IRREDUCIBLE MATRIX REPRESENTATIONS OF FINITE SEMIGROUPS

BY
GÉRARD LALLEMENT AND MARIO PETRICH

Munn [9] has shown that for a semigroup S satisfying the minimal condition on principal ideals, there is a natural one-to-one correspondence between irreducible representations of S and irreducible representations vanishing at zero of its 0-simple (or simple) principal factors; for the case of S finite, see Ponizovskii [11]. On the other hand, Clifford, [3] and [4], has obtained all representations of a completely 0-simple semigroup as "extensions" of those of its maximal subgroups. Combining their results, one can, in principle, obtain all irreducible representations of a semigroup satisfying the minimal conditions on principal left and right ideals and thus of finite semigroups. However, in constructing the representations of a completely 0-simple semigroup $S=\mathscr{M}^0(G; I, \Lambda; P)$, one has to solve the problem in matrix theory of factoring the block matrix

$$\Omega = [\gamma(p_{\lambda i})-\gamma(p_{\lambda 1}p_{1i})]_{\lambda i}, \qquad i \in I\backslash 1, \quad \lambda \in \Lambda\backslash 1,$$

where γ is an irreducible representation of G (see [5, §5.4]).

The main object of this paper is to show that, when dealing with finite semigroups and irreducible representations, it is possible to avoid the factorization problem and give explicit expressions for these representations. Let S be a finite semigroup and J a regular $\mathscr{J}$-class of S. By M_J denote the Schützenberger representation of S by row-monomial matrices over G^0, where G is the Schützenberger group of J (isomorphic to the maximal subgroups of S contained in J) ([5, §§2.4, 3.5], or [12]). For every $x \in S$, let $\Gamma(x)=\gamma[M_J(x)]$, where γ is a proper irreducible representation of G^0 by matrices over a field Φ, and $\gamma[M_J(x)]$ denotes the matrix over Φ obtained by replacing each entry $g_{\lambda\mu}$ of $M_J(x)$ by $\gamma(g_{\lambda\mu})$. Then Γ is a representation of S by matrices over Φ, and we prove (Theorem 1.7) that Γ has a unique nonnull irreducible constituent Γ^* for which $[\Gamma^*(S)]=[\Gamma^*(J)]$, where $[\Gamma^*(T)]$ denotes the linear closure of $\Gamma^*(T)$ (Γ^* is given by (10)). The importance of this constituent Γ^* lies in the fact that every nonnull irreducible representation of S is equivalent to the constituent Γ^* of some representation Γ relative to a suitable $\mathscr{J}$-class of S. This is an analogue to the well-known result in the theory of group representations: every irreducible representation of a group occurs as a constituent of the regular representation [1, 15.2]; this points to the fact that the direct sum of all Schützenberger representations of a semigroup is a suitable analogue of the right regular representation of a group. The proof depends essentially on an analogous property of finite

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0-simple semigroups (in this case Γ^* is the only nonnull constituent of Γ; Theorem 1.4) and an application of a theorem of Munn ([5, 5.28]). Hence the paper is mainly concerned with finite 0-simple semigroups and the determination of their irreducible representations. The explicit form of these representations (see (12)) also yields the solution for the finite case to the problem (stated on p. 168, [5]) of finding the radical of the contracted algebra $\Phi_0[S]$ of a finite 0-simple semigroup $S=\mathscr{M}^0(G; I, \Lambda; P)$ over a field Φ. Indeed, in the Munn algebra $\mathfrak{B}$ (isomorphic to $\Phi_0[S]$, see [5, p. 162])

$$\text{rad } \mathfrak{B} = \{X \in \mathfrak{B} \mid PXP \in (\text{rad } \Phi[G])_{m \times n}\}.$$

An important question from the point of view of algebras is the determination of $\Phi_0[S]/\text{rad } \Phi_0[S]$. In the case of an algebraically closed field, Theorem 3.6 gives a necessary and sufficient condition on the matrix P of $S=\mathscr{M}^0(G; I, \Lambda; P)$ in order that

$$\Phi_0[S]/\text{rad } \Phi_0[S] \cong (\Phi[G]/\text{rad } \Phi[G])_t$$

where t is the invertibility rank of P (see Definition 3.3). Without any restriction on Φ, we prove that

$$\Phi_0[S]/\text{rad } \Phi_0[S] \cong \Phi[G]/\text{rad } \Phi[G]$$

if and only if all the entries of P are elements of G in the class of e mod rad $\Phi[G]$, where e is the identity of G.

Except for the concepts introduced in the paper, we adhere to the terminology and notation of Clifford and Preston [5]. If $\mathfrak{A}$ is an algebra over a field Φ, $(\mathfrak{A})_{m \times n}$ denotes the $\mathfrak{A}$-module of $m \times n$ matrices over $\mathfrak{A}$; in case $m=n$, we write $(\mathfrak{A})_m$. Throughout the whole paper Φ will denote an arbitrary field unless expressly stated otherwise. If M is a matrix over Φ (or over an algebra $\mathfrak{A}$ with identity) whose entries are not explicitly defined, M_{st} denotes its (s, t)-entry. By $I_{m,n}$ $(m \neq n)$ we denote the $m \times n$ matrix whose entries are $(I_{m,n})_{ij}=1$ if $i=j$ and 0 if $i \neq j$; as usual, I_r denotes the $r \times r$ identity matrix. If M is a matrix over Φ, $\overline{M}$ denotes the linear transformation defined by M relative to a given basis. In order to avoid repetition, an irreducible representation is always assumed to be nonnull. If G is a group, whenever we speak of a representation γ of G^0 it is understood that γ is obtained from a representation γ' of G by setting $\gamma'(0)=0$, the zero matrix. In order to simplify our notation γ and γ' will be denoted by the same letter. As a consequence of this convention, the unit representation of the semigroups considered (under which every element is mapped onto the identity of Φ) will be excluded from our consideration. We suppose also that the reader is familiar with the content of Chapter V (§§5.1–5.4) of [5].

1. **The main results.** Let S be a finite semigroup and J a $\mathscr{J}$-class of S (recall that in a finite semigroup $\mathscr{D}=\mathscr{J}$). Denote by M_J the Schützenberger representation of S defined by J (see [5, §3.5]).

DEFINITION 1.1. Let J be a regular $\mathcal{J}$-class of a finite semigroup S and G be its Schützenberger group. If γ is a representation of G^0 by matrices over a field Φ, define $\Gamma(x)$ for every $x \in S$ by $\Gamma(x)=\gamma[M_J(x)]$, where $\gamma[M_J(x)]$ is the matrix obtained by replacing each entry of $M_J(x)$ by its image under γ. Then Γ is a representation of S by matrices over Φ which we call the *standard representation* defined by J and γ.

If S is a finite 0-simple semigroup, S is isomorphic to a Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ over a finite group G, with finite index sets I, Λ and with a $\Lambda \times I$ sandwich matrix $P=(p_{\lambda i})$ [5, §3.2]. Since we are dealing with representations, we can assume without loss of generality, that S coincides with $\mathcal{M}^0(G; I, \Lambda; P)$, that I and Λ have an element 1 in common and that $p_{11}=e$, the identity of G. Thus, whenever we speak of a finite 0-simple semigroup S, then $S=\mathcal{M}^0(G; I, \Lambda; P)$ where $|I|=m$, $|\Lambda|=n$, I, Λ, P satisfying also the preceding requirements. For such a semigroup, the Schützenberger representation defined by $J=S\backslash 0$ is simply

$$M_J(a; i, \lambda) = P(a; i, \lambda),$$

where $P(a; i, \lambda)$ denotes the ordinary product of the matrix P by $(a; i, \lambda)$ considered as an $I \times \Lambda$ matrix over G^0 having the (i, λ)-entry equal to a and 0 elsewhere ([5, Theorem 3.17]). The standard representation defined by $S\backslash 0$ and γ is then

$$\Gamma(a; i, \lambda) = \gamma[P(a; i, \lambda)].$$

The first two propositions are of independent interest illustrating the nature of the standard representation. Recall that Γ is called proper if

(i) $\Gamma(z)=0$ if S has a zero z;

(ii) Γ is not decomposable into two representations, one of which is null.

