are finished.

Suppose, then, that $\dim D[X_1]\leqq\omega+1$. The assumption that

$$\dim D[X_1][X_2,\ldots,X_k] \geqq \omega+k+1$$

then implies, by (A), that there exist elements $\delta_2,\ldots,\delta_k$ in $L(X_1)$ such that $\dim D[X_1][\delta_2,\ldots,\delta_k]\geqq\omega+2$. Let Q be the kernel of the canonical $D[X_1]$-homomorphism ϕ which maps $D[X_1][X_2,\ldots,X_k]$ onto $D[X_1][\delta_2,\ldots,\delta_k]$ in such a way that $\phi(X_i)=\delta_i$ for each i. Then Q must have depth greater than or equal to $\omega+2$, and by Lemma 1, Q has height $k-1$. Hence, there exists a chain of prime ideals of $D[X_1,\ldots,X_k]$ of the form

$$(0)\subset Q_1\subset\cdots\subset Q_{k-2}\subset Q\subset Q_k\subset\cdots\subset Q_{k+\omega+1}\subset D[X_1,\ldots,X_k].$$

If $f(X_1)\in D[X_1]$, then $\phi(f(X_1))=f(X_1)$. Therefore, $Q\cap D[X_1]=(0)$. However, since $h(Q)=k-1$, Lemma 4 implies that $Q_k\cap D[X_i]\neq(0)$ for i, $1\leqq i\leqq k$. We now consider the two cases in which $Q_k\cap D=(0)$ and $Q_k\cap D\neq(0)$.

If $Q_k\cap D=(0)$, then $D[X_1,\ldots,X_k]/Q_k\simeq D[\bar{X}_1,\ldots,\bar{X}_k]$, where $\bar{X}_i=X_i+Q_k$, and since $Q_k\cap D[X_i]\neq(0)$, $\bar{X}_i$ is algebraic over D for each i. But

$$\dim D[\bar{X}_1,\ldots,\bar{X}_k] \geqq \omega+1,$$

so by Corollary 1 there exist elements $t_1,\ldots,t_k$ in L such that $\dim D[t_1,\ldots,t_k]\geqq\omega+1$ and we are finished.

Thus, suppose that $Q_k\cap D\neq(0)$—say $Q_k\cap D=P$, where P is a prime ideal of D such that $h(P)=\mu$. Then $Q_k\supseteq P[X_1,\ldots,X_k]$ and there exists a chain $(0)\subset P_1\subset\cdots\subset P_{\mu-1}\subset P$ of prime ideals of D. Let λ be the maximal length of a proper chain of prime ideals of $D[X_1,\ldots,X_k]$ contained properly between Q_k and $P[X_1,\ldots,X_k]$ (let $\lambda=-1$ if $Q_k=P[X_1,\ldots,X_k]$). We proceed now to show the existence of a chain of prime ideals of $D[X_1,\ldots,X_k]$ which has length greater than or equal to $k+\omega+1$ and which is of the form $Q_{k+\omega+1}\supset\cdots\supset Q_k\supseteq\cdots\supseteq P[X_1,\ldots,X_k]\supset Q'_\gamma\supset\cdots\supset Q'_1\supset(0)$ where either $Q'_1=P'[X_1,\ldots,X_k]$ for some prime ideal P' of D, or $Q'_1\cap D=(0)$ but $Q'_1\cap D[X_1]\neq(0)$. (Here we understand that a proper chain of length λ is contained between Q_k and $P[X_1,\ldots,X_k]$.) We shall say that such a chain has form (C). Thus, we seek a chain of form (C) for which $\gamma+\lambda+\omega+3\geqq k+\omega+1$.

