GEOMETRIC GROUPS AND ·WHITEHEAD TORSION

BY

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The purpose of this paper is to define "geometric groups" and to relate them to various problems in topology. This relation is exhibited through Conjectures I and II. It will be shown that Conjecture I implies Conjecture II and that Conjecture II implies the topological invariance of Whitehead torsion. Conjecture II is true for 2-complexes, and this implies that if K and N are finite connected complexes, $L \subset K$ is a subcomplex with $\dim L \leq 2$, and $f: K \to N$ is a homeomorphism with $f|K-L$ p.w.l., then f is a simple homotopy equivalence. Another corollary is that if K is a 2-complex contained in a p.w.l. manifold M^n, U^n is a compact p.w.l. submanifold, $K \subset U \subset M$, and U ε-deforms to K, then $K \subset U$ is a simple homotopy equivalence and, thus, if $n \geq 6$, U is a regular neighborhood of K. Finally, for any finitely presented group π with $\mathrm{Wh}(\pi) \neq 0$, $\exists$ an h-cobordism W with $\pi_1(W) = \pi$ which is not topologically trivial.

Geometric groups are related to other problems in topology and some of these are mentioned without proof in the appendix. For example, Conjecture I implies that compact ANRs of finite dimension have the homotopy type of finite complexes. Conjecture II has a noncompact analogue and since the difficulties are local, there is essentially nothing new here (Conjecture II is true for infinite 2-complexes). This noncompact form of Conjecture II implies the following: If $f: R^n \to R^n$ is a homeomorphism $(n \geq 5)$ such that $f \times \mathrm{Id}: R^n \times R^k \to R^n \times R^k$ is stable, then f is stable. The final note of the appendix implies the following: Suppose $(a_{i,j})$ is an infinite matrix with integer entries, and that it and its inverse are band matrices, i.e., bounded about the diagonal. Then $(a_{i,j})$ can be diagonalized by row operations. This is a nongeometric analogue of Conjecture II for the infinite complex R^1.

Notation, definitions, and conjectures. Although the results of this paper hold for CW complexes, their use has been avoided whenever possible to simplify the proofs. Unless otherwise stated, the word "complex" means "simplicial complex". If K is a complex, $K \subset R^n$ means that K is topologically embedded in R^n. This is important because in Lemma 5, K and N are homeomorphic complexes embedded in essentially the same manner. If K and N are combinatorially distinct, they cannot each be simplicially embedded. In general, K will be embedded in R^n simply because R^n is a pleasant metric space in which geometric construction may be performed.

Suppose X is any metric space. If $A \subset B \subset X$ then $d(A)$ will be the diameter of A. If $x, y \in X$, then $d(x, y)$ is the diameter of the set $\{x, y\}$, i.e., the distance from x to y. If $\delta > 0$, $V(A, B, \delta)$ is defined to be $\{x \in B : d(A, x) \leq \delta\}$. If $B = X$ and the meaning is clear, this may be written $V(A, \delta)$. The interior of A, where A is considered as a subspace of B, is written Int (A, B). The complement of A in B is written Cm (A, B) and the closure of A in B is written Cl (A, B). If the set B is clear from context, these will be written Int (A), Cm (A), and Cl (A) resp. The statement that A is a δ-deformation retract of B means $\exists$ a map $h: B \times I \to B$ such that $h(x, 0) = x$, $h(x, 1) \in A$, and $d[h(x, I)] < \delta$ for all $x \in B$ and $h(x, t) = x$ for all $x \in A$, $t \in I$.

DEFINITION OF GEOMETRIC GROUP. Suppose X is any metric space. The statement that G is a geometric group associated with X means that $\exists$ a finite collection $\{p_i\}_{i=1}^{v}$ where each $p_i \in X$ and that G is the free group generated by the symbols p_i (free group always means free abelian group). Even if the point p_i is the same as the point p_j, they nevertheless represent distinct generators of G because, when $i \neq j$, they are different symbols. This is a technicality—the reader should think of the points as being distinct. If r is an integer, $1 \leq r \leq v$, and $g_r \in G$, then $g_{i,r}$, $1 \leq i \leq v$, is the unique integer defined by the equation $g_r = g_{1,r}p_1 + g_{2,r}p_2 + \cdots + g_{v,r}p_v$. If $S \subset X$, then $G(S) =$ the subgroup of G generated by $\{p_i : p_i \in S\}$. The symbol "p_i" is used in two distinct ways and the meaning will be clear from context. For example, $p_i \in G(S)$ means $p_i = 1p_i$ is an element of the group G; $p_i \in S$ means p_i is a point of X. Note that $G = G(X)$ and if $S \subset X$, $G = G(S) \oplus G(\mathrm{Cm}\,(S))$. If none of the points p_i are in S, $G(S) = 0 \in G$. If $S_1, S_2 \subset X$ then $G(S_1) \cap G(S_2) = G(S_1 \cap S_2)$ and $G(S_1) + G(S_2) = G(S_1 \cup S_2)$. If A is a subset of G, the carrier of A is the subset of X defined by Ca $(A) = \{p_j : \exists a \in A$ such that when a is expressed as a linear combination of the p_i, the coefficient of p_j is nonzero$\}$. Thus $A \subset G(\mathrm{Ca}\,(A))$.

DEFINITION OF δ-BASIS. Suppose $\{a_j\}_{j=1}^{v}$ is a basis for G (basis means free abelian basis). If $\delta > 0$, the statement that $\{a_j\}$ is a δ-basis means

(1) $d[\mathrm{Ca}\,(a_j)] < \delta$ for $j = 1, 2, \ldots, v$.

(2) Any $p_i \in G$ is a linear combination of $\{a_j : \mathrm{Ca}(a_j) \subset V(p_i, X, \delta)\}$.

The elements a_j determine a set of integers $a_{i,j}$ by the equation $a_j = a_{1,j}p_1 + \cdots + a_{v,j}p_v$. Thus a basis determines a nonsingular integral matrix $(a_{i,j})$. Conversely, given a nonsingular matrix $(a_{i,j})$ of rank v, there is determined a basis $\{a_j\}$ by the same formula above. The matrix $(a_{i,j})$ determines an isomorphism of G onto itself. This automorphism is also written as $(a_{i,j})$ and is determined by the formula $(a_{i,j})p_t = a_t$.

Suppose T is an automorphism on G. The statement that T is δ-blocked means that $\exists$ a finite collection $S_1, S_2, \ldots, S_r$, of disjoint subsets of X with $G = G(S_1 \cup S_2 \cup \cdots \cup S_r)$, $d(S_i) < \delta$, and $T(G(S_i)) = G(S_i)$ for $i = 1, 2, \ldots, r$. A matrix is said to be δ-blocked if the automorphism it determines is δ-blocked. It should be noted that the transformation determined by a matrix depends upon the ordering of the set $\{p_i\}$, i.e., if this set is reindexed, the same matrix will determine a differ-

ent transformation on G which may no longer be δ-blocked. Therefore when a matrix is said to be δ-blocked, it means with respect to some definite ordered basis $\{p_i\}$. It is obvious that if T is δ-blocked, then $\{T(p_i)\}$ forms a δ-basis. Thus the concept of "δ-blocked" is a strong special form of the concept of "δ-basis". The following proposition is appropriate at this stage.

PROPOSITION 0. *Suppose G is a geometric group associated with X and generated by $\{p_i\}_{i=1}^{v}$.*

(a) *If T is an automorphism on G with Ca $[T(p_i)] \subseteq V(p_i, X, \delta)$ and Ca $[T^{-1}(p_i)]$ $\subseteq V(p_i, X, \delta)$, then each of $\{T(p_i)\}$ and $\{T^{-1}(p_i)\}$ is a (2δ)-basis for G.*

(b) *If $\{a_i\}$ is a δ-basis for G, then $\exists$ a δ-basis $\{b_i\}$ obtained by relabeling the a_i, such that the automorphism T defined by $T(p_i) = b_i$ has the properties Ca $[T(p_i)]$ $\subseteq V(p_i, X, \delta)$, Ca $[T^{-1}(p_i)] \subseteq V(p_i, X, \delta)$, and $\{T^{-1}(p_i)\}$ is a (2δ)-basis for G.*

Proof of (a). Suppose Ca $[T(p_i)]$ and Ca $[T^{-1}(p_i)] \subseteq V(p_i, \delta)$. Then $d[\text{Ca } T(p_i)]$ $< 2\delta$. Write $p_i = d_1 T(p_1) + d_2 T(p_2) + \cdots + d_v T(p_v)$, and thus $T^{-1}(p_i) = d_1 p_1 + d_2 p_2 + \cdots + d_v p_v$. Since Ca $[T^{-1}(p_i)] \subseteq V(p_i, \delta)$, $d_j \neq 0 \Rightarrow p_j \in V(p_i, \delta) \Rightarrow T(p_j) \subseteq V(p_i, 2\delta)$. Therefore p_i is a linear combination of $\{T(p_j) : \text{Ca } [T(p_j)] \subseteq V(p_i, 2\delta)\}$ and this shows that $\{T(p_i)\}$ is a (2δ)-basis. By symmetry, $\{T^{-1}(p_i)\}$ is also a (2δ)-basis.

Proof of (b). Suppose $\{a_i\}$ is a δ-basis and, as always, let $(a_{i,j})$ be the matrix determined by the automorphism $(a_{i,j})p_t = a_t$. Since $(a_{i,j})$ is nonsingular, $\exists$ a matrix $(b_{i,j})$ obtained from $(a_{i,j})$ by interchanging columns, such that the diagonal entries $b_{i,i} \neq 0$ for $i = 1, 2, \ldots, v$ (this elementary fact is assumed). As always, let $b_t \in G$ be defined by $b_t = (b_{i,j})p_t = T(p_t)$. (The set $\{b_t\}$ is simply a relabeling of the set $\{a_t\}$.) Since $p_t \in \text{Ca } (b_t)$ and $d[\text{Ca } (b_t)] < \delta$, it follows that Ca $[T(p_t)] = \text{Ca } (b_t) \subseteq V(p_t, \delta)$.

Since $\{b_t\}$ is a δ-basis, $p_t = d_1 b_1 + d_2 b_2 + \cdots + d_v b_v$ where $d_i \neq 0 \Rightarrow \text{Ca } (b_i) \subseteq V(p_t, \delta)$. $T^{-1}(p_t) = d_1 p_1 + \cdots + d_v p_v$ where $d_i \neq 0 \Rightarrow p_i \in \text{Ca } (b_i) \subseteq V(p_t, \delta)$. This shows Ca $[T^{-1}(p_t)] \subseteq V(p_t, \delta)$. Finally, $\{T^{-1}(p_i)\}$ is a (2δ)-basis from part (a).