PROPOSITION 1.2. *The standard representation of $S=\mathcal{M}^0(G; I, \Lambda; P)$ defined by a proper representation of G is proper.*

Proof. Let γ be a proper representation of G of degree r by matrices over Φ. The corresponding standard representation Γ of S is of degree nr. If V is a vector space of dimension nr over Φ, for every $x \in S$, $\Gamma(x)$ is the matrix of the linear transformation $\bar{\Gamma}(x)$ of V, relative to some fixed basis

$$\{e_{\mu s} \mid \mu = 1, 2, \ldots, n; s = 1, \ldots, r\}$$

of V. To establish the proposition, it is sufficient to show that the subspace W of V generated by the ranges of $\bar{\Gamma}(x)$ $(x \in S)$, coincides with V. We have

$$\Gamma(a; i, \lambda) = \begin{array}{c c} & \begin{array}{ccccccc} & & & \lambda & & & \end{array} \\ \begin{array}{c} \\ \\ \mu \\ \\ \\ \end{array} & \begin{bmatrix} 0 & \cdots & 0 & \gamma(p_{1i}a) & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & \cdots & 0 & \gamma(p_{\mu i}a) & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & \cdots & 0 & \gamma(p_{ni}a) & 0 & \cdots & 0 \end{bmatrix} \end{array}.$$

Recall that $\gamma(g)_{st}$ denotes the (s, t)-entry of $\gamma(g)$ $(1 \leqq s \leqq r,\ 1 \leqq t \leqq r)$; then

$$e_{\mu s}\bar{\Gamma}(a; i, \lambda) = \sum_{t=1}^{t=r} \gamma(p_{\mu i}a)_{st}e_{\lambda t}.$$

In particular for $\mu \in \Lambda$ such that $p_{\mu i} \neq 0$ and $a = p_{\mu i}^{-1}$, we obtain

$$e_{\mu s}\bar{\Gamma}(p_{\mu i}^{-1}; i, \lambda) = \sum_{t=1}^{t=r} \gamma(e)_{st}e_{\lambda t} = e_{\lambda s},$$

since $\gamma(e)$ is the $r \times r$ identity matrix. It follows that for every $\lambda \in \Lambda$ and $s = 1, \ldots, r$, $e_{\lambda s}$ is in the range of $\bar{\Gamma}(p_{\mu i}^{-1}; i, \lambda)$. Therefore $W = V$ and Γ is a proper representation of S.

By [5, Theorem 5.43], every proper representation of S extending a proper representation γ of G is obtained from a factorization of the matrix $\Omega = (\Omega_{\lambda i})$, where $\lambda \in \Lambda^* = \Lambda \backslash 1$, $i \in I^* = I \backslash 1$, and $\Omega_{\lambda i} = \gamma(p_{\lambda i}) - \gamma(p_{\lambda 1}p_{1i})$.

Proposition 1.3. *The standard representation of $S = \mathscr{M}^0(G; I, \Lambda; P)$ defined by a proper representation of G of degree r, corresponds to the factorization $\Omega = I_{r(n-1)}\Omega$.*

Proof. The factorization of Ω is obtained by adapting a basis of V (see the proof of Proposition 1.2) to the range and null-space of $\bar{\Gamma}(e; 1, 1)$ [5, p. 178]. If Γ denotes the standard representation defined by a proper representation γ of G of degree r, we have

$$\Gamma(e; 1, 1) = \gamma[P(e; 1, 1)] = \begin{bmatrix} \gamma(e) & 0 & \cdots & 0 \\ \vdots & & & \\ \gamma(p_{\lambda 1}) & 0 & \cdots & 0 \\ \vdots & & & \\ \gamma(p_{n1}) & 0 & \cdots & 0 \end{bmatrix}.$$

In order to obtain an expression for $\bar{\Gamma}(e; 1, 1)$ of the form

$$\Gamma'(e; 1, 1) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

the change of the basis of V is given by the invertible matrix

$$A = \begin{bmatrix} I_r & 0 & & \cdots & & & & 0 \\ -\gamma(p_{21}) & I_r & 0 & \cdots & & & & 0 \\ \vdots & & & & & & & \vdots \\ -\gamma(p_{\lambda 1}) & 0 & \cdots & 0 & I_r & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ -\gamma(p_{n1}) & 0 & \cdots & & & 0 & & I_r \end{bmatrix}.$$

Note that A^{-1} has the same form as A, with $\gamma(p_{\lambda 1})$ instead of $-\gamma(p_{\lambda 1})$ for every $\lambda \in \Lambda^*$. Setting $\Gamma'(a; i, \lambda) = A\Gamma(a; i, \lambda)A^{-1}$ and following Clifford's theory, we obtain a factorization of Ω of the form $\Omega = QR$ by factoring $\Omega_{\lambda i} = Q_\lambda R_i$; Q_λ and R_i are given by

$$\Gamma'(e; i, 1) = \begin{bmatrix} \gamma(p_{1i}) & 0 \\ R_i & 0 \end{bmatrix} \quad \text{and} \quad \Gamma'(e; 1, \lambda) = \begin{bmatrix} \gamma(p_{\lambda 1}) & Q_\lambda \\ 0 & 0 \end{bmatrix}$$

(cf. [5, pp. 178–179]). Since

$$\Gamma'(e; i, 1) = A \begin{bmatrix} \gamma(p_{1i}) & 0 & \cdots & 0 \\ \vdots & & & \\ \gamma(p_{\lambda i}) & 0 & \cdots & 0 \\ \vdots & & & \\ \gamma(p_{ni}) & 0 & \cdots & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} \gamma(p_{1i}) & 0 & \cdots & 0 \\ \vdots & & & \\ \gamma(p_{\lambda i}) - \gamma(p_{\lambda 1} p_{1i}) & 0 & \cdots & 0 \\ \vdots & & & \\ \gamma(p_{ni}) - \gamma(p_{n1} p_{1i}) & 0 & \cdots & 0 \end{bmatrix},$$

we have

$$R_i = \begin{bmatrix} \Omega_{2i} \\ \vdots \\ \Omega_{\lambda i} \\ \vdots \\ \Omega_{ni} \end{bmatrix} \quad \text{and} \quad R = [R_2 \quad \cdots \quad R_i \quad \cdots \quad R_m] = \Omega.$$

Similarly, computing $\Gamma'(e; 1, \lambda)$, we obtain $Q_\lambda = [0 \quad \cdots \quad 0 \quad I_r \quad 0 \quad \cdots \quad 0]$ with I_r in position $(1, \lambda)$ and $Q = I_{r(n-1)}$. Note that the width of the factorization $\Omega = I_{r(n-1)}\Omega$ is $r(n-1)$.

The next theorem is crucial for most results of the paper.

THEOREM 1.4. *Let $S = \mathscr{M}^0(G; I, \Lambda; P)$ be a finite 0-simple semigroup. Let Γ be the standard representation of S defined by an irreducible representation γ of G. Then Γ has only one nonnull irreducible constituent. This irreducible constituent defines an irreducible representation of S which extends γ. Conversely, every irreducible representation of S is equivalent to the representation defined by the nonnull irreducible constituent of the standard representation defined by an irreducible representation γ of G.*

We first prove two lemmas in which $S = \mathscr{M}^0(G; I, \Lambda; P)$ and Γ is the standard representation defined by a representation γ of G of rank r. If $P = (p_{\lambda i})$, $\gamma(P)$ denotes the block matrix $[\gamma(p_{\lambda i})]$.

LEMMA 1.5. *There exists a change of basis of V, defined by a matrix A, such that for every $(a; i, \lambda) \in S$,*

$$(1) \qquad A\Gamma(a; i, \lambda)A^{-1} = \begin{bmatrix} \Gamma^*(a; i, \lambda) & \Delta_{12}(a; i, \lambda) \\ 0 & 0 \end{bmatrix}$$

where $\Gamma^*(a; i, \lambda)$ is a $t \times t$ matrix ($t = \operatorname{rank} \gamma(P)$) and $\Delta_{12}(a; i, \lambda)$ is some $t \times (nr - t)$ matrix.

