LOCALLY COMPACT RINGS HAVING A TOPOLOGICALLY NILPOTENT UNIT

BY
SETH WARNER[1]

What conditions on a locally compact ring with identity imply that it is the topological direct product of finitely many topological algebras over indiscrete locally compact fields? In [11, Theorem 4] it was shown that for an equicharacteristic ring A (an *equicharacteristic* ring A is a commutative ring with identity such that $A/\mathfrak{m}$ has the same characteristic as A for every maximal ideal $\mathfrak{m}$) a necessary and sufficient condition is that A be a semilocal ring none of whose maximal ideals is open (a *semilocal* ring is a commutative ring with identity that has only finitely many maximal ideals). The restriction to equicharacteristic rings is unsatisfying, however, for it rules out cartesian products of algebras over indiscrete locally compact fields of differing characteristics.

Our principal purpose is to find necessary and sufficient conditions for a locally compact ring A with identity to be the topological direct product of finitely many topological algebras over indiscrete locally compact fields (of possibly differing characteristics). That the additive order of each element of A be either infinite or a square-free integer is clearly a necessary condition, and we shall prove (Theorem 8) that this condition together with the following is both necessary and sufficient: the center of A contains an invertible element that is topologically nilpotent. En route, we shall obtain a structure theorem for commutative locally compact rings having an invertible element that is topologically nilpotent: such a ring is the topological direct product of finite-dimensional topological algebras over the field of real numbers and certain locally compact rings that arise naturally from commutative algebra. We shall investigate these rings—suitably topologized total quotient rings of one-dimensional Macaulay rings—in §1, and use the results obtained to investigate locally compact rings in §2.

1. **Topological quotient rings of one-dimensional Macaulay rings.** Let B be a commutative topological ring with identity. For each $b \in B$, let $L_b: x \to bx$. The set S of all cancellable elements $b \in B$ such that L_b is an open mapping from B into B is clearly a multiplicative subsemigroup. We topologize B_S, the quotient ring of B relative to S [14, p. 46], by declaring the filter of neighborhoods of zero in B to be a fundamental system of neighborhoods of zero in B_S. In particular, B is open in B_S; consequently B_S is a topological group under addition [1, p. 12], and multiplication

Received by the editors February 7, 1968 and, in revised form, May 3, 1968.
[1] Research supported by NSF grant GP-5933.

is continuous at (0, 0). Moreover, for each $s \in S$, $L_{s^{-1}}$ is continuous at zero in B_S since L_s is an open mapping from B into B; hence B_S is a topological ring [1, p. 75]. It is easy to see that if M is any multiplicative subsemigroup of B consisting of cancellable elements, then B_M, topologized by declaring the filter of neighborhoods of zero in B to be a fundamental system of neighborhoods of zero in B_M, is a topological ring if and only if $M \subseteq S$. We are thus led to the following definition.

DEFINITION. Let B be a commutative topological ring with identity, and let A be the quotient ring of B relative to the multiplicative subsemigroup consisting of all cancellable elements b such that L_b is an open mapping from B into B. The *B-topology* on A is the topology obtained by declaring the filter of neighborhoods of zero in B to be a fundamental system of neighborhoods of zero in A. The *topological quotient ring* of B is the topological ring A equipped with the B-topology.

We wish to investigate the topological quotient ring of a local noetherian ring B topologized with its natural topology (the powers of the maximal ideal of B form a fundamental system of neighborhoods of zero for the natural topology). By Theorem 1, we need only consider one-dimensional Macaulay rings (we recall that a one-dimensional local noetherian ring is a Macaulay ring if and only if its maximal ideal contains a cancellable element [15, p. 397; 14, Corollary 3, p. 214, and Remark, p. 215]).