If $\mu+\lambda\geqq k-1$, then

$$\begin{aligned}Q_{k+\omega+1}\supset\cdots\supset Q_k\supseteq\cdots\supseteq P[X_1,\ldots,X_k]\supset P_{\mu-1}[X_1,\ldots,X_k]\\ \supset\cdots\supset P_1[X_1,\ldots,X_k]\supset(0)\end{aligned}$$

(where a proper chain of length λ is included between Q_k and $P[X_1, \ldots, X_k]$) is such a chain since $(\mu-1)+\lambda+\omega+3=\mu+\lambda+\omega+2 \geqq k-1+\omega+2=k+\omega+1$.

Suppose, then, that $\mu+\lambda<k-1$ and consider the domain D_P. For $1 \leqq i \leqq k$, let $Q_i^e=Q_iD_P[X_1, \ldots, X_k]$, and for $1 \leqq i \leqq \mu$, let $P_i^e=P_iD_P$, where we set $Q_{k-1}=Q$ and $P_\mu=P$. Then $(0) \subset Q_1^e \subset \cdots \subset Q_k^e$ is a chain of prime ideals of D_P, $Q_i^e \cap D_P=(0)$ for $1 \leqq i \leqq k-1$, $Q_k^e \supseteq (P^e)[X_1, \ldots, X_k]$, $Q_k^e \cap D_P=P^e$, and λ is the maximal length of a chain of prime ideals of $D_P[X_1, \ldots, X_k]$ contained properly between Q_k^e and $P^e[X_1, \ldots, X_k]$. By Lemma 4 there is a chain of prime ideals $P^e[X_1, \ldots, X_k]=\Gamma_{k+1} \subset \Gamma_k \subset \cdots \subset \Gamma_1 \subset D_P[X_1, \ldots, X_k]$ such that $\Gamma_i \cap D_P=P^e$ for each i and such that $Q_k^e=\Gamma_s$ for some s, $1 \leqq s \leqq k+1$. From Lemma 4, and by choice of λ, it follows that $\lambda=k-s$; that is, $s+\lambda=k$. We now have the chain $(0) \subset Q_1 \subset \cdots \subset Q_{k-1} \subset \Gamma_s \subset \cdots \subset \Gamma_1 \subset D_P[X_1, \ldots, X_k]$ of prime ideals of $D_P[X_1, \ldots, X_k]$, from which it follows that $\dim D_P[X_1, \ldots, X_k] \geqq s+k-1>s+\lambda+\mu=k+\mu$. Let γ be chosen so that $s+k-1=k+\gamma+1$. Then $\gamma \geqq \mu=\dim D_P$ and, by choice of s, we have $s \leqq k+1$, so that $k+\gamma+1=s+k-1 \leqq 2k$. Consequently, we have $\mu \leqq \gamma \leqq k-1$ and $\dim D_P[X_1, \ldots, X_k] \geqq k+\gamma+1$. Therefore, by Lemma 5, there exists a chain of prime ideals of $D_P[X_1, \ldots, X_k]$ of the form $P^e[X_1, \ldots, X_k] \supset Q_\gamma'' \supset \cdots \supset Q_1'' \supset (0)$, where either $Q_1''=P''[X_1, \ldots, X_k]$ for some prime ideal P'' of D_P, or $Q_1'' \cap D_P=(0)$ but $Q_1'' \cap D_P[X_1] \neq (0)$. Let $Q_i'=Q_i'' \cap D[X_1, \ldots, X_k]$ for each i, $1 \leqq i \leqq \gamma$. Then $Q_{\omega+k+1} \supset \cdots \supset Q_k \supseteq \cdots \supseteq P[X_1, \ldots, X_k] \supset Q_\gamma' \supset \cdots \supset Q_1' \supset (0)$ is a chain of prime ideals of $D[X_1, \ldots, X_k]$ having form (C), for if $Q_1''=P''[X_1, \ldots, X_k]$ for some prime ideal P'' of D_P, then $Q_1'=P'[X_1, \ldots, X_k]$, where $P'=P'' \cap D$. On the other hand, if $Q_1'' \cap D_P=(0)$ but $Q_1'' \cap D_P[X_1] \neq (0)$, then $Q_1' \cap D=(0)$ but $Q_1' \cap D[X_1] \neq (0)$. Further, $s+k-1=k+\gamma+1=\lambda+s+\gamma+1$ so that $k-1=\lambda+\gamma+1$. It then follows that $k+\omega+1=\lambda+\gamma+\omega+3$.