DEFINITION OF δ-MAP. If $G = G(X)$ is a geometric group and H is a free abelian group, a δ-map $f: G \to H$ is a homomorphism onto satisfying (1) and (2) below (properties (3) and (4) follow from (2)).

(1) If $S \subseteq X$, $f[G(S)] \cap f[G(\text{Cm } V[S, \delta])] = 0$.

(2) If S_1 and S_2 are sets with $X = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$, then for any $g \in G(S_1)$ and any nonzero integer n with $f(ng) \in f[G(S_2)]$, $\exists \bar{g} \in G[V(S_1, \delta) \cap V(S_2, \delta)]$ with $f(g) = f(\bar{g})$.

(3) If S_1 and S_2 are two sets with $X = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$, then

$$f[G(S_1)] \cap f[G(S_2)] \subseteq f(G[V(S_1, \delta) \cap V(S_2, \delta)]).$$

(4) If $S \subseteq X$ then $\exists$ a subgroup $G' \subseteq G$ such that $G(S) \subseteq G' \subseteq G[V(S, \delta)]$ and $H/f(G')$ has no torsion.

OBSERVATION. Condition (2) implies conditions (3) and (4).

Proof. Suppose (2) is true. Condition (3) follows by setting $n = 1$. To show (4) is true, let $S_2 = S \subseteq X$, $S_1 = \text{Cm } (S_2, X)$, $\pi: H \to H/f[G(S_2)]$ be the projection, and T

be the torsion subgroup of $H/f[G(S_2)]$. Let $A \subset G$ be a finite set such that $\pi f(A) = T$. Since $H = f[G(S_1)] + f[G(S_2)]$ and $\pi f[G(S_2)] = 0$, A may be chosen so that $A \subset G(S_1)$. Now by (2), $\exists B \subset G[V(S_1, \delta) \cap V(S_2, \delta)]$ with $f(B) = f(A)$. The proof is completed by setting $G' =$ the subgroup of G generated by $G(S_2) \cup B$. Then $G(S_2) \subset G' \subset G[V(S_2, \delta)]$, $\pi f(G') = T$ and $H/f(G')$ has no torsion.

The connection between δ-bases and δ-maps is as follows: If $G = G(X)$ is a geometric group, $\{a_i\}$ is a δ-basis of G, A is a subgroup of G generated by some subset of $\{a_i\}$, and $S \subset X$ with $A \subset G(S)$, then the map $\pi: G(S) \to G(S)/A$ is a (2δ)-map (see statements 1, 2, 3 of Theorem 2).

CONJECTURE I_n. If K is a finite complex, dim $K \leq n$, and $\varepsilon > 0$, $\exists \delta > 0$ such that: If G is any geometric group associated with K, H is a free abelian group, and $f: G \to H$ is a δ-map, then $\exists \{g_i\}_{i=1}^r \subset G$ such that

   (1) $\{f(g_i)\}_{i=1}^r$ is a free abelian basis for H,

   (2) $d[\text{Ca}(g_i)] < \varepsilon$ for $i = 1, 2, \ldots, r$,

   (3) If $S \subset K$, then $f[G(S)] \subset$ the subgroup of H generated by $\{f(g_i) : \text{Ca}(g_i) \subset V(S, K, \varepsilon)\}$.

CONJECTURE II_n. If K is a finite complex, dim $K \leq n$, and $\varepsilon > 0$, $\exists \delta > 0$ such that: If G is any geometric group associated with K, generated by $\{p_i\}_{i=1}^v$, and $\{a_i\}_{i=1}^v$ is any δ-basis for G, then $\exists$ a sequence $(\gamma^0), (\gamma^1), \ldots, (\gamma^{n+1})$ of $v \times v$ matrices satisfying

   (i) Each (γ^i) is ε-blocked for $i = 1, 2, \ldots, n+1$, and (γ^0) is obtained from the identity matrix by interchanging columns.

   (ii) $(\gamma^{n+1})(\gamma^n) \cdots (\gamma^1) = (a_{i,j})(\gamma^0)$ where $(a_{i,j})$ is the matrix associated with the basis $\{a_i\}$.

The connection between Conj. I and Conj. II is as follows: If $G = G(K)$ is a geometric group, $\{a_i\}$ is a δ-basis of G, A is a subgroup of G generated by some subset of $\{a_i\}$, and $S \subset K$ with $A \subset G(S)$, then as mentioned previously, the map $\pi: G(S) \to G(S)/A$ is a (2δ)-map. Assuming Conj. I, $\exists \{g_i\}$ such that $\{\pi(g_i)\}$ is a basis for $G(S)/A$. Then $\{\pi(g_i)\}$ together with the basis for A forms a $(2\delta + \delta_1)$-basis for $G(S)$. Thus it is possible to "block off" the basis for G and it is this process which allows the proof that Conj. $I_n \Rightarrow$ Conj. II_{n+1}. Each of these conjectures becomes nontrivial in a hurry—when $n = 1$ for example. However the following proposition is trivial.

PROPOSITION 1. *Conjectures* I_0 *and* II_0 *are true.*

PROOF OF CONJ. I_0. Suppose K is a 0-complex, $K = \{x_1, x_2, \ldots, x_n\}$, and $\varepsilon > 0$. Let $\delta > 0$ such that $d(x_i, x_j) > \delta$ whenever $i \neq j$. Let $G = G(K)$ be generated by $\{p_i\}_{i=1}^v$ and $f: G \to H$ a δ-map. Let $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ where G_i is the subgroup of G generated by $\{p_j : p_j = x_i\}$ (remember that the p_i may not be distinct). Since f is a δ-map, $H = f(G_1) \oplus \cdots \oplus f(G_n)$. Let C_i be a finite subset of G_i such that $f(C_i)$ is a basis for $f(G_i)$. Then $\bigcup_{i \leq i \leq n} C_i$ is the desired subset of G. The proof of Conj. II_0 is equally trivial and is omitted.

Conjecture II implies the topological invariance of Whitehead torsion. The

purpose of this section is to prove Lemma 5 below. The five lemmas presented are obvious and the proofs are elementary (but tedious) exercises in general topology. The first lemma shows how a δ-basis arises from a geometric situation.

LEMMA 1. *Hypothesis.* $K \subset R^{2k+5}$ *is a connected k-complex and* $Y \subset K$ *is a connected subcomplex.* $M \subset R^{2k+5}$ *is a CW complex containing* Y *as a subcomplex and* $M \cap K = Y$. (M, Y) *contains cells only in dimensions* $k+1$ *and* $k+2$: *the* $(k+1)$-*cells are* $(k+1)$-*spheres* $\{S_1, S_2, \ldots, S_v\}$ *attached to* Y *at points* $\{p_1, p_2, \ldots, p_v\}$; *the* $(k+2)$-*cells* $\{D_1, D_2, \ldots, D_v\}$ *are attached by maps* $f_i: \partial D_i \to (Y \cup S_1 \cup S_2 \cup \cdots \cup S_v)$. Y *is a* δ-*deformation retract of* M, $d(S_i) < \delta$, *and* $d(D_i) < \delta$ *for* $i = 1, 2, \ldots, v$. G *is the geometric group associated with* Y *that is generated by the symbols* $\{p_i\}_{i=1}^v$. G *also has a basis* $\{a_i\}_{i=1}^v$ *determined as follows*: *Let* γ_i *be a generator of* $H_{k+1}(S_i \cup Y, Y) \approx Z$ *and* Δ_j *be a generator of* $H_{k+2}(D_j, \partial D_j)$. *Then the image of* Δ_j *under the map*

$$(f_j)_* \, \partial: H_{k+2}(D_j, \partial D_j) \to H_{k+1}(\partial D_j) \to H_{k+1}(Y \cup S_1 \cup S_2 \cup \cdots \cup S_v, Y)$$

is $a_{1,j}\gamma_1 + a_{2,j}\gamma_2 + \cdots + a_{v,j}\gamma_v$. *Then* $a_j \in G$ *is defined by* $a_j = a_{1,j}p_1 + a_{2,j}p_2 + \cdots + a_{v,j}p_v$.

Conclusion. The set $\{a_j\}_{j=1}^v$ *is a* (3δ)-*basis for* G.

Proof. Since $d(D_j) < \delta < 3\delta$, it follows immediately that $d[\text{Ca}\,(a_j)] < 3\delta$. Since S_i deforms to Y in $V(p_i, M, 2\delta)$, it follows that S_i deforms to Y in

$$Y \cup (S_1 \cup S_2 \cup \cdots \cup S_v) \cup [D_j : D_j \subset V(p_i, M, 3\delta)\}.$$

Therefore $p_i \in G$ is generated by a combination of $\{a_j : \text{Ca}\,(a_j) \subset V(p_i, Y, 3\delta)\}$. This completes the proof.

Suppose $f: K \to N$ is a homeomorphism and M_f is its mapping cylinder. Then M_f is homeomorphic to $K \times I$, K is a deformation retract of M_f, and this deformation, when projected onto N, is the identity. If h is a map approximating f, then K is a deformation retract of M_h, and this deformation, when projected onto N, approximates the identity. The next lemma is a relative version of that fact.

LEMMA 2. *Let* $N \subset R^n$ *be a finite complex. If* $\varepsilon > 0$, $\exists \delta > 0$ *such that whenever*

(i) K *is any finite complex and* W *is any finite subcomplex of some subdivision of* K,

(ii) $f: K \to N$ *is a homeomorphism which is p.w.l. on* W, *and*

(iii) $h: K \to N$ *is a p.w.l. map with* $h|W = f|W$ *and*

$$d(h(x), f(x)) < \delta \quad \text{for } x \in K,$$

then $\exists$ *a map* $H: M_h \times I \to M_h$ *such that*

(a) $H(z, 0) = z$ *and* $H(z, 1) \in K$ *for all* $z \in M_h$ *and* $H(x, u) = x$ *for all* $x \in K \subset M_h$ *and* $u \in [0, 1]$,

(b) $d[\pi_N(H[z, I])] < \varepsilon$ *for all* $z \in M_h$ *and* $\pi_N[(x, t), u] = f(x) = h(x)$ *for* $x \in W$ *and* $t, u \in [0, 1]$.