THEOREM 1. *The topological quotient ring of a local noetherian ring B is a proper overring of B if and only if B is a one-dimensional Macaulay ring.*

Proof. Necessity. By hypothesis, the maximal ideal $\mathfrak{m}$ of B contains a cancellable element b such that L_b is an open mapping from B into B. In particular, Bb is open and hence contains a power of $\mathfrak{m}$, so Bb is a primary ideal and $\mathfrak{m}$ is its radical [14, Corollary 1, p. 153]; consequently, $\dim B \leqq 1$ [15, Theorem 20, p. 288]. If $\dim B = 0$, then B would be a primary ring, so $\mathfrak{m}^n = (0)$ for some $n \geqq 1$ [14, p. 204], and hence $\mathfrak{m}$ would contain no cancellable elements, a contradiction. Therefore $\dim B = 1$, so B is a one-dimensional Macaulay ring. The sufficiency of the condition is part of Theorem 2.

The author is indebted to the referee for suggesting the statement of (4) of Theorem 2 and for a consequent simplification of the author's original proof of Theorem 3.

THEOREM 2. *Let A be the topological quotient ring of a one-dimensional Macaulay ring B. Let $\mathfrak{m}$ be the maximal ideal of B, $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ the prime ideals of the zero ideal of B.*

(1) *If b is a cancellable element of B, then L_b is an open mapping from B into B; thus algebraically A is the total quotient ring of B.*

(2) *A is a noetherian ring whose maximal ideals are $A\mathfrak{p}_1, \ldots, A\mathfrak{p}_s$, and these are also the only proper prime ideals of A.*

(3) *The Jacobson radical of A is nilpotent.*

(4) *A is artinian.*

(5) *The set G of all invertible elements of A is open, and $x \to x^{-1}$ is continuous on G.*

Proof. As $\mathfrak{m}$ contains cancellable elements, to show that A properly contains B it suffices to establish (1). Let b be a cancellable element belonging to $\mathfrak{m}$, and let $Bb = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ be a primary representation of Bb. By [14, Corollary 3, p. 214], b belongs to the complement of $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$. Thus as the only proper prime ideals of B are $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$, and $\mathfrak{m}$, the radical of each $\mathfrak{q}_i$ is $\mathfrak{m}$, and therefore $\mathfrak{m}^n \subseteq \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m = Bb$ for some $n \geqq 1$ [14, p. 200]. Thus for each $r \geqq 0$, $\mathfrak{m}^r b = \mathfrak{m}^r Bb \supseteq \mathfrak{m}^{r+n}$, so L_b is an open mapping from B into B. If $b \notin \mathfrak{m}$, then b is invertible in B, so L_b is a homeomorphism from B onto B.

By [14, pp. 223–225], A is a noetherian ring whose only proper prime ideals are $A\mathfrak{p}_1, \ldots, A\mathfrak{p}_s$, and $A\mathfrak{p}_k \cap B = \mathfrak{p}_k$ for each $k \in [1, s]$. Consequently, each $A\mathfrak{p}_k$ is both a maximal ideal and a minimal prime ideal of A. The Jacobson radical $\mathfrak{r}$ of A is thus $A\mathfrak{p}_1 \cap \cdots \cap A\mathfrak{p}_s$; by [14, Theorem 7, p. 211], each $A\mathfrak{p}_k$ is a prime ideal of the zero ideal of A, so a suitably high power $\mathfrak{r}^n$ of $\mathfrak{r}$ is contained in each (isolated) primary component of the zero ideal; hence $\mathfrak{r}^n = (0)$.

Thus A is a semilocal noetherian ring whose radical is nilpotent. Hence (4) holds, for such a ring is necessarily artinian. To show this, it suffices by [14, Theorem 2, p. 203] to show that a proper prime ideal $\mathfrak{p}$ of A is one of the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. If not, $\mathfrak{p} \cap \mathfrak{m}_i \subset \mathfrak{m}_i$ for each $i \in [1, n]$, so there exists $c_i \in \mathfrak{m}_i$ such that $c_i \notin \mathfrak{p}$. As $\mathfrak{p}$ is prime, $c = c_1 \cdots c_n \notin \mathfrak{p}$. But as $c \in \mathfrak{r}$, c is a nilpotent element not belonging to the prime ideal $\mathfrak{p}$, a contradiction. Thus A is an artinian ring. By [10, Theorem 8], (5) holds.

We shall say that a local noetherian ring A is *aligned* if the set of prime ideals of A, ordered by inclusion, is totally ordered. Thus a one-dimensional aligned local noetherian ring has precisely two proper prime ideals, one contained in the other.