Lemma 6. *Suppose that* $\dim D=h$ *and* $\dim D[t_1, \ldots, t_k] \leqq \omega$ *for* $\{t_1, \ldots, t_k\} \subseteq L$. *Then if P is a proper prime ideal of D, $P[X_1, \ldots, X_k]$ has depth less than or equal to $\omega+k-1$ in $D[X_1, \ldots, X_k]$, and if Q is a prime ideal of $D[X_1, \ldots, X_k]$ such that $Q \cap D=(0)$ but $Q \cap D[X_1] \neq (0)$, then Q has depth less than or equal to $\omega+k-1$ in $D[X_1, \ldots, X_k]$.*

Proof. If P is a proper prime ideal of D, then by Lemma 3 we have

$$\dim (D/P)[s_1, \ldots, s_k] \leqq \omega-h(P)$$

for any set of elements $\{s_1, \ldots, s_k\}$ contained in the quotient field of D/P. From assumption (B) it then follows that $\dim (D/P)[X_1, \ldots, X_k] \leqq \omega+k-h(P)$. But $(D/P)[X_1, \ldots, X_k] \cong D[X_1, \ldots, X_k]/P[X_1, \ldots, X_k]$, so that $P[X_1, \ldots, X_k]$ has depth less than or equal to $\omega+k-h(P)$.

Suppose that Q is a prime ideal of $D[X_1, \ldots, X_k]$ such that $Q \cap D=(0)$ but $Q \cap D[X_1] \neq (0)$—say $Q \cap D[X_1]=Q'$. Then $Q \supseteq (Q')[X_2, \ldots, X_k]$ and

$$D[X_1, \ldots, X_k]/(Q')[X_2, \ldots, X_k] \cong (D[X_1]/Q')[X_2, \ldots, X_k].$$

But $D[X_1]/Q' \cong D[\bar{X}_1]$, where $\bar{X}_1 = X_1 + Q'$, and $\bar{X}_1$ is algebraic over D. Since $\dim D[t] \leqq \omega$ for $t \in L$, it follows from Corollary 1 that $\dim D[\bar{X}_1] \leqq \omega$. Moreover, by Lemma 2, there exists a nonzero element d in D such that $\bar{X}_1$ is integral over $D[1/d]$. But L is the quotient field of $D[1/d]$ and $\dim D[1/d][t_1, \ldots, t_{k-1}] \leqq \omega$ for $\{t_1, \ldots, t_{k-1}\} \subseteq L$. Therefore, by Theorem 3, we have $\dim D[\bar{X}_1][s_1, \ldots, s_{k-1}] \leqq \omega$ for any set of elements $\{s_1, \ldots, s_{k-1}\}$ of the quotient field of $D[\bar{X}_1]$. It now follows from assumption (A) that $\dim D[\bar{X}_1][X_2, \ldots, X_k] \leqq \omega + k - 1$. Consequently, $Q'[X_2, \ldots, X_k]$ must have depth less than or equal to $\omega + k - 1$ so that Q also has depth less than or equal to $\omega + k - 1$ as we wished to show.

We now complete the proof of Theorem 5.

By assumption the ideal Q_1' in a chain having form (C) has depth greater than or equal to $k + \omega$. However, Q_1' has one of the forms described in Lemma 6 so it follows that $\dim D[t_1, \ldots, t_k] \geqq \omega + 1$ for some set $\{t_1, \ldots, t_k\} \subseteq L$.

Statement (1) of Theorem 5 now follows by induction.