Proof. Note that

$$\Gamma(a; i, \lambda) = \gamma[P(a; i, \lambda)] = \gamma(P)\gamma[(a; i, \lambda)].$$

Let $t = \text{rank}\,\gamma(P)$ and let A be the matrix of a change of basis of the vector space on which $\bar{\gamma}(P)$ acts adapted to the nullspace of $\bar{\gamma}(P)$. Then $A\Gamma(a; i, \lambda)A^{-1} = A\gamma(P)\gamma[(a; i, \lambda)]A^{-1}$, where $A\gamma(P)$ is an $nr \times mr$ matrix having 0 in the $nr-t$ last rows. It follows that $A\Gamma(a; i, \lambda)A^{-1}$ has the form (1) and Γ^* defines a representation of S of degree t.

LEMMA 1.6. $\text{rank}\,\gamma(P) = \text{rank}\,\Omega + r$ *where r is the degree of γ.*

Proof. Suppose that $\gamma(P)$ has rank t. Then $\gamma(P)$ has a set Z of t linearly independent rows. Since $\gamma(p_{11}) = \gamma(e) = I_r$, we may suppose that Z contains the first r rows of $\gamma(P)$. Let T be the subset of $\Lambda \times \{1, \ldots, r\}$ which serves as the index set of rows in Z. We will show that the corresponding rows in Ω are linearly independent. Assume that for scalars $\alpha_{\lambda s}$ we have, for every $i \in I^*$, $k = 1, 2, \ldots, r$,

$$\sum_{(\lambda,s)\in T;\lambda\neq 1} \alpha_{\lambda s}[\gamma(p_{\lambda 1}) - \gamma(p_{\lambda 1}p_{1i})]_{sk} = 0.$$

Then

$$\sum_{(\lambda,s)\in T;\lambda\neq 1} \alpha_{\lambda s}\left\{[\gamma(p_{\lambda i})]_{sk} - \sum_{u=1}^{r} [\gamma(p_{\lambda 1})]_{su}[\gamma(p_{1i})]_{uk}\right\} = 0. \tag{2}$$

Since $\gamma(p_{\lambda 1}) - \gamma(p_{\lambda 1}p_{11}) = 0$ $(p_{11} = e)$, (2) is valid for every $i \in I$. Hence the left-hand side of (2) is a linear combination of the rows of $\gamma(P)$ indexed by the set T. These rows are linearly independent; since the different $\alpha_{\lambda s}$ appear as coefficients in (2), it follows that $\alpha_{\lambda s} = 0$ for every $(\lambda, s) \in T$, $\lambda \neq 1$. Therefore $\text{rank}\,\Omega \geqq t - r$.

Conversely, suppose that Ω has rank t'. Similarly as above, let Z' be a set of t' linearly independent rows of Ω and $T' \subseteq \Lambda^* \times \{1, 2, \ldots, r\}$ be the index set of rows in Z'. Let $T_1 = T' \cup (\{1\} \times \{1, 2, \ldots, r\})$ and suppose that for elements $\alpha_{\lambda s}$ of Φ and for every $i \in I$, $k = 1, 2, \ldots, r$, we have

$$\sum_{(\lambda,s)\in T_1} \alpha_{\lambda s}[\gamma(p_{\lambda i})]_{sk} = 0. \tag{3}$$

In particular for $i = 1$, (3) gives

$$\sum_{(\lambda,s)\in T_1} \alpha_{\lambda s}[\gamma(p_{\lambda 1})]_{su} = 0 \qquad (u = 1, 2, \ldots, r). \tag{4}$$

Multiplying each equality in (4) by $[\gamma(p_{1i})]_{uk}$ and adding, we obtain

$$\sum_{(\lambda,s)\in T_1} \alpha_{\lambda s}\left\{\sum_{u=1}^{r} [\gamma(p_{\lambda 1})]_{su}[\gamma(p_{1i})]_{uk}\right\} = \sum_{(\lambda,s)\in T_1} \alpha_{\lambda s}[\gamma(p_{\lambda 1}p_{1i})]_{sk} = 0.$$

Performing similar operations on (4) for every $k = 1, 2, \ldots, r$, $i \in I$, we get

$$\sum_{(\lambda,s)\in T_1} \alpha_{\lambda s}[\gamma(p_{\lambda 1}p_{1i})]_{sk} = 0 \qquad (k = 1, 2, \ldots, r; i \in I). \tag{5}$$

Subtracting (5) from (3) yields

$$(6) \qquad \sum_{(\lambda,s)\in T_1} \alpha_{\lambda s}[\gamma(p_{\lambda i})-\gamma(p_{\lambda 1}p_{1i})]_{sk} = 0 \qquad (k = 1, 2, \ldots, r;\, i \in I^*).$$

By the definition of T' these equalities imply $\alpha_{\lambda s}=0$ for every $(\lambda, s) \in T'$. Then, for $\lambda=1$ and $i=1$, (5) becomes

$$\sum_{s=1}^{r} \alpha_{1s}[\gamma(e)]_{sk} = 0 \qquad (k = 1, 2, \ldots, r),$$

which implies $\alpha_{1s}=0$ for every $s=1, 2, \ldots, r$. Therefore rank $\gamma(P) \geqq t'+r$. Consequently, if t and t' are the ranks of $\gamma(P)$ and Ω, respectively, then $t=t'+r$.

We are now in a position to prove Theorem 1.4.

Proof of 1.4. By [5, Theorem 5.51] we get all the irreducible representations of S as the basic extensions of irreducible representations of G. If γ is an irreducible representation of G of degree r, and if the extending matrix Ω of γ has rank t', then γ possesses to within equivalence exactly one extension Γ_0 of degree $r+t'$ (Γ_0 is obtained by a basic factorization of Ω; see [5, 5.46]). Let Γ^* be the nonnull constituent of the standard representation Γ of S defined by γ (see formula (1)). By Lemma 1.5, the degree of Γ^* is rank $\gamma(P)$. Moreover,

$$\Gamma(a; 1, 1) = \begin{bmatrix} \gamma(a) & 0 & \cdots & 0 \\ \cdots & & \cdots & \\ \gamma(p_{\lambda 1}a) & 0 & \cdots & 0 \\ \cdots & & \cdots & \\ (\gamma p_{n1}a) & 0 & \cdots & 0 \end{bmatrix};$$

thus by the change of basis indicated in the proof of Proposition 1.3 (matrix A), one obtains

$$A\Gamma(a; 1, 1)A^{-1} = \begin{bmatrix} \gamma(a) & 0 \\ 0 & 0 \end{bmatrix}.$$

Since Γ^* is a nonnull constituent of Γ, its restriction to the $\mathscr{H}$-class H_{11} is equivalent to a representation of the form

$$(a; 1, 1) \to \begin{bmatrix} \gamma(a) & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus Γ^* extends γ in Clifford's sense. It follows from Lemma 1.6 ($t=r+t'$) that Γ_0 (the basic extension of γ) and Γ^* are equivalent. Hence all the irreducible representations of S are obtained as the nonnull constituents of standard representations defined by various irreducible representations of G.

The next theorem describes all irreducible representations of a finite semigroup.

THEOREM 1.7. *Let S be a finite semigroup. Let Γ be the standard representation defined by a regular $\mathcal{J}$-class J of S and an irreducible representation γ by matrices over Φ of the Schützenberger group G of J. Then Γ has a unique nonnull irreducible constituent Γ^* such that $[\Gamma^*(S)]$ coincides with $[\Gamma^*(J)]$ where $[\Gamma^*(T)]$ $(T\subseteq S)$ denotes the linear closure of $\Gamma^*(T)$. Conversely, every irreducible representation of S is equivalent to the constituent Γ^* of a standard representation Γ defined above.*

Proof. First observe that the Schützenberger representation relative to a regular $\mathcal{J}$-class J coincides on J with the Schützenberger representation of the principal factor $Q(J)$ relative to J. Consequently the standard representation restricted to J and the standard representation of $Q(J)$ restricted to J coincide. Since J is a regular $\mathcal{J}$-class of S, we have that $Q(J)\cong\mathcal{M}^0(G; I, \Lambda; P)$ for some sandwich matrix P. Even though there are different choices of P, in what follows, they lead to equivalent representations since any two such matrices P are equivalent (see [5, Corollary 3.12]).