THEOREM 3. *Let A be the total quotient ring of a one-dimensional Macaulay ring B. Then A is the direct product of ideals $A_1, \ldots, A_n$, where each A_k is the total quotient ring of a one-dimensional aligned Macaulay ring B_k. Moreover, A, equipped with the B-topology, is the topological direct product of $A_1, \ldots, A_n$, where each A_k is equipped with the B_k-topology.*

Proof. By Theorem 2, A is artinian, and hence A is the direct product of local artinian rings $A_1, \ldots, A_n$ [14, Theorem 3, p. 205]. For each $k \in [1, n]$, let f_k be the projection on A_k along $\sum_{j \neq k} A_j$, and let $B_k = f_k(B)$. Let S be the multiplicative semigroup of cancellable elements of B. If $s \in S$, then s is invertible in A, so $f_k(s)$ is invertible in A_k and hence cancellable in B_k. Thus $f_k(S)$ is contained in the set of cancellable elements of B_k. Clearly every member of A_k is the quotient of an element of B_k and one of $f_k(S)$. Thus A_k is contained in the total quotient ring of B_k. Hence if y is a cancellable element of B_k, then y is also a cancellable element of A_k, and consequently y is invertible in A_k, for as A_k is a local artinian ring, every element of A_k is either invertible or nilpotent. Therefore A_k is the total quotient ring of B_k.

Let $\mathfrak{m}$ be the maximal ideal of B. Then as B_k is a homomorphic image of B, B_k is a local noetherian ring whose maximal ideal is $f_k(\mathfrak{m})$, and $\dim B_k \leqq \dim B = 1$. But if x is a cancellable element of B belonging to $\mathfrak{m}$, then $f_k(x)$ is a cancellable element of B_k belonging to $f_k(\mathfrak{m})$, as we saw above. Thus $\dim B_k \neq 0$, as a zero-dimensional local noetherian ring has a nilpotent maximal ideal. Consequently, B_k is a one-dimensional Macaulay ring. If $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are the distinct associated prime ideals of the zero ideal of B_k, then $A_k\mathfrak{p}_1, \dots, A_k\mathfrak{p}_r$ are the distinct proper prime ideals of A_k by Theorem 2, since $A_k\mathfrak{p}_j \cap B_k = \mathfrak{p}_j$; hence $r = 1$ as A_k is local, so B_k is a one-dimensional aligned Macaulay ring.

For each $k \in [1, n]$, let $\mathcal{T}_k$ be the topology induced on A_k by the B-topology of A. Let e_k be the identity element of A_k. Then the projection f_k is the function $x \to xe_k$, so f is continuous. Hence A, equipped with the B-topology, is the topological direct product of $(A_1, \mathcal{T}_1), \dots, (A_n, \mathcal{T}_n)$. We have left to show that $\mathcal{T}_k$ is identical with the B_k-topology $\mathcal{T}_k'$ of A_k. Let $\mathfrak{m}_k = f_k(\mathfrak{m})$, the maximal ideal of B_k. Clearly $\mathfrak{m}_k^r = \mathfrak{m}^r e_k$ for all $r \geqq 1$. Now

$$\mathfrak{m}^r \subseteq \mathfrak{m}^r e_k + \sum_{j \neq k} A_j,$$

for if $x \in \mathfrak{m}^r$, then $xe_k \in \mathfrak{m}^r e_k$ and $x - xe_k \in \sum_{j \neq k} A_j$; hence $\mathfrak{o} = \mathfrak{m}^r e_k + \sum_{j \neq k} A_j$ is a neighborhood of zero for the B-topology of A, so $f_k(\mathfrak{o}) = \mathfrak{m}^r e_k = \mathfrak{m}_k^r$ is a neighborhood of zero for $\mathcal{T}_k$ as the projection function f_k is an open mapping from A onto A_k, equipped with topology $\mathcal{T}_k$. Hence every $\mathcal{T}_k'$-neighborhood of zero is a $\mathcal{T}_k$-neighborhood of zero. Let $e_k = a_k b_k^{-1}$ where $a_k \in B$, $b_k \in S$. Then