(*The mapping cylinder over W is just* $W \times I$. *Thus* (x, t) *above is in* M_h.)

Proof. Suppose $N \subset R^n$ and $\varepsilon > 0$. Let $0 \subset R^n$ be an open set, $N \subset 0$, and $r : 0 \to N$ a retraction: Let $\delta > 0$ such that if $x, y \in N$ and $d(x, y) < \delta$, and T is the straight line interval joining x to y, then $T \subset 0$ and $d[r(T)] < \varepsilon/3$. This number δ will satisfy the conclusion of the lemma. Let (K, W), f, and h be given satisfying (i), (ii), and (iii). Since $d(f, h) < \delta$, f and h are $\varepsilon/3$-homotopic relative to W, i.e., $\exists g : K \times I \to N \times I$ with

$g(x, 0) = (f(x), 0)$,
$g(x, 1) = (h(x), 1)$,
$d[\pi_N(g[x, I])] < \varepsilon/3$ and
$\pi_1(g(x, t)) = t$ for $x \in K$, $t \in [0, 1]$, and also
$g(x, t) = (f(x), t) = (h(x), t)$ for $x \in W$, $t \in [0, 1]$.

Construct the mapping cylinder $M_g = [(K \times I) \times I \cup_g N \times I]$ and identify M_f with $(K \times 0) \times I \cup N \times 0$, and M_h with $(K \times 1) \times I \cup N \times 1$. Now $(I \times 0) \cup (0 \times I) \cup (I \times 1)$ is a deformation retract of $I \times I$. Any such deformation induces a map $p_1 : M_g \times I \to M_g$ which deforms M_g into

$$[(K \times I \times 0) \cup N \times I] \cup [(K \times 0 \times I) \cup N \times I] \cup [(K \times I \times 1) \cup N \times I]$$
$$= (K \times I \times 0) \cup M_f \cup N \times I = X$$

with $d[\pi_N(p_1(z, I))] < \varepsilon/3$ for all $z \in M_g$. Here π_N represents the natural projection of M_g to $N \times I$ followed by its projection onto N.

Now $N \times I$ deforms to $N \times 0 \subset M_f$, M_f deforms to $(K \times 0 \times 0) \subset (K \times I \times 0)$ because f is a homeomorphism, and $K \times I \times 0$ deforms to $K \times 1 \times 0$ which is the base of M_h. These three deformations performed in sequence yield $p_2 : X \times I \to X$, a deformation of X onto $K \times I \times 0 \subset M_h$ with $\pi_N(z \times I) = \pi_N(z)$ for any $z \in X$.

Define $p : M_h \times I \to M_g$ by

$$p(z, t) = p_1(a, 2t) \qquad\qquad \text{for } 0 \leqq t \leqq \tfrac{1}{2},$$
$$= p_2(p_1(z, 1), 2t - 1) \quad \text{for } \tfrac{1}{2} \leqq t \leqq 1,$$

and note that

$$p(z, 0) = z, \qquad p(z, 1) \in K \times 1 \times 0 \subset M_h \subset M_g,$$
$$d(\pi_N[p(z, I)]) < \varepsilon/3 \quad \text{all } z \in M_h$$

and

$$\pi_N(x, t, u) = x \quad \text{all } (x, t, u) \in W \times I \times I.$$

A retraction $r : M_g \to M_h$ may be defined as follows: For $(x, t, u) \in K \times I \times I$

$$r(x, t, u) = (x, 1, (1 - t) + u) \quad \text{when } (1 - t) + u \leqq 1,$$
$$= (x, 1, 1) \qquad\qquad \text{when } (1 - t) + u \geqq 1.$$

For $(y, t) \in N \times I$, $r(y, t) = (y, 1)$. Note that $(x, 1, 1)$ is identified with $(f(x), 1)$ $\in N \times I$ in $M_h \subset M_g$. Then r satisfies $d(\pi_N(z), \pi_N[r(z)]) < \varepsilon/3$ for $z \in M_g$. Also

$\pi_N[r(z, t, u)] = \pi_N(x, t, u) = h(x) = f(x)$ for $(x, t, u) \in W \times I \times I$. The deformation $H: M_h \times I \to M_h$ desired in the conclusion of the lemma is given by $H = rp$.

Suppose K is a deformation retract of Q. It is a classical theorem of J. H. C. Whitehead that $\exists M \supset K$ such that (M, K) has cells in only two distinct dimensions, M deforms to K, and the Whitehead torsion $\tau(M, K) = \tau(Q, K)$. The next lemma states that if Q δ-deforms to K then M may be chosen to ε-deform to K. This is proved by working through Whitehead's proof and keeping track of distances. Since the proof has no instructional value whatsoever, it is omitted.

LEMMA 3. *Suppose $K \subset R^{2k+5}$ is a finite connected k-complex, $Y \subset K$ is a connected subcomplex, and $\varepsilon > 0$. Then $\exists \delta > 0$ such that if $Q \subset R^{2k+5}$ is any $(k+1)$-complex with $Q \cap K = Y$ which δ-deforms to Y, then $\exists$ a CW complex $M \subset R^{2k+5}$ containing Y as a subcomplex, such that $M \cap K = Y$ and*

(a) *M ε-deforms to Y.*

(b) *(M, Y) has cells only in dimensions $k+1$ and $k+2$ and the $(k+1)$-cells are attached trivially.*

(c) *If σ is a cell of (M, Y) then $d(\varepsilon) < \varepsilon$.*

(d) *The Whitehead torsion $\tau(M \cup K, K) = \tau(Q \cup K, K)$.*

Suppose K is a finite connected complex and $L \subset K$ is a subcomplex. Let $\delta_1 > 0$ such that any set of diameter $< \delta_1$ is contained in the star of some vertex. Suppose G is a geometric group associated with L, generated by $\{p_i\}_{i=1}^v$, and $\{a_i\}_{i=1}^v$ is a δ-basis for G where $0 < \delta < \delta_1$. Then the basis $\{a_i\}$ determines an element of the Whitehead group Wh $(\pi_1(K))$ as follows: Join each p_i to the base point by a path P_i. Join each Ca (a_j) to the base point by a path A_j—this is possible because Ca $(a_j) \subset$ the star of some vertex. For each $p_i \in$ Ca (a_j), $[P_i^{-1}A_j]$ will represent the element of $\pi_1(K)$ determined by the loop $P_i^{-1}A_j$.

The matrix $(a_{ij}[P_i^{-1}A_j])$ determines an element of Wh $(\pi_1(K))$. This element is independent of the ordering of the set $\{p_i\}$, the ordering of the set $\{a_i\}$, the paths P_i and the paths A_j. It will be shown that if dim $L \leq n$ and Conj. II_n is true, then for δ small enough, this element of Wh $(\pi_1(K))$ is 0.

LEMMA 4. *Suppose K is a finite connected complex and L is a subcomplex with dim $L \leq n$. Then if Conjecture II_n is true, $\exists \delta > 0$ such that if G is any geometric group associated with L, and $\{a_i\}$ is any δ-basis for G, then the matrix $(a_{i,j}[P_i^{-1}A_j])$ represents the 0 element in Wh $(\pi_1(K))$.*

Proof. Suppose K is given. The proof is by induction on dim L. Suppose dim $L = 0$. Then let $\delta > 0$ be such that δ is less than the minimum distance between vertices of L. Then for any geometric group G and any δ-basis $\{a_i\}$, each Ca(a_j) = some p_i. Then the path A_j can be chosen equal to the path P_i and therefore the matrix $(a_{i,j}[P_i^{-1}A_j])$ will have only integer entries and can be diagonalized by row and column operations, and will represent the 0 element of Wh $(\pi_1(K))$.

Now suppose by induction that the lemma is true when the dimension of the subcomplex is $\leq n$. Suppose Conjecture II_{n+1} is true and dim $L = n+1$. Let $\varepsilon > 0$ be

such that the conclusion of the lemma holds for the subcomplex L^n, the n-skeleton of L. This means that if G' is any geometric group associated with L^n, then any ε-basis for G' determines 0 in Wh $(\pi_1(K))$.

Let O_1 and O_2 be open subsets of L with $L^n \subset O_1$ and Cl $(O_1) \subset O_2$, and $r: O_2 \to L^n$ a retraction with $d(r(x), x) < \varepsilon/8$ for all $x \in O_2$. Let $\varepsilon_1 > 0$ such that $\varepsilon_1 < \varepsilon/16(n+2)$, $V(L^n, L, 4(n+2)\varepsilon_1) \subset O_1$, and $V(O_1, L, 4(n+2)\varepsilon_1) \subset O_2$.

By assumption, Conjecture II_{n+1} is true. Thus $\exists \delta$, $0 < \delta < \varepsilon_1$, such that the conclusion of Conjecture II is true for the complex L and the positive number ε_1. This is to be the "δ" in the conclusion of this lemma. Let G be a geometric group associated with L, generated by $\{p_i\}_{i=1}^v$ and containing a δ-basis $\{a_i\}_{i=1}^v$. It must be shown that the basis $\{a_i\}$ determines the 0 element of Wh $(\pi_1(K))$.

The paths P_i and A_j are to be selected satisfying the following: If σ is an $(n+1)$-simplex of L, $p_i \in \text{Int } \sigma$, and Ca $(a_j) \subset \text{Int } \sigma$, then $[P_i^{-1} A_j]$ is the unit of $\pi_1(K)$. This is done by joining σ to the base point by a path and using this single path to determine all P_i and A_j above. If $p_i \in L^n$, then P_i is chosen arbitrarily. If Ca (a_j) is not contained in the interior of some $(n+1)$-simplex of L, then A_j is chosen arbitrarily.