If A is the matrix of a change of basis adapted to the null space of $\gamma(P)$, then for every $x \in S$,

$$A\Gamma(x)A^{-1} = \begin{bmatrix}\Gamma^*(x) & \Delta_{12}(x)\\ \Delta_{21}(x) & \Delta_{22}(x)\end{bmatrix}, \tag{7}$$

where $\Gamma^*(x)$ is a $t\times t$ matrix ($t=\operatorname{rank}\gamma(P)$). We will show that Γ^* has the required properties.

If $x\in J$, by Lemma 1.5, $\Delta_{21}(x)=\Delta_{22}(x)=0$. Extending Γ^* to the principal factor $Q(J)$ by letting $\Gamma^*(0)=0$, Γ^* becomes an irreducible representation of $Q(J)$ by Theorem 1.4. By [5, Lemma 5.32], there exists $e\in\Phi[J]$ such that $\Gamma^*(e)=I_t$. In order to establish that Γ^* is a representation of S, we show that for every $x\in S$, $\Gamma^*(x)=\Gamma^*(\boldsymbol{xe})$ where $x\to\boldsymbol{x}$ is the natural homomorphism of $\Phi_0[S^1JS^1]$ onto $\Phi_0[Q(J)]$. If $z\in\Phi_0[Q(J)]$, then $z=\sum_{b\in J}\beta_b b$; we define $M_J(z)$ by $M_J(z)=\sum_{b\in J}\beta_b M_J(b)$. Hence for $x\in S$, $M_J(\boldsymbol{xe})$ is a matrix over $\Phi[G]$ obtained from the expression for $\boldsymbol{xe}$ as a linear combination of elements of $Q(J)$. Note that $M_J(\boldsymbol{xe})$ is the zero matrix if and only if $\boldsymbol{xe}=0$, the zero of $Q(J)$. Writing $e=\sum_{a\in J}\alpha_a a$ with $\alpha_a\in\Phi$, we obtain $\boldsymbol{xe}=\sum_{a\in J}\alpha_a\boldsymbol{xa}$ and thus

$$M_J(\boldsymbol{xe}) = \sum_{a\in J}\alpha_a M_J(\boldsymbol{xa}).$$

For $a\in J$ and $x\in S$, $\boldsymbol{xa}=0$ in $Q(J)$ implies $M_J(\boldsymbol{xa})=0$ and also $M_J(xa)=0$; if $\boldsymbol{xa}\neq 0$ in $Q(J)$, then $M_J(\boldsymbol{xa})=M_J(xa)$ so that in any case $M_J(xa)=M_J(\boldsymbol{xa})$. Consequently

$$M_J(\boldsymbol{xe}) = \sum_{a\in J}\alpha_a M_J(\boldsymbol{xa}) = \sum_{a\in J}\alpha_a M_J(xa) = M_J(x)\sum_{a\in J}\alpha_a M_J(a) = M_J(x)M_J(e).$$

It follows that

$$\Gamma(\boldsymbol{xe}) = \gamma[M_J(\boldsymbol{xe})] = \gamma[M_J(x)M_J(e)] = \gamma[M_J(x)]\gamma[M_J(e)] = \Gamma(x)\Gamma(e)$$

and $A\Gamma(xe)A^{-1} = A\Gamma(x)A^{-1}A\Gamma(e)A^{-1}$. Thus by (7),

$$\begin{bmatrix} \Gamma^*(xe) & \Delta_{12}(xe) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Gamma^*(x) & \Delta_{12}(x) \\ \Delta_{21}(x) & \Delta_{22}(x) \end{bmatrix} \begin{bmatrix} \Gamma^*(e) & \Delta_{12}(e) \\ 0 & 0 \end{bmatrix}.$$

Since $\Gamma^*(e) = I_t$, this implies for every $x \in S$,

$$\Gamma^*(xe) = \Gamma^*(x). \tag{8}$$

Observing that $\Gamma^*(xe) = \Gamma^*(ex)$, a straightforward computation shows that $\Gamma^*(x)\Gamma^*(y) = \Gamma^*(xy)$ (see [5, p. 173], where a similar computation is performed). Thus Γ^* is a representation of S; it is irreducible since it is irreducible on $Q(J)$. Moreover $[\Gamma^*(S)] = [\Gamma^*(J)]$. For any irreducible constituent Γ_i of Γ distinct from Γ^*, $\Gamma_i(J) = 0$ by Theorem 1.4; hence Γ^* is the unique *nonnull* constituent Γ_i of Γ such that $[\Gamma_i(S)] = [\Gamma_i(J)]$.

To prove the converse, let Γ_0 be an irreducible representation of S. By ([5, Theorem 5.33]), the apex J of Γ_0 is a regular $\mathcal{J}$-class, and there exists an irreducible representation Γ' of the principal factor $Q(J)$ and $e \in \Phi[J]$ such that $\Gamma'(e) = I_t$ (here t is the degree of Γ_0); furthermore for every $x \in S$,

$$\Gamma_0(x) = \Gamma'(xe). \tag{9}$$

Γ' is equivalent to the irreducible constituent of the standard representation of $Q(J)$ defined by an irreducible representation γ of the group G of J. Let Γ_1 be the standard representation of S defined by J and γ. Using the same notation as in the first part of the proof and the results of this part, we have (cf. (8))

$$\Gamma_1^*(x) = \Gamma_1^*(xe). \tag{8_1}$$

Theorem 1.4 implies that Γ_1^* and Γ' are equivalent representations of $\Phi_0[Q(J)]$. In view of (9) and (8_1), it follows that Γ_0 and Γ_1^* are equivalent, which establishes the second assertion.

The preceding proof yields a general formula for an irreducible representation Γ^* of S defined by its apex J and an irreducible representation γ of the group of J. For every $x \in S$,

$$\Gamma^*(x) = I_{t,nr}A\gamma[M_J(x)]A^{-1}I_{nr,t}, \tag{10}$$

where r is the degree of γ, $t = \operatorname{rank} \gamma(P)$ (P is a matrix of $Q(J)$), and A the matrix of a change of basis adapted to the null-space of $\gamma(P)$.

Remark. If S is a finite 0-*simple* semigroup, the standard representation Γ, defined by an irreducible γ, has only one nonnull constituent Γ^*. If S is not 0-simple, then Γ has, in general, nonnull constituents distinct from Γ^*. The following example illustrates this situation. Let $T = \mathcal{M}^0(\{e\}; I, \Lambda; P)$ where

$$P = \begin{bmatrix} e & 0 \\ 0 & e \\ e & e \end{bmatrix}$$

(thus $|I|=2$, $|\Lambda|=3$), and let S be the semigroup T with an identity u adjoined. The standard representation Γ of S defined by $T\backslash 0$ and the unique representation γ of $\{e\}$ ($\gamma(e)=1$) is

$$\Gamma(e;i,\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}(1;i,\lambda); \qquad \Gamma(u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $(1;i,\lambda)$ is the 2×3 matrix whose (i,λ)-entry equals 1 and the others are 0. Γ has two constituents: Γ^* of degree 2 and Δ of degree 1, viz.,

$$\Gamma^*(e;i,\lambda) = (1;i,\lambda)\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}; \qquad \Gamma^*(u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$\Delta(e;i,\lambda) = 0; \qquad \Delta(u) = 1.$$

2. **The radical of the algebra of a finite 0-simple semigroup.** Let $S=\mathscr{M}^0(G;I,\Lambda;P)$ be a finite 0-simple semigroup. For a given irreducible representation γ of G of degree r over Φ, as before we denote by A the $nr\times nr$ matrix defining a change of basis of the vector space on which $\bar{\gamma}(P)$ acts, adapted to the null-space of $\bar{\gamma}(P)$. Let B be the $mr\times mr$ matrix defining a change of basis of the vector space of $\bar{\gamma}(P)$ adapted to the range of $\bar{\gamma}(P)$. Then

$$A\gamma(P)B^{-1} = \begin{bmatrix} U_\gamma & Z \\ 0 & 0 \end{bmatrix}, \tag{11}$$

where U_γ is an invertible $t\times t$ matrix ($t=\operatorname{rank}\gamma(P)$). A specialization of (10) gives the irreducible representation Γ^* of S extending γ, viz., for every $(a;i,\lambda)\in S$,

$$\Gamma^*(a;i,\lambda) = I_{t,nr}A\gamma[P(a;i,\lambda)]A^{-1}I_{nr,t}. \tag{12}$$

By ([5, Lemma 5.17]) the (contracted) algebra $\Phi_0[S]$ is isomorphic to the Munn algebra $\mathfrak{B}=\mathscr{M}(\Phi[G];I,\Lambda;P)$. Recall that $\mathfrak{B}$ is the vector space over Φ of all $m\times n$ matrices over $\Phi[G]$ with multiplication $\circ$ defined by $X\circ Y=XPY$ for every $X, Y\in\mathfrak{B}$.