$$(b_k e_k)\mathfrak{m}_k^r = b_k e_k \mathfrak{m}^r e_k = (b_k e_k)\mathfrak{m}^r = a_k \mathfrak{m}^r \subseteq \mathfrak{m}^r,$$

so $(b_k e_k)\mathfrak{m}_k^r \subseteq \mathfrak{m}^r \cap A_k$. Now $b_k e_k = f_k(b_k)$ is cancellable in B_k as we saw earlier, so $(b_k e_k)\mathfrak{m}_k^r$ is a neighborhood of zero for topology $\mathcal{T}_k'$ by (1) of Theorem 2. Thus $\mathfrak{m}^r \cap A_k$ is a $\mathcal{T}_k'$-neighborhood of zero. Hence every $\mathcal{T}_k$-neighborhood of zero is a $\mathcal{T}_k'$-neighborhood of zero. Thus $\mathcal{T}_k = \mathcal{T}_k'$.

2. **Application to locally compact rings.** An element b of a topological ring is *topologically nilpotent* if $\lim_n b^n = 0$. We recall that a local noetherian ring A, equipped with its natural topology, is compact if and only if A is complete and its residue field is finite [10, Theorem 7]. Moreover, if a compact ring A is algebraically a local noetherian ring, then the given topology of A is its natural topology [10, Theorem 4].

THEOREM 4. *Let A be a commutative locally compact ring with identity. The following statements are equivalent*:

1°. *A is totally disconnected and possesses an invertible element that is topologically nilpotent.*

2°. *A is the topological direct product of a sequence $(A_k)_{1 \leqq k \leqq n}$ of locally compact ideals, where each A_k is the topological quotient ring of a compact, one-dimensional, aligned Macaulay ring B_k.*

If A satisfies 1° *and* 2°, *then A is semilocal and its Jacobson radical is nilpotent.*

Proof. Let A be the topological quotient ring of a compact, one-dimensional aligned Macaulay ring B. As B is compact, A is locally compact; as the open additive subgroups of A form a fundamental system of neighborhoods of zero, A is totally disconnected. As B is a one-dimensional Macaulay ring, its maximal ideal contains a cancellable element b; hence b is invertible in A and $\lim_n b^n=0$. Also A has a nilpotent maximal ideal as A is artinian by Theorem 2. Thus if 2° holds, then 1° holds, A is semilocal, and its Jacobson radical is nilpotent.

To show that 1° implies 2°, let A be totally disconnected, and let c be an invertible, topologically nilpotent element of A. By [8, Lemma 4], A contains a compact open subring B'. As $\lim_n c^n=0$, there exists $s \geqq 1$ such that $c^s \in B'$; as c^s is invertible, $x \rightarrow c^{-s}x$ is a topological automorphism of the additive group A. Thus $c^{-s}B'$ is a compact additive group that contains the additive subgroup of A generated by 1. Therefore there is a compact open subring B of A that contains 1 [8, Lemma 5]. As $\lim c^n=0$, $c^m \in B$ for some $m \geqq 1$; as c^m is invertible, we may, by replacing c with c^m if necessary, assume that $c \in B$.