Since Conjecture II_{n+1} is true and $\{a_i\}$ is a δ-basis, $(a_{i,j}) = (\alpha^{n+2})(\alpha^{n+1}) \cdots (\alpha^1)$ where each (α^i) is ε_1-blocked. (Conjecture II gives this after $(a_{i,j})$ is altered by interchanging columns. Since interchanging columns does not effect the induced element in Whitehead torsion, we suppose that $(a_{i,j})$ itself satisfies this equation.) Let $S_1, S_2, \ldots, S_r$ be disjoint subsets of L such that $G = G(S_1 \cup S_2 \cup \cdots \cup S_r)$, $(\alpha^1)[G(S_t)] = G(S_t)$, and $d(S_t) < \varepsilon_1$ for $t = 1, 2, \ldots, r$. Define a new matrix $(\bar{\alpha}^1)$ by the transformation formula: $(\bar{\alpha}^1)|G(S_t) = (\alpha^1)|G(S_t)$ when $S_t \subset \text{Cm}(O_1, L)$ and $(\bar{\alpha}^1)|G(S_t)$ = the identity, otherwise. Define similarly $(\bar{\alpha}^2), \ldots, (\bar{\alpha}^{n+2})$, and let $(\bar{\alpha}) = (\bar{\alpha}^{n+2}) \cdot (\bar{\alpha}^{n+1}) \cdots (\bar{\alpha}^1)$. Finally, let $(\beta_{i,j}) = (a_{i,j}[P_i^{-1} A_j])(\bar{\alpha})^{-1}$ and $(b_{i,j}) = (a_{i,j})(\bar{\alpha})^{-1}$.

Note that for any $(n+1)$-simplex σ of L, $(a_{i,j})[G(\sigma \cap \text{Cm}(O_1))] \subset G(\text{Int } \sigma)$. This is true because $V(L^n, L, 4(n+2)\varepsilon_1) \subset O_1$ and $(a_{i,j})$ is the product of $(n+2)$ matrices, each of which is ε_1-blocked. Therefore $p_t \in \text{Cm}(O_1)$ implies $(a_{i,j}[P_i^{-1} A_j])p_t = (a_{i,j})p_t = a_t$, i.e., no fundamental group elements are involved. Since $(\bar{\alpha})^{-1}$ is the identity on $G(O_1)$, $(\bar{\alpha})^{-1}$ does not effect the matrix $(a_{i,j}[P_i^{-1} A_j])$ whenever nontrivial elements of the fundamental group appear.

The basic properties of $(\beta_{i,j})$ are:

(a) $(\beta_{i,j})$ represents the same element of Wh $(\pi_1(K))$ as does $(a_{i,j}[P_i^{-1} A_j])$.

(b) $(\beta_{i,j})$ arises from the basis $\{(b_{i,j})p_t\}$.

Property (a) follows from the fact that $(\bar{\alpha})^{-1}$ has integer entries and is thus 0 in the Whitehead group. Property (b) is obvious from the remarks above. The basic properties of $(b_{i,j})$ are:

(1) Whenever $p_t \in G[\text{Cm}(O_2)]$, $(b_{i,j})p_t = p_t$. (In this case $b_{i,t} = \beta_{i,t} = 0$ when $i \neq t$ and $b_{t,t} = \beta_{t,t} = 1$.)

(2) $(b_{i,j})p_t$ and $(b_{i,j})^{-1}p_t \subset G[V(p_t, L, 4(n+2)\varepsilon_1)] \subset G[V(p_t, L, \varepsilon/4)]$.

(3) $(b_{i,j})[G(O_2)] = G(O_2)$.

Property (1) follows from $(a_{i,j})^{-1}(p_t) = (\bar{a})^{-1}(p_t)$ for each $p_t \in G[\mathrm{Cm}\ (O_2)]$. Property (2) follows from the fact that $(b_{i,j})$ and $(b_{i,j})^{-1}$ are the products of $2(n+2)$ matrices, each of which is ε_1-blocked, and the fact that $\varepsilon_1 < \varepsilon/16(n+2)$. Property (3) follows from (1).

If row i and column j of a matrix are each composed of zeros except for a "one" in the (i, j) place, then this row and column may be removed from the matrix without changing its class in the Whitehead group. Thus if $(\beta'_{i,j})$ is the matrix determined by $(\beta_{i,j})$ restricted to $G(O_2)$, then $(\beta'_{i,j})$ represents the same class in Wh $(\pi_1(K))$ as does $(\beta_{i,j})$. Let $(b'_{i,j})$ be the matrix determined by $(b_{i,j})$ restricted to $G(O_2)$. Then $(b'_{i,j})$ still satisfies (2) and (3) above.

Let G' be the geometric group associated with L^n and generated by the symbols $\{r(p_i) : p_i \in G(O_2)\}$. Now $(b'_{i,j})$, considered as a transformation of G' onto itself, satisfies $(b'_{i,j})[r(p_t)]$ and $(b'_{i,j})^{-1}[r(p_t)] \subset V[r(p_t), L^n, \varepsilon/2]$. This follows from (2) and the fact that $d(r(x), x) < \varepsilon/8$ for $x \in O_2$. Therefore by Proposition 0, part (a), $(b'_{i,j})$ arises from an ε-basis. By induction and the manner in which ε was chosen, it follows that $(b'_{i,j})$ determines the 0 element in Wh $(\pi_1(K))$. It follows from (b) above that $(b'_{i,j})$ determines the element $(\beta'_{i,j})$, and the proof is complete.

The purpose of the four preceding lemmas is to prove Lemma 5 below. Lemma 5 is a relative version of the classical conjecture that homeomorphisms induce simple homotopy equivalences. Here is the idea of the proof: Suppose $f: K \to N$ is a homeomorphism. Let $h: K \to N$ be a p.w.l. map which approximates f. Using a short mapping cylinder (instead of one of length "one") and Lemma 2, the mapping cylinder M_h will deform to its base K through a small deformation. By Lemma 3, it may be assumed that (M_h, K) has cells in only two dimensions. Using Lemma 1, this determines a geometric group associated with K and a δ-basis. It follows from Lemma 4 and the assumption that Conjecture II is true, that this basis induces the 0 element in Wh $(\pi_1(K))$.

LEMMA 5. *Suppose Conjecture* II_n *is true,* K *is a finite connected k-complex, and* L *is a connected subcomplex of K with* dim $L \leqq n$. *Then there exists a subcomplex X of some subdivision of K,* $L \subset$ Int (X, K), *such that if N is any finite complex and $f: K \to N$ is any homeomorphism with $f |$ Cm (Int X, K) p.w.l., then f is a simple homotopy equivalence.*

Proof. Suppose K is topologically embedded in R^{2k+1}. By Lemma 4, there exists $\varepsilon_1 > 0$ such that G is any geometric group associated with L and $\{a_i\}$ is any ε_1-basis for G, then the matrix $(a_{i,j}[P_i^{-1}A_j])$ represents the 0 element in Wh $(\pi_1(K))$. Let K_1 be a subdivision of K, $Y = \mathrm{St}\ (L_1, K_1)$, and $r: Y \to L$ be a retraction. Suppose the subdivision K_1 is fine enough that $d(r(y), y) < \varepsilon_1/4$ for $y \in Y$. Let K_2 be the first derived subdivision of K_1 and $X = \mathrm{St}\ (L_2, K_2)$. This X will satisfy the conclusion of the lemma.

By Lemma 3, there exists $\varepsilon_2 > 0$ such that if Q is any finite $(k+1)$-complex which ε_2-deforms to Y, then there exists a CW complex M, $Y \subset M \subset R^{2k+5}$ such that

(a) M ($\varepsilon_1/6$)-deforms to Y.

(b) (M, Y) has cells only in dimensions $k+1$ and $k+2$.

(c) If σ is a cell of (M, Y), then $d(\sigma) < \varepsilon_1/6$.

(d) The Whitehead torsion $\tau(M \cup K, K) = \tau(Q \cup K, K)$.

Let $W = \mathrm{Cm}\,(\mathrm{Int}\,X, K)$ and suppose N is a finite complex and $f\colon K \to N$ is a homeomorphism such that $f|W$ is a p.w.l. map. It will be shown that f is a simple homotopy equivalence. Let $c > 0$ and define an embedding $\alpha\colon N \to R^{2k+1} \times R$ by $\alpha(x) = (f^{-1}(x), c)$, i.e., N is embedded as a translate of K. For simplicity, we disregard α and consider N already embedded in this manner. One important point is that the homeomorphism

$$f\colon K \subset R^{2k+1} \times R \to N \subset R^{2k+1} \times R$$

is now defined by $f(x, 0) = (x, c)$. For later purposes, we consider K and $N \subset R^{2k+5}$.

Let $h\colon K \to N$ be a p.w.l. map approximating f such that $h|W = f|W$. Using Lemma 2 and choosing c small enough, it is a straightforward construction to embed M_h in R^{2k+5} such that

(1) The base of M_h is K.

(2) The "top" of M_h is N.

(3) M_h contains a subcomplex Q such that $Q \cap K = Y$, M_h collapses to $K \cup Q$, and $Y \cup Q$ ε_2-deforms to Y (Q will be mapping cylinder of $h|Y$).

The number ε_2 was chosen so that there exists a CW complex M, $Y \subset M \subset R^{2k+5}$, which ($\varepsilon_1/6$)-deforms to Y and has cells only in dimensions $k+1$ and $k+2$. By Lemma 1, this determines a geometric group associated with Y and an ($\varepsilon_1/2$)-basis. When retracted onto L, this gives a geometric group associated with L and an ε_1-basis. Using Conjecture II_n, the number ε_1 was chosen so that this basis determines the 0 element in $\mathrm{Wh}\,(\pi_1(K))$. This completes the proof.

Conjecture I_1 and Conjecture II_2 are true. The purpose of this section is to prove Theorems 1 and 2 below.

THEOREM 1. *Conjecture I_1 is true.*

Proof. Suppose K is a 1-dimensional finite simplicial complex and $\varepsilon > 0$. Since each component may be considered independently, suppose that K is connected. Let K_1 be a subdivision of K such that for each vertex α of K_1, $d[\mathrm{St}\,(\alpha, K_1)] < \varepsilon/2$.

Let K_2, K_3, and K_4 be the second, third, and fourth barycentric subdivisions of K_1. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the vertices of K_1 and $\beta_1, \beta_2, \ldots, \beta_m$ be the barycenters of the 1-simplexes of K_1. Define

$$U_i = \mathrm{St}\,(\alpha_i, K_2), \qquad A_j = \mathrm{St}\,(\beta_j, K_4),$$
$$V_i = \mathrm{St}\,(U_i, K_4), \qquad B_j = \mathrm{St}\,(A_j, K_4),$$
$$W_i = \mathrm{St}\,(V_i, K_4), \qquad C_j = \mathrm{St}\,(B_j, K_4),$$
$$X_i = \mathrm{St}\,(W_i, K_4), \qquad D_j = \mathrm{St}\,(C_j, K_4).$$

If $\beta_j \in$ St (α_i, K_1), then $X_i \cap A_j =$ one point. If $\beta_j \notin$ St (α_i, K_1), then $X_i \cap A_j = \varnothing$. Similar statements hold for $W_i \cap B_j$, $V_i \cap C_j$, and $U_i \cap D_j$.