Assume now that $\dim D[t_1, \ldots, t_m] = \omega$ for some $\{t_1, \ldots, t_m\} \subseteq L$. From (1) it follows that $\dim D[X_1, \ldots, X_m] \leqq \omega + m$. But if $\dim D[X_1, \ldots, X_m] = \alpha + m$, where $\alpha \leqq \omega$, it follows from Theorem 2 that $\dim D[s_1, \ldots, s_m] \leqq \alpha$ for $\{s_1, \ldots, s_m\} \subseteq L$. In particular, $\dim D[t_1, \ldots, t_m] = \omega \leqq \alpha$, so that $\alpha = \omega$. Statement (2) of Theorem 5 now follows.

This completes the proof of Theorem 5.

In [3] Jaffard defines the *valuative dimension*, denoted by $\dim_v D$, of the domain D to be the maximal rank of the valuation overrings of D. With this notation and terminology, we now relate many of the results of this paper in the following theorem.

THEOREM 6. *Let D be a finite-dimensional integral domain with identity having quotient field L, and let ω be a positive integer such that $\omega \geqq \dim D$. Then the following statements are equivalent.*

(1) $\dim_v D = \omega$.

(2) *Each overring of D has dimension less than or equal to ω and ω is minimal.*

(3) *For any nonnegative integer m, $\dim D[t_1, \ldots, t_m] \leqq \omega$ for $\{t_1, \ldots, t_m\} \subseteq L$, and for $m \geqq \omega - 1$ there exists $\{t_1, \ldots, t_m\} \subseteq L$ such that $\dim D[t_1, \ldots, t_m] = \omega$.*

(4) *For any nonnegative integer m, $\dim D[X_1, \ldots, X_m] \leqq m + \omega$ and for $m \geqq \omega - 1$ equality holds.*

(5) $\dim D[X_1, \ldots, X_\omega] = 2\omega$.

(6) $\dim D[t_1, \ldots, t_\omega] \leqq \omega$ *for any set $\{t_1, \ldots, t_\omega\} \subseteq L$, and there exists a set $\{s_1, \ldots, s_\omega\} \subseteq L$ such that $\dim D[s_1, \ldots, s_\omega] = \omega$.*

Proof. It was shown in the proof of Theorem 1 that if D' is an overring of D such that $\dim D' = k$, then there exists a valuation overring V of D such that $\dim V \geqq k$. This fact together with Theorem 1 shows that (1) and (2) are equivalent.

To show that (2) implies (3), it clearly suffices to show that for any positive integer $m \geqq \omega - 1$, there exists $\{t_1, \ldots, t_m\} \subseteq L$ such that $\dim D[t_1, \ldots, t_m] = \omega$.

However, it follows from the proof of Theorem 1 that if there exists an overring D' of D such that $\dim D'=\omega$, then $\dim D[t_1, \ldots, t_{\omega-1}] \geqq \omega$ for some $\{t_1, \ldots, t_{m-1}\} \subseteq L$. Thus, equality holds and for any $m \geqq \omega-1$, $\dim D[t_1, \ldots, t_m]=\omega$, where $t_{\omega-1}=t_\omega=\cdots=t_m$.

That (3) implies (4) is an immediate consequence of Theorem 5 and certainly (4) implies (5). If (5) holds, then by Theorem 2 we have $\dim D[t_1, \ldots, t_\omega] \leqq \omega$ for $\{t_1, \ldots, t_\omega\} \subseteq L$. But if $\dim D[t_1, \ldots, t_\omega] \leqq k$ for any $k \leqq \omega$, then it follows from Theorem 5 that $\dim D[X_1, \ldots, X_\omega] \leqq k+\omega$ so that $k \geqq \omega$. Thus $k=\omega$, and it follows that $\dim D[s_1, \ldots, s_\omega]=\omega$ for some $\{s_1, \ldots, s_\omega\} \subseteq L$. Therefore (6) holds.

It is immediate from Theorem 1 that (6) implies (2) and Theorem 6 is proved.