Let $\mathfrak{r}$ be the Jacobson radical of B. By [7, Corollary of Theorem 13], $\mathfrak{r}$ is closed, and by [7, Theorem 16], $B/\mathfrak{r}$ is topologically isomorphic to the cartesian product $\prod F_\alpha$ of a family of finite fields (each equipped with the discrete topology). Clearly $c+\mathfrak{r}$ is a topologically nilpotent element of $B/\mathfrak{r}$. But the zero element is certainly the only topologically nilpotent element of $\prod F_\alpha$, as each F_α is a discrete field. Hence $c \in \mathfrak{r}$. Consequently, as $x \rightarrow cx$ is a homeomorphism from A onto A, Bc is an open ideal of B that is contained in $\mathfrak{r}$. Thus $\mathfrak{r}$ is open, so $B/\mathfrak{r}$ is compact and discrete and hence finite. Consequently, $B/\mathfrak{r}$ is isomorphic to the cartesian product of a finite family of finite fields, so B has only finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. As $\mathfrak{r}$ is a topologically nilpotent ideal [7, Theorem 14], B is suitable for building idempotents [11, Lemma 4]. Hence there exist orthogonal idempotents $e_1, \ldots, e_n$ in B such that $e_1+\cdots+e_n=1$ and, for each $k \in [1, n]$, $e_k \equiv 1 \bmod \mathfrak{m}_k$, $e_j \equiv 0 \bmod \mathfrak{m}_k$ if $j \neq k$ [4, Proposition 5, p. 54]. For each $k \in [1, n]$ let A_k be the ideal Ae_k. As $x \rightarrow xe_k$ is continuous, A is the topological direct product of the ideals $A_1, \ldots, A_n$ [1, Proposition 2, p. 72]. Also $B_k=Be_k$ is a compact open subring of A_k, since $Be_k=Ae_k \cap B$. Clearly $\mathfrak{m}_k$ is the only maximal ideal of B_k, and $c_k=ce_k$ is an invertible element of A_k that belongs to $\mathfrak{m}_k$ and is topologically nilpotent. Thus $x \rightarrow c_k^2x$ is a homeomorphism from A_k onto A_k, so $B_kc_k^2$ is an open ideal of B_k contained in $\mathfrak{m}_k^2$; hence $\mathfrak{m}_k^2$ is open. By [7, Theorem 20], B_k is a local noetherian ring and the topology it inherits from A is its natural topology.

We wish next to show that A_k is the topological quotient ring of B_k. Every element of A_k belongs to the topological quotient ring, for if $a \in A_k$, then $ac_k^n \rightarrow 0$, so $ac_k^m \in B_k$ for some $m \geqq 0$, whence $a=(ac_k^m)c_k^{-m}$ belongs to the topological quotient ring of B_k as $x \rightarrow c_k^m x$ is a homeomorphism from A_k onto A_k. Conversely, let b be a cancellable element of B_k such that $x \rightarrow bx$ is an open mapping from B_k into B_k. As c_k is an invertible, topologically nilpotent element of A_k and as B_k is a compact open subring of A_k, $(B_kc_k^n)_{n \geqq 1}$ is clearly a fundamental system of neighborhoods of zero in A_k. Therefore as $B_k b$ is open, there exists m such that $B_k c_k^m \subseteq B_k b$, whence $c_k^m = zb$ for some $z \in B_k$. As c_k^m is invertible in A_k, so is b; thus $b^{-1} \in A_k$. Consequently, A_k is the topological quotient ring of B_k. Since $c_k \in \mathfrak{m}_k$, c_k^{-1} is an element of A_k not belonging to B_k. Thus by Theorem 1, B_k is a one-dimensional Macaulay ring. The desired conclusion now follows from Theorem 3.

We recall two classical theorems of algebra: the radical of a finite-dimensional algebra over a field is nilpotent (as the radical of an algebra is an algebra ideal [4, Theorem 1, p. 18], the classical argument [4, Theorem 1, p. 38] in the context of algebras requires only the descending chain condition on algebra right ideals); a finite-dimensional algebra over a field has only finitely many regular maximal ideals (a consequence of Wedderburn's Theorem on semisimple finite-dimensional algebras).

The following theorem is essentially due to Jacobson and Taussky [5, Theorem 3].

THEOREM 5. *Let A be a connected locally compact ring. Then A contains a compact ideal $\mathfrak{h}$ such that $A\mathfrak{h} = (0)$ and $A/\mathfrak{h}$ is a finite-dimensional topological algebra over the field $\boldsymbol{R}$ of real numbers. Moreover, A contains only finitely many regular maximal ideals.*

Proof. By the Pontrjagin-van Kampen Theorem [13, p. 110], the additive topological group A is the topological direct sum of a subgroup topologically isomorphic to $\boldsymbol{R}^n$ for some $n \geqq 0$ and a compact group $\mathfrak{h}$. By [8, Theorem 1], $A\mathfrak{h} = (0)$, so $\mathfrak{h}$ is a closed ideal of A. Thus $A/\mathfrak{h}$ is a locally compact ring that, considered as an additive topological group, is topologically isomorphic to $\boldsymbol{R}^n$. By [5, Lemma 2], we may regard $A/\mathfrak{h}$ as an n-dimensional topological $\boldsymbol{R}$-algebra. Consequently, $A/\mathfrak{h}$ has only finitely many regular maximal ideals. Every regular maximal ideal of A contains $\mathfrak{h}$ since $\mathfrak{h}$ is a nilpotent ideal. Hence A has only finitely many regular maximal ideals.