Let $\delta > 0$ be such that if L is any subcomplex of K_4, then $V(L, K, \delta) \subset$ Open St (L, K_4). Suppose $G = G(K)$ is a geometric group associated with K, H is a free abelian group, and $f: G \to H$ is a δ-map. The proof consists in showing that this G, H, and δ-map f satisfy the conclusion of Conjecture I_1.

Let $T_{i,j} = \{g \in G[\text{Int } (W_i) \cap \text{Int } (D_j)]:$ there exists an integer $n \neq 0$ with $f(ng) \in f(G(V_i))\}$. The basic properties of $T_{i,j}$ are:

(1) $T_{i,j} = 0$ when $W_i \cap D_j = \varnothing$. Thus for a fixed j, there are at most two integers i with $T_{i,j} \neq 0$.

(2) The set $T_{i,j}$ is a subgroup of $G[\text{Int } (W_i) \cap \text{Int } (D_j)]$.

(3) $T_{i,j} \supset G[V_i \cap \text{Int } (D_j)]$.

(4) Any $t \in G[\text{Int } (W_i) \cap \text{Int } (D_j)]$ can be written uniquely as $t = t_1 + t_2$ where $t_1 \in G[V_i \cap \text{Int } (D_j)]$ and $t_2 \in G[\text{Int } (W_i) \cap \text{Int } (C_j)]$. If $t \in T_{i,j}$ then t_1 and $t_2 \in T_{i,j}$. It follows from (3) that any $t_1 \in G[V_i \cap \text{Int } (D_j)]$ belongs to $T_{i,j}$. Since $T_{i,j}$ is a group, $t_2 \in T_{i,j}$.

STATEMENT 1. *Suppose* $V_i \cap C_j \neq \varnothing$, $g \in G(\text{Int } (C_j))$, *and* $\exists$ *an integer* $n \neq 0$ *such that* $f(ng) \in f[G(V_i) + T_{i,j}]$. *Then* $f(g) \in f(T_{i,j})$.

Proof. Suppose the hypothesis. Then $\exists v \in G(V_i)$ and $t \in T_{i,j} \ni f(ng) = f(v+t)$. Let $t = t_1 + t_2$ where $t_1 \in G[V_i \cap \text{Int } (D_j)]$ and $t_2 \in G[\text{Int } (W_i) \cap \text{Int } (C_j)]$. Now $t_1 \in G(V_i)$ and by property (4) above, $t_2 \in T_{i,j}$. Thus v and t may be chosen so that $v \in G(V_i)$ and $t \in T_{i,j} \cap G[\text{Int } (W_i) \cap \text{Int } (C_j)]$. Thus $f(v) = f(ng - t)$ where $v \in G(V_i)$ and $(ng - t) \in G(\text{Int } (C_j)) \subset G(\text{Cm } (V_i))$. Therefore by property (2) of a δ-map, $\exists u \in G(\text{Int } (W_i) \cap \text{Int } (D_j))$ with $f(u) = f(ng - t) = f(v)$. Since $v \in G(V_i)$, it follows from the definition of $T_{i,j}$ that $u \in T_{i,j}$. Now $f(ng) = f(u+t)$ where $u+t \in T_{i,j}$. Again using the definition of $T_{i,j}$, $\exists$ an integer $p \neq 0 \ni f(p(u+t)) = f(png) \in f(V_i)$. Since Ca $(g) \subset$ Cm (V_i), it follows from property (2) of a δ-map that $\exists \bar{u} \in G(\text{Int } (W_i) \cap \text{Int } (D_j))$ with $f(\bar{u}) = f(g)$. Since $f(pn\bar{u}) = f(png) \in f(V_i)$, it follows that $\bar{u} \in T_{i,j}$. Thus $f(g) \in f(T_{i,j})$ and Statement 1 is proved.

STATEMENT 2. *Let* j *be an integer,* $1 \leq j \leq m$, *and* a *and* b *be the two integers* $\ni W_a \cap D_j \neq \varnothing$ *and* $W_b \cap D_j \neq \varnothing$. *Then if* $g \in G(C_j)$ *and* $\exists$ *an integer* $n \neq 0 \ni f(ng) \in f[T_{a,j} + G(\text{Cm } [\text{Int } (D_j)]) + T_{b,j}]$ *then* $\exists t_a \in T_{a,j}$ *and* $t_b \in T_{b,j}$ *with* $f(t_a + t_b) = f(g)$.

Proof. Let $f(h) = f(ng)$ with $h = h_1 + h_2 + h_3$ and $g = g_1 + g_2 + g_3$ where

$$g_1 \in G(\text{Int } (W_a) \cap C_j), \quad h_1 \in T_{a,j},$$
$$g_2 \in G(B_j), \quad\quad\quad h_2 \in G(\text{Cm } [\text{Int } (D_j)]),$$
$$g_3 \in G(\text{Int } (W_b) \cap C_j), \quad h_3 \in T_{b,j}.$$

Now $f(ng_2) = f(h - ng_1 - ng_3)$ where $ng_2 \in G(B_j)$ and $(h - gn_1 - ng_3) \in G[\text{Cm } (B_j)]$. Thus by property (2) of a δ-map, $\exists u \in G[C_j \cap X_a]$, $v \in G[C_j \cap X_b]$ with $f(u+v) = f(g_2)$.

Furthermore $f(h_2)=f(ng-h_1-h_3)$ where $h_2 \in G[\mathrm{Cm} \ (\mathrm{Int} \ D_j)]$, $(ng-h_1-h_3)$ $\in G[\mathrm{Int} \ D_j]$. Again by property (2) of a δ-map, $\exists x \in G(V_a)$, $y \in G(V_b)$ with $f(x+y)$ $=f(h_2)$.

Collecting the above results, $f(n[g_1+u+v+g_3])=f(h_1+x+y+h_3)$ or

$$f(n[g_1+u]-[h_1+x]) = f([y+h_3]-n[v+g_3])$$

where $g_1, u, h_1, x \in G(X_a)$ and $y, h_3, v, g_3 \in G(X_b)$. Since the distance from X_a to X_b is greater than δ, it follows from property (1) of δ-map that $f[G(X_a)] \cap f[G(X_b)]=0$ and thus $f(n[g_1+u])=f(h_1+x)$ and $f(n[v+g_3])=f(y+h_3)$. Now $[g_1+u] \in G(C_j)$ and $[h_1+x] \in T_{a,j}+G(V_a)$ and it follows from Statement 1 that $\exists t_a \in T_{a,j} \ni f(t_a)$ $=f(g_1+u)$. Similarly, $\exists t_b \in T_{b,j} \ni f(t_b)=f(v+g_3)$. Therefore

$$f(t_a+t_b) = f(g_1+u+v+g_3) = f(g_1+g_2+g_3) = f(g).$$

This proves Statement 2.

For any given i, $1 \le i \le k$, define G_i' by $G_i'=G(U_i) \oplus [\bigoplus_{(i,j)\in\alpha} T_{i,j}]$ where α $=\{(i,j) : U_i \cap D_j \ne \varnothing\}$. Then $G(U_i) \subset G(V_i) \subset G_i' \subset G(W_i)$ and $G_i' \cap [+_{j \ne i} G_j']=0$. Let $G'=\bigoplus_{i=1}^{k} G_i'$.

STATEMENT 3. *$H/f(G')$ has no torsion.*

Proof. Suppose $g \in G$ and for some integer $n>1$, $f(ng) \in f(G')$. Show $f(g)$ $\in f(G')$. Now $g=h'+g_1+g_2+\cdots+g_m$ where $h' \in \bigoplus_{i=1}^{k} G(V_i) \subset G'$ and $g_j \in G(C_j)$. Thus $f[n(g_1+g_2+\cdots+g_m)] \in f(G')$ and it must be shown that $f(g_1+g_2+\cdots+g_m)$ $\in f(G')$. Since $f(ng) \in f(G')$ and $f(nh') \in f(G')$, $\exists g' \in G'$ with

$$f(ng_1) = f(g'-n(g_2+g_3+\cdots+g_m)).$$

It now follows from Statement 2 that $f(g_1) \in f(G')$. Similarly, $f(g_j) \in f(G')$, and therefore $f(g_1+g_2+\cdots+g_m) \in f(G')$.

STATEMENT 4. *Let $\pi: H \to H/f(G')$ be the projection. Then*

$$H/f(G') = \bigoplus_{j=1}^{m} \pi f[G(C_j)].$$

Proof. It must be shown, for example, that $\pi f[G(C_1)] \cap (+_{j=2}^{m} \pi f[G(C_j)])=0$. Suppose $\pi f(c_1)=\pi f(c_2+c_3+\cdots+c_m)$ where $c_j \in G(C_j)$. This means $\exists g' \in G'$ with $f(c_1)=f(c_2+\cdots+c_m+g')$. Therefore by Statement 2 $f(c_1) \in f(G')$ and $\pi f(c_1)=0$. This proves Statement 4.

Now $f(G_i')$ is a subgroup of the free abelian group H and thus $f(G_i')$ is free abelian. Let F_i be a finite subset of $G_i' \ni f(F_i)$ is a free abelian basis of $f(G_i')$. According to Statement 3, $H/f(G')$ is a free abelian group and according to Statement 4, $H/f(G')=\bigoplus \pi f[G(C_j)]$. Let E_j be a finite subset of $G(C_j) \ni \pi f(E_j)$ is a free basis for $\pi f[G(C_j)]$.

STATEMENT 5. *The set $f[(\bigcup E_j) \cup (\bigcup F_i)]$ is a free basis for H satisfying the conclusion of the theorem.*

Proof. The fact that it is a free basis follows from the exact sequence $0 \to f(G')$ $\to H \to H/f(G') \to 0$ and the fact that $f(\bigcup F_i)$ is a free basis for $f(G')$ and $\pi f(\bigcup E_j)$ is a free basis for $H/f(G')$.

If $g \in (\bigcup E_j)$ or $g \in (\bigcup F_i)$ then $d[\mathrm{Ca}\,(g)] < \varepsilon$ because, for some vertex α_t, $\mathrm{Ca}\,(g) \subset \mathrm{St}\,(\alpha_t, K_1)$ and $d[\mathrm{St}\,(\alpha_t, K_1)] < \varepsilon/2 < \varepsilon$.