REMARK 2. If we take $\omega=\dim D$, then for any nonnegative integer m and $\{t_1, \ldots, t_m\} \subseteq L$, we have $\dim D[t_1, \ldots, t_m]=\omega$. Thus from Theorem 6,

$$\dim D[X_1, \ldots, X_m] = m+\dim D$$

for all m if and only if $\dim D=\dim_v D$.

II. Suppose now that D is integrally closed. Let $\{V_\alpha\}$ be the set of all valuation overrings of D, and let A be an ideal of D. Then $\tilde{A}=\bigcap_\alpha AV_\alpha$ is an ideal of D called the *completion* of A. If X is an indeterminate over D and $f \in D[X]$, then we denote by A_f the ideal of D generated by the coefficients of f. We now define the Kronecker function ring of D as follows:

$$D^k = \{f/g \mid f, g \in D[X], \tilde{A}_f \subseteq \tilde{A}_g\}.$$

In [4], Krull shows that D^k is an integral domain having quotient field $L(X)$ and that $D^k \cap L=D$. He further showed that D^k is a Bezout domain, where a *Bezout domain* is defined to be a domain in which each finitely generated ideal is principal.

Now let V be a valuation overring of D and let v be a valuation associated with V. If $f \in L[X]-\{0\}$, $f=f_0+f_1X+\cdots+f_nX^n$, we define $v^*(f)=\min_{0 \leqq i \leqq n}\{v(f_i) \mid f_i \neq 0\}$. Then v^* defines a valuation on $L(X)$ having the same value group as v. In particular, v and v^* have the same rank. We call v^* the *trivial extension of* v *to* $L(X)$, and if V^* is the valuation ring of $L(X)$ associated with v^*, then V^* is called the *trivial extension of* V *to* $L(X)$. Krull has shown in [4, p. 560] that if $\{V_\alpha\}$ is the collection of valuation overrings of D, then $\{V_\alpha^*\}$ is the collection of valuation overrings of D^k.

An integral domain R with identity is said to be a *Prüfer domain* provided each finitely generated nonzero ideal of R is invertible. In particular, a Bezout domain if a Prüfer domain, so D^K is a Prüfer domain. Therefore, $\dim_v D^K=\dim D^K$ [3, p. 56]. But from the previous remarks we see that $\dim_v D=\dim_v D^K$. We have thus proved the following result.

THEOREM 7. *Let D be an integrally closed domain with identity and let D^K be the Kronecker function ring of D. Then* $\dim_v D=\dim D^K$.

COROLLARY 2. *If D is an integral domain with identity having integral closure $\bar{D}$, the statement that* $\dim (\bar{D})^K=\omega$ *is equivalent to each of the statements* (1)–(6) *of Theorem* 6.

III. Let D be an n-dimensional integral domain with identity having quotient field L. We have seen that each overring of D has dimension less than or equal to n if and only if $\dim D[t_1, \ldots, t_n] \leqq n$ for each subset $\{t_1, \ldots, t_n\} \subseteq L$. For any positive integer n, we now show the existence of an integral domain D such that $\dim D = n$, and such that $\dim D[t_1, \ldots, t_m] \leqq n$ for any positive integer $m < n$ and for each subset $\{t_1, \ldots, t_m\} \subseteq L$, but such that $\dim V = n+1$ for some valuation overring V of D([2]). We first state the following results which are proved in [2].

LEMMA 7. *Let $\{V_1, \ldots, V_k\}$ be a collection of valuation rings having quotient field L, and suppose that $V_i \nsubseteq V_j$ for $i \neq j$. If M_i is the maximal ideal of V_i, then $\bigcap_{j \neq 1} M_j \nsubseteq V_i$ for any i.*

LEMMA 8. *Let $\{V_1, \ldots, V_k\}$ be as in Lemma 7 and suppose that each V_i contains some fixed field F. If $D = F + M$, where $M = M_1 \cap \cdots \cap M_k$, then D is a quasi-local domain with maximal ideal M and if P is a nonmaximal prime ideal of D, then $P = Q \cap D$, where Q is a nonmaximal prime ideal of V_i for some i, $1 \leqq i \leqq k$.*

Now let n be an arbitrary positive integer, let K be a field, and let $L = K(X_1, \ldots, X_{n+1})$. We may construct valuation rings V_1 and V_2 on L such that:

(a) V_1 has rank one and $V_1 = K(X_1, \ldots, X_n) + M_1$, where M_1 is the maximal ideal of V_1, and $X_{n+1} \in M_1$.