THEOREM 2.1. *Let $S=\mathscr{M}^0(G;I,\Lambda;P)$ be a finite 0-simple semigroup, and let $\mathfrak{B}$ be the Munn algebra isomorphic to $\Phi_0[S]$. Then*

$$\operatorname{rad}\mathfrak{B} = \{X\in\mathfrak{B} \mid PXP\in(\operatorname{rad}\Phi[G])_{n\times m}\}.$$

Proof. Recall that an element $x=\sum_{(a;i,\lambda)\in S}\alpha_{(a;i,\lambda)}(a;i,\lambda)$ of $\Phi_0[S]$ is in rad $\Phi_0[S]$ if and only if $\Gamma^*(x)=0$ for every irreducible representation Γ^* of $\Phi_0[S]$. By (12), and with the same meaning for γ, r, t, we have

$$\Gamma^*(x) = I_{t,nr}A\Big\{\sum_{(a;i,\lambda)\in S}\alpha_{(a;i,\lambda)}\gamma[P(a;i,\lambda)]\Big\}A^{-1}I_{nr,t}. \tag{13}$$

Setting $g_{i\lambda}=\sum_{a\in G}\alpha_{(a;i,\lambda)}a$, the isomorphism θ of $\Phi_0[S]$ onto $\mathfrak{B}$ is given by $x\theta=[g_{i\lambda}]_{i\in I;\lambda\in\Lambda}=X$. Introducing X in (13) yields

$$\Gamma^*(x) = I_{t,nr}A\gamma(P)\gamma(X)A^{-1}I_{nr,t}.$$

Thus the following conditions are equivalent:

(i) $x\in\text{rad }\Phi_0[S]$;

(ii) for every irreducible representation γ of G with $X=x\theta$,

$$I_{t,nr}A\gamma(P)\gamma(X)A^{-1}I_{nr,t} = 0. \tag{14}$$

Partitioning A^{-1} into two submatrices $A^{-1}=[C \quad D]$ where C is an $nr\times t$ matrix, (14) is equivalent to

$$A\gamma(P)\gamma(X)[C \quad 0] = 0, \tag{15}$$

where $[C \quad 0]$ is an $nr\times nr$ matrix; the equivalence of (14) and (15) follows from the fact that the multiplication of (14) by $I_{nr,t}$ on the left and by $I_{t,nr}$ on the right gives (15); similarly (15) implies (14). We show next that (15) is equivalent to

$$\gamma(PXP) = 0. \tag{16}$$

Multiplying (15) by A^{-1} on the left and by

$$\begin{bmatrix} U_\gamma & Z \\ 0 & 0 \end{bmatrix} B$$

on the right yields

$$\gamma(P)\gamma(X)[C \quad 0]\begin{bmatrix} U_\gamma & Z \\ 0 & 0 \end{bmatrix} B = 0.$$

Further,

$$[C \quad 0]\begin{bmatrix} U_\gamma & Z \\ 0 & 0 \end{bmatrix} = [C \quad D]\begin{bmatrix} U_\gamma & Z \\ 0 & 0 \end{bmatrix} = A^{-1}\begin{bmatrix} U_\gamma & Z \\ 0 & 0 \end{bmatrix},$$

which together with the preceding formula and (11) implies

$$0 = \gamma(P)\gamma(X)A^{-1}\begin{bmatrix} U_\gamma & Z \\ 0 & 0 \end{bmatrix} B = \gamma(P)\gamma(X)\gamma(P) = \gamma(PXP).$$

Thus (15) implies (16). Conversely, (16) implies

$$A\gamma(P)\gamma(X)A^{-1}\begin{bmatrix} U_\gamma & Z \\ 0 & 0 \end{bmatrix} B = 0,$$

which in turn yields

$$A\gamma(P)\gamma(X)A^{-1}\begin{bmatrix} U_\gamma & Z \\ 0 & 0 \end{bmatrix} BB^{-1}\begin{bmatrix} U_\gamma^{-1} & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

and thus (15) holds. From the equivalence of (14) and (16) it follows that

$$\text{rad } \Phi_0[S] \cong \text{rad } \mathfrak{B} = \{X \in \mathfrak{B} \mid PXP \in (\text{rad } \Phi[G])_{n \times m}\}.$$

COROLLARY 2.2.

$$\text{rad } \mathfrak{B} = \{X \in \mathfrak{B} \mid \textit{for all } M, N \in \mathfrak{B}, M \circ X \circ N \in (\text{rad } \Phi[G])_{m \times n}\}.$$

Proof. If $PXP \in (\text{rad } \Phi[G])_{n \times m}$, then for all $M, N \in \mathfrak{B}$,

$$M \circ X \circ N = MPXPN \in (\text{rad } \Phi[G])_{m \times n}$$

since rad $\Phi[G]$ is an ideal of $\Phi[G]$. Conversely, if $PXP \notin (\text{rad } \Phi[G])_{n \times m}$, then $c_{\lambda i} \notin \text{rad } \Phi[G]$ for some entry of PXP. Letting $M=(e; i, \lambda)$, we get the (i, λ)-entry of $MPXPM$ equal to $c_{\lambda i}$, so that $M \circ X \circ M \notin (\text{rad } \Phi[G])_{m \times n}$.

REMARKS. In fact, we have proved more: if $M \circ X \circ M \in (\text{rad } \Phi[G])_{m \times n}$ for all $M \in S$, then $X \in \text{rad } \mathfrak{B}$ (here S is considered as a subset of $\mathfrak{B}$).

So, if char $\Phi=0$, letting $\mathfrak{A}=\Phi_0[S]$ and $\mathfrak{R}=\text{rad } \Phi_0[S]$, then $\mathfrak{A}\mathfrak{R}\mathfrak{A}=0$ and conversely, if for $x \in \mathfrak{A}$, $\mathfrak{A}x\mathfrak{A}=0$, then $x \in \mathfrak{R}$ (Teissier [14] and Munn [5, p. 168]). Furthermore, by the preceding remark $a \in \mathfrak{A}$ is properly nilpotent if and only if $(ax)^2=0$ for every $x \in \mathfrak{A}$.

In the case that S is left simple, $|\Lambda|=n=1$. In $\mathfrak{B}$, if $PX \in (\text{rad } \Phi[G])_{1 \times 1} = \text{rad } \Phi[G]$, then clearly $PXP \in (\text{rad } \Phi[G])_{1 \times m}$. Conversely, $PXP \in (\text{rad } \Phi[G])_{1 \times m}$ implies that each entry of $PXP \in \text{rad } \Phi[G]$. Since $P=[ee \ \cdots \ e]$ each entry of PXP is equal to PX; thus $PX \in \text{rad } \Phi[G]$. Hence, for S left simple

$$\begin{aligned} \text{rad } \mathfrak{B} &= \{X \in \mathfrak{B} \mid PX \in \text{rad } \Phi[G]\} \\ &= \{X \in \mathfrak{B} \mid \text{for all } M \in \mathfrak{B}, M \circ X \in (\text{rad } \Phi[G])_{m \times 1}\}. \end{aligned}$$

In particular, for char $\Phi=0$ (considered by Teissier [14]), $\mathfrak{A}\mathfrak{R}=0$, and conversely $\mathfrak{A}x=0$ implies $x \in \mathfrak{R}$ (i.e., the radical is the right annihilator of $\mathfrak{A}$).