A *Cohen algebra* [12] is a local algebra over a field whose maximal ideal has codimension one. The following is our structure theorem for commutative locally compact rings possessing an invertible, topologically nilpotent element.

THEOREM 6. *Let A be a commutative locally compact ring with identity. The following statements are equivalent*:

1°. *A contains an invertible element that is topologically nilpotent.*

2°. *A is semilocal, and none of its maximal ideals is open.*

3°. *A is the topological direct product of a sequence $(A_k)_{1 \leqq k \leqq n}$ of ideals where each A_k is either a locally compact finite-dimensional Cohen algebra over $\boldsymbol{R}$ or the field $\boldsymbol{C}$ of complex numbers or the topological quotient ring of a compact one-dimensional aligned Macaulay ring.*

Proof. 1° implies 2°: If A is connected, then A is a finite-dimensional topological algebra over $\boldsymbol{R}$ by Theorem 5 as A has an identity element; consequently, A is semilocal, and none of its maximal ideals is open (as a topological vector space over $\boldsymbol{R}$ contains no proper open subspaces). We shall assume, therefore, that A is not connected. Let $\mathfrak{c}$ be the connected component of zero in A. Then $A/\mathfrak{c}$ is totally disconnected and has an invertible element that is topologically nilpotent, so by Theorem 4, $A/\mathfrak{c}$ is semilocal. Consequently, A has only finitely many maximal ideals that contain $\mathfrak{c}$. As $\mathfrak{c}$ is an ideal of A, for each regular maximal ideal $\mathfrak{n}$ of $\mathfrak{c}$ there is a unique maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{m} \cap \mathfrak{c} = \mathfrak{n}$ [9, Chapter I, Exercise 6(c)]; conversely, if $\mathfrak{m}$ is a maximal ideal of A that does not contain $\mathfrak{c}$, then $\mathfrak{m} \cap \mathfrak{c}$ is a regular maximal ideal of $\mathfrak{c}$ as $\mathfrak{c}/(\mathfrak{m} \cap \mathfrak{c})$ is canonically isomorphic to $(\mathfrak{c}+\mathfrak{m})/\mathfrak{m} = A/\mathfrak{m}$. Thus $\mathfrak{m} \to \mathfrak{c} \cap \mathfrak{m}$ is a bijection from the set of maximal ideals of A not containing $\mathfrak{c}$ onto the set of regular maximal ideals of $\mathfrak{c}$, a finite set by Theorem 5. Hence A is semilocal. If b is an invertible element of A such that $\lim b^n = 0$, then for no $n \geqq 0$ does b^n belong to a proper ideal of A, as b^n is invertible; hence A has no open maximal ideals.

To show that 2° implies 3°, we shall first show that if A is totally disconnected in addition to satisfying 2°, then A has an invertible element that is topologically nilpotent and the Jacobson radical of A is nilpotent. The argument of [12, Lemma 3] shows that A contains a compact open subring B whose radical R is open. Therefore A is a Q-ring, and hence all the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of A are closed [7, Theorem 2]. By hypothesis, none of them contains an interior point, so as each $\mathfrak{m}_i$ is closed, $\bigcup_{i=1}^n \mathfrak{m}_i$ also contains no interior point. Therefore the complement G of $\bigcup_{i=1}^n \mathfrak{m}_i$ is dense, and in particular, $G \cap R \neq \varnothing$. But each element of G is invertible, and each element of R is topologically nilpotent [7, Theorem 14]. Thus A has an invertible element that is topologically nilpotent, and consequently by Theorem 4 its radical is nilpotent.