It remains to be shown that if $S \subset K$, then $f[G(S)] \subset$ the subgroup of H generated by $f(R)$ where $R = \{c \in (\bigcup E_j) \cup (\bigcup F_i) : \mathrm{Ca}\,(c) \subset V(S, \varepsilon)\}$. Now any $g \in G(S)$ can be written as $g = g_1 + g_2 + \cdots + g_t$ where $g_v \in [\text{some } G(C_j)] \cap G(S)$ or $g_v \in [\text{some } G(\mathrm{Int}\,V_i)] \cap G(S) \subset [\text{some } G'_i] \cap G(S)$. It will be shown that each $g_v \in$ the subgroup of H generated by $f(R)$ and the theorem will follow.

Suppose first that $g_v \in G(\mathrm{Int}\,V_i) \cap G(S)$. Now $S \cap (\mathrm{Int}\,V_i) \neq \varnothing$ (assuming $g_v \neq 0$). Since $d[\mathrm{St}\,(\alpha_i, K_1)] < \varepsilon/2$, it follows that $V(S, \varepsilon) \supset \mathrm{St}\,(\alpha_i, K_1) \supset \mathrm{Ca}\,(G'_i)$ $\supset \mathrm{Ca}\,(F_i)$. Therefore $F_i \subset R$ and the result follows because $f(g_v) \in$ the subgroup generated by $f(F_i)$.

For the other case, suppose $g_v \in G(C_j) \cap G(S)$. Then $\pi f(g_v) = \pi f(n_1 e_1 + \cdots + n_r e_r)$ where each $e_u \in E_j$. Then $f(g_v - [n_1 e_1 + \cdots + n_r e_r]) \in f(G')$. Since $(g_v - [n_1 e_1 + \cdots + n_r e_r]) \in G(C_j)$, it follows from Statement 2 that $\exists t_a \in T_{a,j}$ and $t_b \in T_{b,j} \ni f(t_a + t_b)$ $= f(g_v - [n_1 e_1 + \cdots + n_r e_r])$. Now $f(t_a)$ and $f(t_b)$ are in the subgroups generated by $f(F_a)$ and $f(F_b)$ respectively. Thus $f(g_v)$ is in the subgroup generated by $f(F_a \cup F_b \cup E_j)$. Since $\mathrm{Ca}\,(E_a \cup E_b \cup E_j) \subset V(\mathrm{Ca}\,(g_v), \varepsilon)$, this completes the proof of Statement 5 and the theorem.

THEOREM 2. *Conjecture* $\mathrm{I}_n \Rightarrow$ *Conjecture* II_{n+1} *for* $n = 0, 1, 2, \ldots$.

Sketch of proof. Since the proof is hidden in a mass of details, a crude sketch of the ideas involved is presented. Suppose Conj. I_n and Conj. II_n are true and show Conj. II_{n+1} is true. Consider the simplest case possible—an $(n+1)$-simplex σ with $d(\sigma) < \varepsilon$. Suppose $G(\sigma)$ is a geometric group with a δ-basis $\{a_i\}_{i=1}^v$. Let Bdry $\sigma \subset \Delta_1$ $\subset \Delta_2 \subset \sigma$ where Δ_2 retracts to Bdry σ. Define $A = \{a_i : a_i \in G(\Delta_2)\}$. The basic step in the proof is showing that $\pi : G(\Delta_2) \to G(\Delta_2)/A$ is a δ_1-map (actually Δ_2 must be retracted onto Bdry σ so that the inductive hypothesis will apply—however at this point we simply think of Δ_2 as being n-dimensional). Applying Conj. I_n to the map π yields an extension of A to a δ_2-basis of $G(\Delta_3)$. The transformation (c) determined by this basis is the product of ε-blocked matrices (Conj. II_n) and (c) agrees with $(a_{i,j})$ on $G(\Delta_1)$. This means that if $(a_{i,j}) = (d)(c)$, then (d) is the identity on $G(\Delta_1)$ and thus ε-blocked. In the general case the complex will have $(n+1)$-simplexes $\sigma_1, \sigma_2, \ldots, \sigma_r$. As before, (d) will be the identity on some neighborhood Δ_1 of the n-skeleton and will carry each $G(\sigma_i)$ into itself. Thus (d) will still be ε-blocked.

Proof. The proof is by induction on n. Suppose Conj. I_n is true for some $n \geq 0$. Suppose by induction that Conj. II_n is true and show that Conj. II_{n+1} is true. Suppose that K is a finite simplicial complex with dim $K = n+1$ and that $\varepsilon > 0$. Let K_1 be a subdivision of K with $d[\mathrm{St}\,(\alpha, K_1)] < \varepsilon$ for each vertex α of K_1. Let K_1^n be the

n-skeleton of K_1, 0 an open set, $K_1^n \subset 0 \subset K$ and $r: 0 \to K_1^n$ a retraction. Let $\delta_2 > 0$ be such that Conj. II holds true for the complex K_1^n and the positive number $\varepsilon/2$. Let $\delta_1 > 0$ be such that Conj. I holds true for the complex K_1^n and the positive number $\delta_2/8$. Let Δ_1, Δ_2 be closed sets with $K_1^n \subset \text{Int } \Delta_1$, $\Delta_1 \subset \text{Int } \Delta_2$, and $d(x, r(x)) < \min(\varepsilon/4, \delta_2/16, \delta_1/8)$ for all $x \in \Delta_2$. Let $\delta > 0$ be such that $\delta < \min(\delta_2/4, \delta_1/4)$ and $V(K_1, K, \delta) \subset \text{Int } \Delta_1$, and $V(\Delta_1, K, \delta) \subset \text{Int } \Delta_2$. It will be shown that this δ satisfies the conclusion of Conj. II for the complex K and the positive number ε. Let $\{a_i\}_{i=1}^v$ be any δ-basis for G. Since the conclusion of Conj. II allows relabeling this basis, it will be assumed that $p_i \in \text{Ca}(a_i)$ [see proof of Proposition 0, part (b)].

Let $A \subset G$ be the subgroup generated by $\{a_i : \text{Ca}(a_i) \subset \Delta_2\}$ and $\pi: G \to G/A$ be the projection.

STATEMENT 1. $G(\Delta_2)/A$ *has no torsion and is thus a free abelain group.*

This follows from the fact that G/A has no torsion because A is generated by a subset of a free basis.

STATEMENT 2. *If* $g_1, g_2 \in G(\Delta_2)$ *and* $V(\text{Ca}(g_1), K, \delta) \cap V(\text{Ca}(g_2), K, \delta) = \varnothing$, *then* $\pi(g_1) = \pi(g_2) \Rightarrow \pi(g_1) = 0$.

Proof. Suppose $\pi(g_1) = \pi(g_2)$. Then $g_1 = g_2 + n_1 a_1 + \cdots + n_v a_v$ where $n_i \neq 0 \Rightarrow a_i \in A$. Due to property (2) of the δ-basis $\{a_i\}$, $g_1 = g_{1,1} a_1 + \cdots + g_{v,1} a_v$ where $g_{i,1} \neq 0 \Rightarrow \text{Ca}(a_i) \subset V(\text{Ca}(g_1), K, \delta)$ and $g_2 = g_{1,2} a_1 + \cdots + g_{v,2} a_v$ where $g_{i,2} \neq 0 \Rightarrow \text{Ca}(a_i) \subset V(\text{Ca}(g_2), K, \delta)$. Since the a_i form a free basis and $g_{1,1} a_1 + \cdots + g_{v,1} a_v = g_{1,2} a_1 + \cdots + g_{v,2} a_v + n_1 a_1 + \cdots + n_v a_v$ the coefficients must be the same on each side of the equation. Since $V(\text{Ca}(g_1), K, \delta) \cap V(\text{Ca}(g_2), K, \delta) = \varnothing$, $g_{i,1} = g_{i,2} \Rightarrow g_{i,1} = 0$. Now $a_i \notin A \Rightarrow n_i = 0 \Rightarrow g_{i,1} = g_{i,2} \Rightarrow g_{i,1} = g_{i,2} = 0$. Therefore g_1 and $g_2 \in A$ and $\pi(g_1) = \pi(g_2) = 0$.

STATEMENT 3. *Suppose* $S_1 \cup S_2 = \Delta_2$ *and* $S_1 \cap S_2 = \varnothing$. *If* $g \in G(S_2)$ *and* $\exists$ *an integer* $n \neq 0$ *such that* $\pi(ng) \in \pi[G(S_1)]$, *then* $\exists \bar{g} \in G[V(S_1, \Delta_2, \delta) \cap V(S_2, \Delta_2, \delta)]$ *with* $\pi(\bar{g}) = \pi(g)$.

Proof. Suppose $h \in G(S_1)$ and $\pi(ng) = \pi(h)$. Then $ng = h + n_1 a_1 + \cdots + n_v a_v$ where $n_i \neq 0 \Rightarrow a_i \in A$. Also $g = c_1 a_1 + \cdots + c_v a_v$ where $c_i \neq 0 \Rightarrow \text{Ca}(a_i) \subset V(S_2, K, \delta)$ and $h = d_1 a_1 + \cdots + d_v a_v$ where $d_i \neq 0 \Rightarrow \text{Ca}(a_i) \subset V(S_1, K, \delta)$. Write

$$g = \left(\mathop{+}_{i \in \alpha} c_i a_i\right) + \left(\mathop{+}_{j \in \beta} c_j a_j\right)$$

where

$$\alpha = \{j : 1 \leq i \leq v, \, c_i \neq 0, \text{ and Ca}(a_i) \subset V(S_1, K, \delta)\},$$
$$\beta = \{j : 1 \leq j \leq v, \, c_j \neq 0, \text{ and Ca}(a_j) \nsubseteq V(S_1, K, \delta)\}.$$

From the equation

$$n(c_1 a_1 + \cdots + c_v a_v) = d_1 a_1 + \cdots + d_v a_v + n_1 a_1 + \cdots + n_v a_v$$

and the fact that $Ca\,(a_i) \notin V(S_1, K, \delta) \Rightarrow d_i = 0$, it follows that for each $j \in \beta$, $nc_j = n_j$. Thus for each $j \in \beta$, $n_j \neq 0$ and $a_j \in A$. Therefore $\pi(+_{j \in \beta}\, c_j a_j) = 0$ and $\pi(+_{i \in \alpha}\, c_i a_i) = \pi(g)$. Now write $(+_{i \in \alpha}\, c_i a_i) = \bar{g} + g'$ where

$$Ca\,(\bar{g}) \subset [V(S_1, K, \delta) \cap \Delta_2] = V(S_1, \Delta_2, \delta),$$

$$Ca\,(g') \subset [V(S_1, K, \delta) \cap Cm\,(\Delta_2)] \subset Cm\,(\Delta_2).$$

With this notation, $\pi(g') = \pi(g - \bar{g})$, i.e., $g' = g - \bar{g} + a$ where $a \in A$. But $(g - \bar{g} + a) \in G(\Delta_2)$ and therefore $g' = 0$. This gives $\pi(\bar{g}) = \pi(g)$ where $Ca\,(\bar{g}) \subset V(S_1, \Delta_2, \delta)$ and $Ca\,(\bar{g}) \subset V(Ca\,(g), K, \delta) \subset V(S_2, K, \delta)$. Since $Ca\,(\bar{g}) \subset \Delta_2$, the second inclusion may be written $Ca\,(\bar{g}) \subset V(S_2, \Delta_2, \delta)$. This proves Statement 3.