Returning to the general case of a finite 0-simple semigroup, writing for $X \in \mathfrak{B}$, $X=(x_{j\mu})$, the relation $PXP \in (\text{rad } \Phi_0[G])_{n \times m}$ is equivalent to the system

$$\sum_{j,\mu} p_{\lambda j} x_{j\mu} p_{\mu i} \in \text{rad } \Phi[G] \qquad (i \in I, \ \lambda \in \Lambda).$$

Thus if char $\Phi=0$ and $p_{\lambda i}=e$ for every $i \in I$, $\lambda \in \Lambda$, in the Munn algebra the radical is given by the subalgebra of $I \times \Lambda$ matrices over $\Phi[G]$ whose entries $x_{j\mu}$ satisfy $\sum_{j,\mu} x_{j\mu}=0$. Translated into the semigroup algebra, this yields the following corollary, generalizing the result of Teissier [13] concerning finite left simple semigroups.

We call a completely (0)-simple semigroup S a *rectangular group* (*with zero*) if the matrix P of S has all its entries equal to e. Such a semigroup is isomorphic to the Cartesian product $G \times E$ of a group and a rectangular band ($(G \times E)^0$).

COROLLARY 2.3. *If S is a finite rectangular group and if Φ has characteristic* 0, *then*

$$\operatorname{rad} \Phi[S] = \left\{x = \sum_{(a;i,\lambda)} \alpha_{(a;i,\lambda)}(a;\, i,\, \lambda) \,\middle|\, \textit{for every } a \in G,\ \sum_{i,\lambda} \alpha_{(a;i,\lambda)} = 0\right\}.$$

3. **Quotient algebras.** The next problem we consider is the determination of the quotient algebra $\Phi_0[S]/\operatorname{rad}\Phi_0[S]$ where S is a finite 0-simple semigroup. First of all, as a direct corollary of Theorem 2.1, we obtain the result of Munn [8], Ponizovskii [10], concerning the semisimplicity of $\Phi_0[S]$.

THEOREM 3.1. (*Cf.* [5, *Theorem* 5.20].) *Let $S=\mathscr{M}^0(G; I, \Lambda; P)$ be a finite* 0-*simple semigroup and Φ be a field. Then $\Phi_0[S]$ is semisimple if and only if the characteristic of Φ does not divide the order of G and P is nonsingular as a matrix over $\Phi[G]$.*

Proof. We identify $\Phi_0[S]$ with the Munn algebra $\mathfrak{B}$. Assume that $\Phi_0[S]$ is semisimple. Then by Theorem 2.1, $PXP \in (\operatorname{rad}\Phi[G])_{n\times m}$ implies $X=0$. If P is singular, by ([5, Theorem 5.11 and Corollary 5.10]) there exists a nonzero $m\times n$ matrix X over $\Phi[G]$ such that PX or $XP=0$; in either case, $PXP=0$ with $X\neq 0$, a contradiction. Thus P is nonsingular and $m=n$. If $A \in (\operatorname{rad}\Phi[G])_{m\times m}$, then $A=P(P^{-1}AP^{-1})P \in (\operatorname{rad}\Phi[G])_{m\times m}$, so that $P^{-1}AP^{-1} \in \operatorname{rad}\mathfrak{B}$ by Theorem 2.1. The hypothesis then implies $P^{-1}AP^{-1}=0$ and thus $A=0$. Hence $\operatorname{rad}\Phi[G]=0$ (take for A a matrix having a single nonzero entry). By Maschke's theorem the characteristic of Φ does not divide the order of G. The converse follows easily from Theorem 2.1.

REMARK. With the hypothesis of Theorem 3.1, for every irreducible representation γ of G, $\gamma(P)$ is invertible. By (12), it follows immediately that the standard representation defined by γ is irreducible (Γ^* is equivalent to the standard representation). By Theorem 1.4 all the irreducible representations of S are given exactly by the standard representations (see [5, Theorem 5.28]).

We now answer the following question. When is $\Phi_0[S]/\operatorname{rad}\Phi_0[S]$ isomorphic to $\Phi[G]/\operatorname{rad}\Phi[G]$? In his dissertation, Munn has proved that this holds if S is a rectangular group and Φ has characteristic 0 (see [5, p. 168]), the next theorem gives necessary and sufficient conditions.

THEOREM 3.2. *Let $S=\mathscr{M}^0(G; I, \Lambda; P)$ be a finite* 0-*simple semigroup and Φ a field. Then $\Phi_0[S]/\operatorname{rad}\Phi_0[S]$ is isomorphic to $\Phi[G]/\operatorname{rad}\Phi[G]$ if and only if*

$$p_{\lambda i} - e \in \operatorname{rad}\Phi[G] \qquad (\lambda \in \Lambda,\ i \in I)$$

where e is the identity of G.

Proof. Suppose that $\Phi_0[S]/\operatorname{rad}\Phi_0[S]$ and $\Phi[G]/\operatorname{rad}\Phi[G]$ are isomorphic. For X in the Munn algebra $\mathfrak{B}$ of S, $PXP \in (\operatorname{rad}\Phi[G])_{n\times m}$ is equivalent to the system

$$\sum_{\kappa,\nu} p_{\lambda\kappa}x_{\kappa\nu}p_{\nu i} \in \operatorname{rad}\Phi[G] \qquad (\lambda \in \Lambda,\ i \in I). \tag{17}$$

For a fixed couple λ, i let $\mathfrak{B}_{i\lambda}$ denote the subspace of $\mathfrak{B}$ formed by the matrices $X \in \mathfrak{B}$ whose entries $x_{k\nu}$ satisfy (17).

We show next that the codimension over Φ of $\mathfrak{B}_{i\lambda}$ is equal to the dimension over Φ of $\Phi[G]/\text{rad } \Phi[G]$. In case $\Phi[G]$ is semisimple (17) becomes

$$(17_1) \qquad \sum_{k,\nu} p_{\lambda k} x_{k\nu} p_{\nu i} = 0 \qquad (\lambda \in \Lambda,\ i \in I).$$

Expressing every unknown $x_{k\nu}$ of (17_1) as a linear combination over Φ of elements of G, (17_1) yields $o(G)$ (the order of G) independent linear equations over Φ (the unknowns are now the coefficients of the $x_{k\nu}$). Thus

$$\text{codim}_\Phi \mathfrak{B}_{i\lambda} = o(G) = \dim_\Phi (\Phi[G]).$$

In case $\Phi[G]$ is not semisimple, a similar argument applies to the equation

$$\sum_{k,\nu} \bar{p}_{\lambda k} \bar{x}_{k\nu} \bar{p}_{\nu i} = \bar{0}$$

over $\Phi[G]/\text{rad } \Phi[G]$. Indeed, an element $x = \sum_{i=1}^{h} \xi_i g_i \in \Phi[G]$ is in rad $\Phi[G]$ if and only if the ξ_i are solutions of a system of r independent linear equations

$$\sum_{i=1}^{h} a_j^i \xi_i = 0, \qquad j = 1, 2, \ldots, r,$$

with $r = \text{codim}_\Phi \text{ rad } \Phi[G] = \dim_\Phi (\Phi[G]/\text{rad } \Phi[G])$. Writing (17) in the form of a system of linear equations in the coefficients $\eta_i^{k\nu}$ of

$$x_{k\nu} = \sum_{i=1}^{h} \eta_i^{k\nu} g_i,$$

yields a new system with unknowns $\eta_i^{k\nu}$ which is still of rank r. Thus

$$\begin{aligned} \text{codim}_\Phi \mathfrak{B}_{i\lambda} &= \dim_\Phi (\Phi[G]/\text{rad } \Phi[G]) = \dim_\Phi (\Phi_0[S]/\text{rad } \Phi_0[S]) \\ &= \text{codim}_\Phi (\text{rad } \Phi_0[S]). \end{aligned}$$

Returning to (17) whose solutions determine rad $\Phi_0[S]$, by the property just established, rad $\Phi_0[S]$ is completely determined by the solutions of any one of the equations (17). Thus

$$(18) \qquad \sum_{k,\nu} p_{1k} x_{k\nu} p_{\nu 1} \in \text{rad } \Phi[G]$$

is equivalent to each of

$$(19) \qquad \sum_{k,\nu} p_{\lambda k} x_{k\nu} p_{\nu i} \in \text{rad } \Phi[G], \qquad (\lambda \in \Lambda,\ i \in I,\ (\lambda, i) \neq (1, 1)).$$