Next we shall show that, in general, the radical $\mathfrak{r}$ of A is nilpotent. Let $\mathfrak{c}$ be the connected component of zero in A. Either $A/\mathfrak{c}$ is a zero ring, or $A/\mathfrak{c}$ is a totally disconnected semilocal ring none of whose maximal ideals is open. By the preceding, the radical of $A/\mathfrak{c}$ is nilpotent. But $(\mathfrak{r}+\mathfrak{c})/\mathfrak{c}$ is contained in the radical of $A/\mathfrak{c}$ [4, Proposition 1, p. 10]. Thus $\mathfrak{r}^m \subseteq \mathfrak{c}$ for some $m \geqq 1$. The proof of the latter half of [12, Lemma 6] now establishes that $\mathfrak{r}$ is nilpotent. Consequently by [12, Lemma 2], A is the topological direct product of a sequence $(A_k)_{1 \leqq k \leqq n}$ of ideals where each A_k is a locally compact local ring whose maximal ideal is nilpotent and not open. By [12, Lemma 7], each A_k is either connected or totally disconnected. If A_k is totally disconnected, then A_k has an invertible element that is topologically nilpotent by what we proved above, and hence by Theorem 4, A_k is the topological quotient ring of a compact, one-dimensional, aligned Macaulay ring as A_k is local. If A_k is connected, then A_k is a finite-dimensional topological algebra over $\boldsymbol{R}$ by Theorem 5 (as A has an identity element), so by [12, Lemma 5], A_k is a Cohen algebra over either $\boldsymbol{R}$ or $\boldsymbol{C}$ as these are the only connected locally compact fields.

To show that 3° implies 1°, it suffices by Theorem 4 to show that if A_k is a locally compact Cohen algebra over $\boldsymbol{R}$ or $\boldsymbol{C}$, then A_k possesses an invertible element that is topologically nilpotent. But A_k has an identity element e_k; if λ is any nonzero real number such that $|\lambda| < 1$, then λe_k is invertible and topologically nilpotent. Thus 3° implies 1°.

COROLLARY. *Let A be a locally compact integral domain. Statements* 1° *and* 2° *of Theorem* 6 *are equivalent to the following assertion: A is an indiscrete locally compact field.*

Proof. By Theorem 6, 1° implies that A is a local artinian ring and hence is a field. Also by 1°, A is indiscrete. By 1° and [8, Theorem 8], A is therefore an indiscrete locally compact field.

THEOREM 7. *Let A be a commutative locally compact ring with identity. The following statements are equivalent*:

1°. *A contains an invertible element that is topologically nilpotent, and the additive order of each element of A is either infinite or a square-free integer.*

2°. *A is semilocal, none of its maximal ideals is open, and the additive order of each element of A is either infinite or a square-free integer.*

3°. *A is the topological direct product of finitely many topological algebras over indiscrete locally compact fields.*

4°. *A is the topological direct product of finitely many finite-dimensional Cohen algebras over indiscrete locally compact fields.*

Proof. 3° implies 1°: Let A be the topological direct product of ideals $(A_k)_{1 \leqq k \leqq n}$ where each A_k is a topological algebra over an indiscrete topological field F_k. As A has an identity element, each A_k has an identity element e_k. As the topology of F_k is defined by a proper absolute value [6, Theorem 8], F_k contains a nonzero element λ_k such that $\lambda_k^m \to 0$, whence $(\lambda_k e_k)^m \to 0$. Thus $\sum_{k=1}^n \lambda_k e_k$ is an invertible, topologically nilpotent element of A. Certainly 3° also implies that the additive order of each element of K is either infinite or a square-free integer. By Theorem 6, 1° and 2° are equivalent, and clearly 4° implies 3°.

It remains, therefore, to show that 1° and 2° imply 4°. By Theorem 6 and (5) of Theorem 2, we may assume that A is local and totally disconnected and that $x \to x^{-1}$ is continuous on the set of all invertible elements of A. The characteristic m of A is then either zero or a prime, for otherwise $m = pq$ where p and q are relatively prime; consequently, A would be the direct product of the ideals

$$\{x \in A : px = 0\} \quad \text{and} \quad \{x \in A : qx = 0\},$$

so A would have at least two maximal ideals, a contradiction. As observed in the proof of Theorem 4, there is a compact open subring B of A that contains the identity element of A and an invertible, topologically nilpotent element a. We also observed in the proof of Theorem 6 that by virtue of 2°, the maximal ideal $\mathfrak{m}$ of A is closed (but not open). Thus $A/\mathfrak{m}$ is an indiscrete locally compact field.