Now summarizing the proof so far:

STATEMENTS 1, 2, 3. *The function* $\pi: G(\Delta_2) \to G(\Delta_2)/A$ *is a* (2δ)-*map.*

Now consider the geometric group G' associated with the complex K_1^n and generated by the symbols $r(p_i)$ where $p_i \in \Delta_2$. Even though $r(p_i)$ and $r(p_j)$ may represent the same point of K_1^n, they are distinct free generators of G' whenever $i \neq j$. Since p_i is a point of a complex and also an element of a group, the function r can also be considered as a retraction and a group homomorphism $r: G(\Delta_2) \to G'$. This is done by defining $r(n_1 p_1 + \cdots + n_v p_v) = n_1 r(p_1) + n_2 r(p_2) + \cdots + n_v r(p_v)$ whenever $n_1 p_1 + \cdots + n_v p_v \in G(\Delta_2)$. Then r is an isomorphism, $r : G(\Delta_2) \approx G'$ and A' is defined by $A' = r(A)$. The projection map $\pi': G' \to G'/A'$ is given by $\pi' = r\pi r^{-1}$.

STATEMENT 1'. *G'/A' has no torsion and is thus a free abelian group.*

This follows from Statement 1 and the fact that $G(\Delta_2)/A \approx G'/A'$.

STATEMENT 2'. *If $g_1, g_2 \in G'$ and $V(Ca\,(g_1), K_1^n, \delta_1/2) \cap V[Ca\,(g_2), K_1^n, \delta_1/2] = \varnothing$ then $\pi'(g_1) = \pi'(g_2) \Rightarrow \pi'(g_1) = 0$.*

STATEMENT 3'. *Suppose $S_1 \cup S_2 = K_1^n$ and $S_1 \cap S_2 = \varnothing$. If $g \in G'(S_2)$ and $\exists$ an integer $n \neq 0$ such that $\pi(ng) \in \pi[G'(S_1)]$, then $\exists \bar{g} \in G[V(S_1, K_1^n, \delta_1) \cap V(S_2, K_1^n, \delta_1)]$ with $\pi'(\bar{g}) = \pi'(g)$.*

STATEMENTS 1', 2', 3'. *The function* $\pi': G' \to G'/A'$ *is a* δ_1-*map.*

The proofs of Statements 2' and 3' follow immediately from Statements 2 and 3 and the facts that $d(x, r(x)) < \delta_1/8$ and $\delta < \delta_1/4$. In order to simplify the proof and keep the notation under control, we suppose that there is only one $(n+1)$-simplex σ in K_1. The general case is no different except each $(n+1)$-simplex must be handled independently.

STATEMENT 4'. *$\exists$ a collection $\{d_1', d_2', \ldots, d_s'\} \subset G'$ such that:*
(a) $\{\pi'(d_i')\}_{i=1}^s$ *is a free basis for* G'/A'.
(b) $d(Ca\,(d_i')) < \delta_2/8$ *for* $i = 1, 2, \ldots, s$.
(c) *If* $S \subset K_1^n$, *then* $\pi'(G'(S)) \subset$ *the subgroup of H generated by*

$$\{\pi'(d_i') : Ca\,(d_i') \subset V(S, K_1^n, \delta_2/8)\}.$$

The collection $\{d_1', d_2', \ldots, d_s'\} \cup \{r(a_i) : a_i \in A\}$ is a δ_2-basis for G'.

Proof. By assumption, Conjecture I is true for n-dimensional complexes. The number δ_1 was chosen so that the conclusion of Conj. I holds for the positive number $\delta_2/8$, the complex K_1^n, and δ_1-maps from geometric groups associated with K_1^n. Statement 4' is the conclusion of Conj. I applied to the δ_1-map $\pi' : G' \to G'/A'$. The fact that the collection $\{d_1', d_2', \ldots, d_s'\} \cup \{r(a_i) : a_i \in A\}$ is a δ_2-basis for G' follows immediately from (b), (c), the fact that $\{a_i\}$ forms a $(\delta_2/4)$-basis for G, and the fact that $d(r(x), x) < \delta_2/16$ for all $x \in \Delta_2$.

STATEMENT 5'. *Given any ordering of $\{r(p_i) : p_i \in G(\Delta_3)\}$, $\exists$ an ordering of the δ_2-basis $\{d_1', d_2', \ldots, d_s'\} \cup \{r(a_i) : a_i \in A\}$ such that the automorphism T' which sends the first basis onto the second, is the product of $(n+1)$ $(\varepsilon/2)$-blocked automorphisms:* $T' = T_{n+1}' T_n' \cdots T_1'$.

Proof. The number δ_2 was chosen so that the conclusion of Conj. II holds for the complex K_1^n and the positive number $\varepsilon/2$. Statement 5' is simply the conclusion of Conj. II.

Now define $d_i \in G(\Delta_3)$ by $d_i = r^{-1}(d_i')$. Since A generates $G(\Delta_1)$ and $\{\pi(d_i)\}$ generates $G(\Delta_2)/A$, the d_i may be restricted to $G(\Delta_2 - \Delta_1)$, i.e., assume $d_i \in G(\Delta_2 - \Delta_1)$. Let $T = r^{-1} T' r$ and $T_i = r^{-1} T_i' r$.

STATEMENTS 4, 5. *Summary of the proof to this point. Given any ordering of $\{p_i : p_i \in G(\Delta_3)\}$, $\exists$ an ordering of the basis $\{d_1, d_2, \ldots, d_s\} \cup \{a_i : a_i \in A\}$ such that the automorphism $T : G(\Delta_2) \to G(\Delta_2)$ which sends the first basis to the second, is the product of $(n+1)$ ε-blocked automorphisms:* $T = T_{n+1} T_n \cdots T_1$.

Proof of Statements 4, 5. If $S \subset K_1^n$ and $d(S) < \varepsilon/2$, then $d[r^{-1}(S) \cap \Delta_2] < \varepsilon$, because $d(r(x), x) < \varepsilon/4$ for all $x \in \Delta_2$. The T_i are ε-blocked because the T_i' are $(\varepsilon/2)$-blocked. Thus the proof follows from Statements 4', 5'.

Extend T and T_i to all of G by defining $T|G[Cm(\Delta_2)] = T_i|G[Cm(\Delta_2)] = $ the identity, and note that T_i is still ε-blocked. Let (c) be the matrix which represents T under the natural ordering of $\{p_i\}_{i=1}^v$. If $a_t \in G(\Delta_2)$, it is in the basis which determined T, i.e., $\exists p_r$ with $(c)p_r = a_t$. Therefore, $\exists$ a matrix (Y) obtained from the identity by interchanging columns, such that $(c)(Y)p_t = a_t$ for each t with $a_t \in G(\Delta_2)$. Let (d) be the unique matrix such that $(a_{i,j}) = (d)(c)(Y)$.

STATEMENT 6. $(d)a_t = a_t$ *for all* $a_t \in G(\Delta_2)$.

Proof. $(a_{i,j})p_t = a_t = (c)(Y)p_t$ for all t with $a_t \in G(\Delta_2)$.

STATEMENT 7. $d(p_t) = p_t$ *for all* $p_t \in G(\Delta_1)$.

Proof. This follows from Statement 6 and the fact that $\{a_i : a_i \in G(\Delta_2)\}$ generates $G(\Delta_1)$. This shows that the matrix (d) is ε-blocked and since (c) is the product of $(n+1)$ ε-blocked matrices, and $(a_{i,j})(Y)^{-1} = (d)(c)$, the proof is complete.

COROLLARY 1. *Suppose K is a finite connected complex and L is a connected subcomplex with $\dim L \leq 2$. Then $\exists$ a subcomplex X of some subdivision of K,*

$L \subset \text{Int } (X, K)$, *such that if N is any complex and $f: K \to N$ is any homeomorphism with $f|\text{Cm (Int } [X, K], K)$ p.w.l., then f is a simple homotopy equivalence.*

Proof. The proof is immediate from Lemma 5 and the fact that Conj. II_2 is true. (The hypothesis that L be connected was merely a convenience—Lemma 5 and Corollary 1 are true without this restriction.)

COROLLARY 2. *Suppose M is a p.w.l. n-manifold and $K \subseteq M$ is a connected 2-complex topologically embedded in M. Then $\exists \delta > 0$ such that if U is any complex topologically embedded in M, $K \subset U \subset M$, K is a subcomplex of U, and U δ-deforms to K, then $K \subset U$ is a simple homotopy equivalence. If in addition K and U are subcomplexes of M, U is an n-manifold with boundary, $K \subset \text{Int } (U)$, and $n \geq 6$, then U is a regular neighborhood of K.*

Proof. It follows from Lemmas 3 and 4 that $\exists \delta > 0$ such that if U δ-deforms to K, then $K \subset U$ is a simple homotopy equivalence. Suppose K and U are subcomplexes of M, U is an n-manifold with boundary, $K \subset \text{Int } (U)$, and $n \geq 6$. Let U_1 be a regular neighborhood of K, $U_1 \subset \text{Int } U$. Then $U - \text{Int } (U_1)$ is an H-cobordism. Since $K \subset U_1$ and $K \subset U$ are simple homotopy equivalences, and $\pi_1(U_1) \approx \pi_1(\partial U_1)$, it follows that $\partial U_1 \subset U - \text{Int } (U_1)$ is a simple homotopy equivalence. Therefore $U - \text{Int } (U_1)$ is p.w.l. homeomorphic to $\partial U_1 \times I$ ([10], [11], [12]). Thus U is a regular neighborhood of K.