(b) V_2 has rank n, $V_2 = K + M_2$, where M_2 is the maximal ideal of V_2, $X_1/X_{n+1} \in M_2$, and if $M_2 = P_1 \supset P_2 \supset \cdots \supset P_n \supset P_{n+1} = (0)$ is the chain of prime ideals of V_2, then $X_i \in P_i - P_{i+1}$ for each i, $1 \leqq i \leqq n$.

We have $X_1/X_{n+1} \in V_2 - V_1$ and $1/X_1 \in V_1 - V_2$. Thus, by Lemma 8, if $D = K + M$, where $M = M_1 \cap M_2$, then D is a quasi-local domain with maximal ideal M, and D has quotient field L since M does. Further, $X_i X_{n+1} \in (P_i \cap D) - (P_{i+1} \cap D)$ for each i, $1 \leqq i \leqq n$, so it follows from Lemma 8 that $\dim D = n$.

Suppose that V is a nontrivial valuation overring of D. Then $V \supseteq M_1 \cap M_2$. so by Lemma 7 either $V \subseteq V_i$ or $V \supseteq V_i$ for $i = 1$ or 2. If $V \supseteq V_1$, then $V = V_1$ since $\dim V_1 = 1$. If $V \subseteq V_2$, then $V \supseteq M_2$, and $V \supseteq K$ since $V \supseteq D$. Therefore, $V \supseteq K + M_2 = V_2$, so that equality holds. Thus, if V is a nontrivial valuation overring of D, either $V \subseteq V_1$ or $V \supseteq V_2$.

Let m be a positive integer, $m < n$, and let $\{t_1, \ldots, t_m\} \subseteq L$. Then $D[t_1, \ldots, t_m]$ is the homomorphic image of $D[Y_1, \ldots, Y_m]$, $Y_1, \ldots, Y_m$ indeterminates over D, so it follows from Lemma 4 that if $P_1 \subset P_2 \subset \cdots \subset P_s$ is a chain of prime ideals of $D[t_1, \ldots, t_m]$ such that $P_i \cap D = M$ for each i, then $s \leqq m+1 \leqq n$. Further, let D' be an overring of D such that $\dim D' \geqq n+1$, and let $(0) \subset P_1' \subset \cdots \subset P_{n+1}' \subset D'$ be a chain of prime ideals of D'. Then there exists a valuation overring V of D', and a chain $(0) \subset Q_1 \subset \cdots \subset Q_{n+1} \subset V$ of prime ideals of V such that $Q_i \cap D' = P_i'$, $1 \leqq i \leqq n+1$ [5, p. 37]. Since $\dim V \geqq n+1$, $V \nsupseteq V_2$. Therefore, $V \subseteq V_1$, so it follows

([2]) The method for constructing such an example was suggested by William Heinzer.

that $M_1 \subseteq Q_1$. Thus $P_1' \cap D = (Q_1 \cap D') \cap D \supseteq (M_1 \cap D') \cap D = M$, and consequently, $P_i' \cap D = M$ for each i, $1 \leqq i \leqq n+1$. From what we have shown it follows that dim $D[t_1, \ldots, t_m] \leqq n$ for $\{t_1, \ldots, t_m\} \subseteq L$. But we may construct a valuation ring V_3 on L such that V_3 has rank $n+1$ and $V_3 = K + M_3$, where M_3 is the maximal ideal of V_3 and $M_3 \supseteq M_1$. Then $V_3 \supseteq K + M_1 \supseteq D$.

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