If $p_{1t} = 0$ for some $t \in I$, then for every $\nu \in \Lambda$, $x_{t\nu}$ does not appear in (18). From the equivalence of (18) and (19), for every $\nu \in \Lambda$, $x_{t\nu}$ cannot appear in (19). But $x_{t\nu}$ certainly appears in (19) relative to the couple (λ, i) such that $p_{\lambda t} \neq 0$ and $p_{\nu i} \neq 0$. Thus $p_{1t} \neq 0$, and a similar proof shows that $p_{\lambda k} \neq 0$ for every $\lambda \in \Lambda$, $k \in I$. S is then a 0-simple semigroup without zero divisors. Assuming that P has been normalized so that for every $\lambda \in \Lambda$, $p_{\lambda 1}=e$, and for every $i \in I$, $p_{1i}=e$, (18) and (19), respectively, become

$$\sum_{k,\nu} x_{k\nu} \in \operatorname{rad} \Phi[G], \tag{20}$$

$$x_{11}+\sum_{(k,\nu)\neq(1,1)} p_{\lambda k}x_{k\nu}p_{\nu i} \in \operatorname{rad} \Phi[G] \qquad (\lambda \in \Lambda,\ i \in I,\ (\lambda, i) \neq (1, 1)). \tag{21}$$

If $p_{\mu t}-e \notin \operatorname{rad} \Phi[G]$ for a certain couple (μ, t), $\mu \neq 1$, $t \neq 1$, the relation

$$\sum_{(k,\nu)\neq(1,1)} (p_{\mu k}x_{k\nu}p_{\nu i}-x_{k\nu}) \in \operatorname{rad} \Phi[G] \qquad (i \in I,\ i \neq 1) \tag{22}$$

obtained by subtracting (20) from (21) written for $\lambda=\mu$, is not satisfied by all $x_{k\nu} \in \Phi[G]$ (i.e., (22) is not an identity), since the coefficient of x_{t1}, which is equal to $p_{\mu t}-e$, is not in rad $\Phi[G]$. Since (20) and (21) are equivalent, (20) and (22) are also. But (20) contains x_{11} and (22) does not, which is a contradiction. Thus $p_{\mu t}-e \in \operatorname{rad} \Phi[G]$ for every $\mu \in \Lambda$, $t \in I$.

The converse part will follow from Theorem 3.6 which deals with a more general situation.

REMARK. In the course of the proof of Theorem 3.2, we have shown that $\dim_\Phi (\Phi[G]/\operatorname{rad} \Phi[G])$ equals the rank of the system of linear equations over Φ obtained by writing (17) in terms of the unknown coefficients $\eta_i^{k\nu}$ of $x_{k\nu}=\sum_{i=1}^{n} \eta_i^{k\nu} g_i$ for an arbitrarily fixed pair k, ν. We will make use of this remark in the proof of Theorem 3.6.

In order to state the next theorem in a convenient form, we introduce new definitions concerning matrices over an algebra $\mathfrak{L}$ with identity e over Φ.

DEFINITION 3.3. Let P be an $n \times m$ matrix over an algebra $\mathfrak{L}$ with identity over Φ. The *invertibility rank* or *i-rank* of P is the largest integer r such that P has an invertible $r \times r$ submatrix M.

In general, the *i*-rank of P is less than or equal to the usual rank of P as defined in [2, p. 166]. Note that even if all the entries of P are different from 0, P may have *i*-rank 0. In the next definition, by a *permutational matrix* we mean a matrix having only one nonzero entry, equal to e, in each row and in each column.

DEFINITION 3.4. Let P be an $n \times m$ matrix over $\mathfrak{L}$ of *i*-rank $t>0$ and let M be a $t \times t$ invertible submatrix of P. Denote by A and B the permutational matrices of degrees n and m, respectively, such that

$$APB^{-1} = \begin{bmatrix} M & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and set

$$Q = B^{-1}\begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix}A.$$

P is called a Φ*-matrix* if

$$PQP-P \in (\text{rad } \mathfrak{L})_{n\times m}. \tag{23}$$

In the case of matrices over a field Φ ($\cong \Phi[\{e\}]$), the i-rank coincides with the usual rank, and it can easily be shown that every nonnull matrix over Φ is a Φ-matrix. In general, to show that P is a Φ-matrix one has to find a submatrix M satisfying the requirements of Definition 3.4. However, it will follow from the proof of the next lemma, that if P is a Φ-matrix, then P has property (23) with Q relative to *any* maximal invertible submatrix M. If i-rank of P equals $\min\{|I|, |\Lambda|\}$ (i.e., P is of maximal i-rank), then P is a Φ-matrix. This can be verified directly but it also follows obviously from the next lemma.

LEMMA 3.5. *Let $P=(p_{\lambda i})$ be an $n\times m$ matrix over an algebra $\mathfrak{L}$ with identity over Φ. Suppose that P has invertibility rank t over $\Phi[G]$. Then the following conditions are equivalent*

(i) *P is a Φ-matrix*;

(ii) *For every irreducible representation γ of $\mathfrak{L}$ by $r\times r$ matrices over Φ, the block matrix $\gamma(P)$ obtained by replacing $p_{\lambda i}$ by $\gamma(p_{\lambda i})$ has rank rt over Φ.*

Proof. Suppose that P is a Φ-matrix. Using the notation introduced in Definition 3.4, let $P'=APB^{-1}$. With

$$Q' = \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

we have

$$P'Q'P'-P' = APQPB^{-1}-APB^{-1} = A(PQP-P)B^{-1}.$$

Since P is a Φ-matrix, (23) implies that $P'Q'P'-P' \in (\text{rad } \mathfrak{L})_{n\times m}$. Consequently for every irreducible representation γ of $\mathfrak{L}$ of degree r

$$\gamma(P'Q'P') = \gamma(P')\gamma(Q')\gamma(P') = \gamma(P').$$

But then

$$\text{rank } \gamma(M) \leqq \text{rank } \gamma(P') \leqq \text{rank } \gamma(Q') = \text{rank } \gamma(M).$$

Thus $\text{rank } \gamma(P')=\text{rank } \gamma(M)=rt$. Since $P'=APB^{-1}$, with A and B invertible, it follows that

$$\text{rank } \gamma(P) = \text{rank } \gamma(P') = rt.$$

Conversely, assume that $\gamma(P)$ has rank rt where r is the degree of the irreducible representation γ. Let M be *any* invertible $t\times t$ submatrix of P. With suitable permutational matrices A and B, we have

$$APB^{-1} = \begin{bmatrix} M & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

For $P'=APB^{-1}$, rank $\gamma(P')=$ rank $\gamma(P)=rt$. Thus $\gamma(M)$ is a maximal invertible submatrix of $\gamma(P')$. Since a matrix over Φ is always a Φ-matrix, we have

$$\gamma(P')I_{mr,rt}[\gamma(M)]^{-1}I_{rt,nr}\gamma(P') = \gamma(P'). \tag{24}$$

With

$$Q' = \begin{bmatrix} M^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

we get

$$\gamma(Q') = I_{mr,rt}[\gamma(M)]^{-1}I_{rt,nr},$$

and (24) implies $\gamma(P'Q'P')=\gamma(P')$ for every irreducible representation γ. Hence $P'Q'P'-P' \in (\text{rad } \mathfrak{L})_{n\times m}$. Letting $Q=B^{-1}Q'A$, it follows easily that $PQP-P \in (\text{rad } \mathfrak{L})_{n\times m}$, i.e., P is a Φ-matrix.

THEOREM 3.6. *Let $S=\mathscr{M}^0(G; I, \Lambda; P)$ be a finite 0-simple semigroup and Φ a field. Let t be the invertibility rank of P over $\Phi[G]$. If P is a Φ-matrix, then*

$$\Phi_0[S]/\text{rad } \Phi_0[S] \cong (\Phi[G]/\text{rad } \Phi[G])_t.$$

The converse holds if Φ is algebraically closed.