DEFINITION. If K is a finite connected complex, an abstract regular neighborhood X^n of K is a compact p.w.l. manifold containing K as a subcomplex such that X collapses to K.

COROLLARY 3. *Suppose K is a finite, connected 2-complex and X^n is an abstract regular neighborhood of K, $n \geq 6$, and Y is a p.w.l. manifold and $f: X \to Y$ is a homeomorphism onto. Then f is a simple homotopy equivalence.*

Proof. Since $n \geq 6$, f is isotopic (mod ∂X) to a homeomorphism which, when restricted to K is a p.w.l. embedding ([18] or [19]). Thus suppose f itself has this property. Now $\tau(f) = \tau(i)$ where $i: f(K) \to Y$ is the inclusion. Let $\delta > 0$ satisfy the conclusion of Corollary 2 for the complex $f(K)$. Since Y is homeomorphic to X, $\exists$ a topological embedding $h: Y \to Y$ such that

(1) h is topologically isotopic to Id: $Y \to Y$,

(2) $h|f(K) = \text{Id}: f(K) \to f(K)$, and

(3) $f(K)$ is a δ-deformation retract of $h(Y)$. Now by Corollary 2, $j: f(K) \to h(Y) = U$ is a simple homotopy equivalence. Since $\tau(f) = \tau(i) = \tau(j) = 0$, this completes the proof.

Farrell and Hsiang have constructed h-cobordisms which are not topologically trivial [20]. It follows from the next corollary, that if π is a finitely presented group and $\text{Wh } (\pi) \neq 0$, then $\exists$ a p.w.l. h-cobordism W with $\pi_1(W) = \pi$ and such that W is not topologically trivial. This result has also been obtained by A. Casson.

COROLLARY 4. *Suppose K is a finite connected 2-complex and X^n is an abstract regular neighborhood of K, $n \geq 6$. If W is a p.w.l. h-cobordism, one of whose boundary components is ∂X, then W is homeomorphic to $\partial X \times I$ iff $\tau(W, \partial X) = 0$.*

Proof. If $\tau(W, \partial X) = 0$, then W is p.w.l. homeomorphic to $\partial X \times I$ by p.w.l. Smale theory. Suppose W is homeomorphic to $\partial X \times I$. If $Y = W \cup X$ identified by ∂X, then $\exists$ a homeomorphism $f: X \to Y$ which is homotopic to $\mathrm{Id}: X \to X$. By Corollary 3, $\tau(f) = 0$. Since $\tau(W, \partial X) = \tau(Y, X) = \tau(f) = 0$, the proof is complete.

Appendix. Geometric groups are related to several problems in topology, and some of these are mentioned here. No proofs are presented but brief sketches of the ideas are included.

I. CONJECTURE I AND WALL'S OBSTRUCTION.

LEMMA 6. *Suppose $X \subset R^n$ is a compact connected set which is a retract of some open subset of R^n, and $\varepsilon > 0$. Then $\exists$ a finite $(n+1)$-complex K. $X \subset K \subset R^{2n+3}$, such that K^n ε-deforms to X in $K^{n+1} = K$, every element of $H_{n+1}(K, X) \approx H_{n+1}(K)$ is spherical, and $H_{n+1}(K)$ is generated by $\{f_*[H_{n+1}(S^{n+1})] : f: S^{n+1} \to K$ satisfies $d[f(S^{n+1})] < \varepsilon\}$.*

This is proved by brute force construction and induction. Suppose for example, that K is given as above, and it is desired to construct $L = L^{n+2}$ such that L^{n+1} small-deforms to X in L. L is formed by attaching small $(n+2)$-cells to K. It must be shown that $\pi_{n+1}(K, X)$ is locally finitely generated and that $(n+2)$-cells may be attached in a manner that will allow K to deform to X by a slight movement (this is a difficult geometric construction).

THEOREM 3. *Suppose $X \subset R^n$ is a compact connected set which is a retract of some open subset of R^n. Then Conj. I_{2n+3} implies that X has the homotopy type of a finite complex.*

Let K be as in Lemma 6, $X \subset K \subset R^{2n+3}$, $\tilde{K}$ be the universal covering space of K, and $\pi = \pi_1(K) = \pi_1(X)$. $H_{n+1}(\tilde{K})$ is a projective π-module and its element in the projective class group of $Z(\pi)$ is Wall's obstruction [9]. If $H_{n+1}(\tilde{K})$ is a free π-module, the free generators will allow $(n+2)$-cells to be attached to K to determine L such that $X \subset L$ is a homotopy equivalence. In the covering space this corresponds to killing $H_{n+1}(\tilde{K})$ without creating any $(n+2)$-cycles. Suppose Conj. I_{2n+3} is true and let W be a $(2n+3)$-cell with $X \subset \mathrm{Int}(W)$. Then K can be constructed to lie in W. We consider geometric groups associated with W, *not* with K. $G = G(W)$ will be generated by $\{p_i\}$ constructed as follows: Let $\{P_i\}$ be a collection of small cycles, $P_i \in Z_{n+1}(K)$, which generate $H_{n+1}(K)$, and let $p_i \in K \subset W$ be a point of the carrier of P_i. Let $H = H_{n+1}(K)$. Then $f: G \to H$ defined by $f(p_i) =$ the homology class of P_i is a δ-map. The conclusion of Conj. I gives a free basis for $H_{n+1}(K)$ and each basis element is carried by a small cycle. Since these cycles are small, they lift homeomorphically to the covering space where π acts freely on them.

SUMMARY. Lemma 6 gives a K where $H_{n+1}(K)$ is generated by small cycles. Conj. I is used to determine a free basis composed of small cycles. This free basis lifts to a free π-basis of $H_{n+1}(\widetilde{K})$.

II. NONCOMPACT SMALE THEORY.

CONJECTURE A. Suppose M is a p.w.l. n-manifold $(n \geq 6)$ without boundary topologically embedded as a closed subset of R^{2n+3}. Then if $\varepsilon(x): R^{2n+3} \to (0, 1)$ is continuous, $\exists$ a continuous $\delta(x): R^{2n+3} \to (0, 1)$ such that if C is a p.w.l. manifold topologically embedded as a closed subset of R^{2n+3}, $\partial C =$ two components M and M_1, and C δ-deforms to M and to M_1, then $\exists$ a p.w.l. homeomorphism $h: M \times I \to C$ with $h(x, 0) = x$ and $d[h(x, I)] < \varepsilon(x)$ for $x \in M$.

First consider the special case where M is compact. This conjecture states that the H-cobordism is trivial whether M is simply-connected or not; i.e., it states that $M \subset C$ is a simple homotopy equivalence. This is implied by Conj. II_n and Lemmas 3 and 4. Conj. A also says that C is trivial in a controlled manner, i.e., $d[h(x, I)] < \varepsilon$. This requires reproving Smale theory using δ-blocked matrices. The column operations necessary to diagonalize the matrix can be done locally and handles can be cancelled in a controlled manner. Thus Conj. II_n implies Conj. A for M compact. For noncompact M, Conj. II must be generalized to consider infinitely generated geometric groups associated with infinite complexes. Since Conj. II is local in nature, this is only a slight generalization. Summary: A generalized Conj. II_n implies Conj. A.

III. STABLE HOMEOMORPHISMS.

CONJECTURE B. Suppose $f: R^n \to R^n$ $(n \geq 5)$ is a homeomorphism and $\exists$ an integer k such that $f \times I: R^n \times R^k \to R^n \times R^k$ is stable [15]. Then f is stable.

It suffices to consider the case $k = 1$. A homeomorphism is stable iff it can be approximated by a p.w.l. homeomorphism [16]. Suppose $g: R^n \times R \to R^n \times R$ is a p.w.l. homeomorphism approximating $f \times I$. Move g slightly so that $g(R^n \times 0) \cap R^n \times 0 = \varnothing$. Then $R^n \times 0$ and $g(R^n \times 0)$ bound an H-cobordism and using Conj. A, f may be approximated by a p.w.l. homeomorphism. Thus Conj. A implies Conj. B.

IV. DIAGONALIZING INFINITE MATRICES.

THEOREM 4. *If M is a positive integer, $\exists$ a positive integer N such that: If $a_{i,j}$ and $b_{i,j}$ are integers for $-\infty \leq i, j \leq \infty$, with $a_{i,j} = b_{i,j} = 0$ for $|i - j| > M$ and $(a_{i,j})(b_{i,j}) =$ the identity matrix, then $\exists$ matrices $(c_{i,j})$ and $(d_{i,j})$ and an integer k, $-M \leq k \leq M$, such that*

(1) $(c_{i,j})(d_{i,j}) = (a_{i,j})$,

(2) $c_{i,j} = 0$ *when $\exists n$ with $2nN \leq i \leq 2(n+1)N$ and*

          ($j < 2nN + k$ or $j > 2(n+1)N + k$),

(3) $d_{i,j} = 0$ *when $\exists n$ with $(2n-1)N \leq i \leq 2(n+1)N$ and*

          ($j < 2(n-1)N$ or $j > (2n+1)N$).

This is analogous to Conj. II_1 for the noncompact complex $K=$ Reals. The proofs of Conj. II_1 and Theorem 3 are similar, except Theorem 3 is easier. Here is the idea for constructing $(c_{i,j})$. Divide $(a_{i,j})$ into finite blocks $\ldots A_{-2}, A_{-1}, A_0,$ $A_1, \ldots$, throw away A_i for i odd and replace it by B_i where B_i is an identity matrix slightly smaller than A_i. Now the columns of A_{2i} are part of a basis for an infinitely generated free abelian group and, of course, so are the columns of B_{2i+1}. The approach is to take their union and extend it to a complete basis—this corresponds to filling in the gaps between A_{2i} and B_{2i+1}. Use the fact that the inverse matrix is bounded about the diagonal to show this can be done. This gives $(c_{i,j})$, and $(d_{i,j})$ is defined by $(d_{i,j}) = (c_{i,j})^{-1}(a_{i,j})$. The blocks A_{2i} in $(c_{i,j})$ will cause $(d_{i,j})$ to have blocks of the identity and thus automatically be broken into finite nonsingular blocks.

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