Assume first that the characteristic of A is a prime p. Then $a + \mathfrak{m}$ is an invertible, topologically nilpotent element of $A/\mathfrak{m}$ which therefore cannot be algebraic over the finite prime subfield of $A/\mathfrak{m}$. Hence for every nonzero polynomial g over the prime subfield P of A, $g(a) \notin \mathfrak{m}$ and thus $g(a)$ is invertible. Therefore the subring $P[a]$ of A generated by P and a is a principal ideal domain contained in B that admits a quotient field $P(a)$ contained in A, and with its induced topology, $P(a)$ is a topological field since $x \to x^{-1}$ is continuous on the set of invertible elements of A. By applying Correl's Theorem [3, Theorem 3], as in the proof of [12, Theorem 1], and [12, Lemma 5], we conclude that A is a finite-dimensional Cohen algebra over an indiscrete locally compact field.

Assume finally that the characteristic of A is zero. As $\mathfrak{m}$ is nilpotent by Theorem 4, the characteristic of $A/\mathfrak{m}$ is zero, for otherwise there would exist a nonzero integer p belonging to $\mathfrak{m}$, whence $p^n = 0$ for some $n \geqq 0$, a contradiction. An argument similar to the preceding now establishes the result in this case, for A contains an indiscrete topological field Q that is the quotient field of the subring Z of A generated by 1, and $Z \subseteq B$, so Correl's Theorem is applicable as in the proof of [12, Theorem 2].

THEOREM 8. *Let A be a locally compact ring with identity. The following statements are equivalent*:

1°. *The center of A contains an invertible element that is topologically nilpotent, and the additive order of each element of A is either infinite or a square-free integer.*

2°. *A is the topological direct product of finitely many (necessarily finite-dimensional) topological algebras over indiscrete locally compact fields.*

Proof. The first part of the proof of Theorem 7 shows that 2° implies 1°. To show that 1° implies 2°, let C be the center of A. Then C is locally compact and contains the identity element of A, so by Theorem 7, C is the topological direct product of ideals $Ce_1, \ldots, Ce_n$, where $(e_k)_{1 \leq k \leq n}$ is an orthogonal set of idempotents in C whose sum is 1, and where each Ce_k contains an indiscrete locally compact field F_k whose identity element is e_k. But then A is the topological direct product of the ideals $Ae_1, \ldots, Ae_n$, and each Ae_k is a topological algebra over F_k.

REFERENCES

1. N. Bourbaki, *Topologie générale*, Chapters 3–4, 3rd ed., Hermann, Paris, 1960.

2. ———, *Espaces vectoriels topologiques*, Chapters 1–2, Hermann, Paris, 1953.

3. Ellen Correl, *Topologies on quotient fields*, Duke Math. J. **35** (1968), 175–178.

4. Nathan Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.

5. N. Jacobson and O. Taussky, *Locally compact rings*, Proc. Nat. Acad. Sci. U.S.A. **21** (1935), 106–108.

6. Irving Kaplansky, *Topological methods in valuation theory*, Duke Math. J. **14** (1947), 527–541.

7. ———, *Topological rings*, Amer. J. Math. **69** (1947), 153–183.

8. ———, *Locally compact rings*, Amer. J. Math. **70** (1948), 447–459.

9. ———, *The theory of rings*, mimeographed notes, University of Chicago, 1951.

10. Seth Warner, *Compact noetherian rings*, Math. Ann. **141** (1960), 161–170.

11. ———, *Compact rings*, Math. Ann. **145** (1962), 52–63.

12. ———, *Locally compact equicharacteristic semilocal rings*, Duke Math. J. **35** (1968), 179-189.

13. André Weil, *L'intégration dans les groupes topologiques et ses applications*, Hermann, Paris, 1938.

14. Oscar Zariski and Pierre Samuel, *Commutative algebra*, Vol. I, Van Nostrand, Princeton, N. J., 1958.

15. ———, *Commutative algebra*, Vol. II, Van Nostrand, Princeton, N. J., 1960.

DUKE UNIVERSITY,
DURHAM, NORTH CAROLINA