Proof. (1) We may assume that P has a $t\times t$ invertible submatrix M contained in the first t rows and columns of P. (If not, replacing P by an equivalent matrix P', we obtain an isomorphic copy of S.) Thus

$$PI_{m,t}M^{-1}I_{t,n}P-P \in (\text{rad } \Phi[G])_t. \tag{25}$$

We define a mapping δ of the Munn algebra $\mathfrak{B}=\mathscr{M}(\Phi[G]; I, \Lambda; P)$ into $(\Phi[G]/\text{rad } \Phi[G])_t$ by

$$X\delta = I_{t,n}PXPI_{m,t}M^{-1}+(\text{rad } \Phi[G])_t.$$

Clearly, δ preserves addition and scalar multiplication. As for multiplication,

$$(X\delta)(Y\delta) = I_{t,n}PXPI_{m,t}M^{-1}I_{t,n}PYPI_{m,t}M^{-1}+\text{rad } (\Phi[G])_t.$$

Every irreducible representation γ of $(\Phi[G]/\text{rad } \Phi[G])_t$ canonically induces an irreducible representation of $(\Phi[G])_t$. Denoting both of them by γ, for every irreducible representation γ of $(\Phi[G]/\text{rad } \Phi[G])_t$, we have

$$\gamma[(X\delta)(Y\delta)] = \gamma(I_{t,n}PX)\gamma(PI_{m,t}M^{-1}I_{t,n}P)\gamma(YPI_{m,t}M^{-1}).$$

Since P is a Φ-matrix, (23) yields $\gamma(PI_{m,t}M^{-1}I_{t,n}P)=\gamma(P)$. It follows that

$$\gamma[(X\delta)(Y\delta)] = \gamma[I_{t,n}P(XPY)PI_{m,t}M^{-1}] = \gamma[(X\circ Y)\delta].$$

Since $(\Phi[G]/\text{rad } \Phi[G])_t$ is semisimple, it follows that $(X\delta)(Y\delta)=(X\circ Y)\delta$ and δ is a homomorphism of $\mathfrak{B}$ into $(\Phi[G]/\text{rad } \Phi[G])_t$. Let $A+(\text{rad } \Phi[G])_t$ be an element of $(\Phi[G]/\text{rad }\Phi[G])_t$ and let $X \in \mathfrak{B}$ be defined by $X = I_{m,t}M^{-1}AI_{t,n}$. A straightforward computation shows that

$$X\delta = A + (\text{rad }\Phi[G])_t,$$

which proves that δ is onto. Finally

$$\text{Ker }\delta = \{X \in \mathfrak{B} \mid I_{t,n}PXPI_{m,t}M^{-1} \in (\text{rad }\Phi[G])_t\}.$$

Thus $X \in \text{Ker }\delta$ implies

$$(PI_{m,t}M^{-1}I_{t,n}P)X(PI_{m,t}M^{-1}I_{t,n}P) \in (\text{rad }\Phi[G])_{n\times m},$$

which by (25) in turn implies $PXP \in (\text{rad }\Phi[G])_{n\times m}$. Conversely, $PXP \in (\text{rad }\Phi[G])_{n\times m}$ implies $I_{t,n}PXPI_{m,t}M^{-1} \in (\text{rad }\Phi[G])_t$. Thus

$$\text{Ker }\delta = \{X \in \mathfrak{B} \mid PXP \in (\text{rad }\Phi[G])_{n\times m}\} = \text{rad }\mathfrak{B}$$

by Theorem 2.1. Therefore

$$\Phi_0[S]/\text{rad }\Phi_0[S] \cong (\Phi[G]/\text{rad }\Phi[G])_t. \tag{26}$$

(2) Suppose that Φ is an algebraically closed field and that (26) holds. We will show that for every irreducible representation γ_σ of G ($\sigma = 1, 2, \ldots, s$) of degree r_σ, the matrix $\gamma_\sigma(P)$ has rank $r_\sigma t$ over Φ (recall that t is the i-rank of P). By Lemma 3.5, it will follow that P is a Φ-matrix. Keeping the same notation

$$\begin{aligned}\text{rad }\mathfrak{B} &= \{X \in \mathfrak{B} \mid PXP \in (\text{rad }\Phi[G])_{n\times m}\}\\ &= \{X \in \mathfrak{B} \mid \gamma_\sigma(PXP) = 0, \sigma = 1, 2, \ldots, s\}.\end{aligned}$$

It has been shown (see Remark after the proof of Theorem 3.2) that

$$\dim_\Phi (\Phi[G]/\text{rad }\Phi[G])$$

is equal to the rank of the linear system obtained from (17) by writing $x_{\kappa\nu} = \sum_{i=1}^{n} \eta_i^{\kappa\nu} g_i$. But (17) is equivalent to the system

$$\gamma_\sigma\Big(\sum_{\kappa,\nu} p_{\lambda\kappa}x_{\kappa\nu}p_{\nu i}\Big) = 0 \qquad (i = 1, 2, \ldots, s, \ \lambda \in \Lambda, \ i \in I).$$

Finally, $\dim_\Phi (\mathfrak{B}/\text{rad }\mathfrak{B})$ is equal to the rank k of the system of linear equations over Φ obtained from

$$\gamma_\sigma(P)\gamma_\sigma(X)\gamma_\sigma(P) = 0 \qquad (\sigma = 1, 2, \ldots, s). \tag{27}$$

(Recall that the unknowns are the coefficients $\eta_i^{\kappa\nu}$ of the entries $x_{\kappa\nu}$ of X.)

If t_σ denotes the rank over Φ of $\gamma_\sigma(P)$, let us show that $k = \sum_{\sigma=1}^{s} t_\sigma^2$. For this purpose we introduce the system

$$\gamma_\sigma(P)Z_\sigma\gamma_\sigma(P) = 0 \qquad (\sigma = 1, 2, \ldots, s), \tag{28}$$

where the unknowns are the entries (in Φ) of Z_σ, $\sigma = 1, 2, \ldots, s$. Every solution of (27) yields a solution of (28) by letting $Z_\sigma = \gamma_\sigma(X)$. Conversely, if Z_σ ($\sigma = 1, 2, \ldots, s$)

is a solution of (27), by Burnside's theorem ([6, 27.4] or [1, 13.1]) for every Z_σ there exists $X_\sigma \in \mathfrak{B}$ such that $\gamma_\sigma(X_\sigma)=Z_\sigma$. By the Frobenius-Schur theorem ([1, 10.10]), there exists $X \in \mathfrak{B}$ such that $\gamma_\sigma(X)=\gamma_\sigma(X_\sigma)=Z_\sigma$ for every $\sigma=1, 2, \ldots, s$. Thus the linear systems arising from (27) and (28) are equivalent. Since rank $\gamma_\sigma(P)=t_\sigma$, the rank of the latter is $\sum_{\sigma=1}^s t_\sigma^2$. Hence $k=\sum_{\sigma=1}^s t_\sigma^2$; moreover, for every $\sigma=1, 2, \ldots, s$, $t_\sigma \geqq r_\sigma t$. On the other hand, by [1, 12.7], noting that $\Phi[G]/\text{rad } \Phi[G]$ has dimension $\sum_{\sigma=1}^s r_\sigma^2$ over Φ, we obtain

$$\dim_\Phi (\Phi[G]/\text{rad } \Phi[G])_t = t^2 \sum_{\sigma=1}^{s} r_\sigma^2.$$

From the hypothesis, $\dim_\Phi (\Phi[G]/\text{rad } \Phi[G])_t = \dim_\Phi (\mathfrak{B}/\text{rad } \mathfrak{B}) = k$, so that

$$k = \sum_{\sigma=1}^{s} t_\sigma^2 = t^2 \sum_{\sigma=1}^{s} r_\sigma^2.$$

This together with $t_\sigma \geqq tr_\sigma$, yields $t_\sigma = tr_\sigma$. Thus, for every $\sigma=1, 2, \ldots, s$, $\gamma_\sigma(P)$ has rank $r_\sigma t$ and P is a Φ-matrix.

In conclusion, we list a few open problems. Theorem 3.2 shows that for the case $t=1$, the converse part of 3.5 holds without the assumption that Φ be algebraically closed. Is it possible to drop this assumption also in the case $t>1$? When the sandwich matrix P of a finite 0-simple S is not a Φ-matrix, what is the structure of $\Phi_0[S]/\text{rad } \Phi_0[S]$? In [7, Theorem 5.20], Hewitt and Zuckerman have characterized the radical of the algebra of any finite semigroup over the field of complex numbers. Can Theorem 2.1 be derived from their result?

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Pennsylvania State University,
University Park, Pennsylvania