ON THETA FUNCTIONS AND WEIL'S GENERALIZED POISSON SUMMATION FORMULA

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In his paper, G. Shimura quotes a proposition [2, Proposition 2.5] which is derived directly from a transformation formula of theta functions due to Krazer-Prym and suggests proving this proposition by means of a generalized Poisson summation formula obtained by A. Weil [4, Théorème 4]. The purpose of this paper is to execute his idea. Let us explain briefly the result.

Let V be a $2n$ dimensional real vector space and let D be a discrete subgroup of rank $2n$ in V. Suppose that we have a nondegenerate alternate bilinear form A on $V\times V$ which assumes integral values on $D\times D$. The form A represents an integral cohomology class on the torus $T=V/D$. A complex vector space structure J on V induces a complex structure on the torus, which may be denoted by the same notation. Such a complex structure J is said to be admissible if there is a positive divisor on the complex torus (T, J) whose cohomology class is A. Making use of the theory of theta functions on the complex vector space (V, J), we assign to each admissible complex structure J on T a holomorphic map $\Theta(J)$ of the complex torus (T, J) into a complex projective space P of a certain dimension depending only on the form A, in such a way that the cohomology class A is a rational multiple of the image of the cohomology class of hyperplane sections on P under the cohomology homomorphism $\Theta(J)^*$.

Consider the group $S(D)$ of real linear transformations of V which leave the bilinear form A and the subgroup D invariant. If $\sigma\in S(D)$, then σ induces an isomorphism σ of T onto itself. If J is an admissible complex structure on T, there is a unique admissible complex structure J^σ on T such that $\sigma\colon (T, J^\sigma)\to(T, J)$ is a holomorphic isomorphism of these two complex tori.

Now, the result asserts that if we make a suitable choice of the assignment $\Theta(J)$ to each admissible complex structure J and if σ belongs to a congruence subgroup in $S(D)$ of sufficiently high level with respect to an appropriate base of D, then the following diagram is commutative:

$$\begin{tikzcd}
(T, J^\sigma) \arrow[rr, "\sigma"] \arrow[dr, "\Theta(J^\sigma)"'] & & (T, J) \arrow[dl, "\Theta(J)"] \\
& P &
\end{tikzcd}$$

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Our task is not difficult, because there is available a paper by P. Cartier [1], which views theta functions in the framework of the unitary representation of a certain nilpotent Lie group which is a fundamental tool used by A. Weil to obtain his generalization of the Poisson summation formula.

In what follows, we denote by $\boldsymbol{C}$ the field of complex numbers, by $\boldsymbol{R}$ the field of real numbers and by $\boldsymbol{Z}$ the ring of integers.

The referee of the present work kindly suggests the author to refer, for other proofs of the above result, to the following two articles: J.-I. Igusa, *On the graded ring of theta-constants*, Amer. J. Math. **86** (1964), 219–246 and D. Mumford, *On the equations defining abelian varieties*, (III), Invent. Math. **3** (1967), 215–244.

1. Notations and theorems.

1.1. Let V be a $2n$ dimensional real vector space and let D be a discrete subgroup of rank $2n$. We assume that our vector space V is equipped with a nongenerate alternate bilinear form A on $V\times V$ which assumes integral values on $D\times D$. This setting (V, D, A) is fixed throughout this paper. An endomorphism J (acting on the right of the vector space V) satisfying the following conditions:

(i) $J^2=-E$,

(ii) $A(u\cdot J, v\cdot J)=A(u, v), \qquad u, v\in V,$

(iii) $A(u, u\cdot J)>0, \qquad u\in V, u\neq 0,$

is called *admissible* complex structure for (V, A). We denote by V_J the complex vector space determined by J, and by T_J the complex torus V_J/D. The alternate bilinear form A defines canonically an integral cohomology class of the complex torus T_J, which provides a polarization of T_J. The hermitian form H_J on $V_J\times V_J$ defined by

$$H_J(u, v) = A(u, v\cdot J)+(-1)^{1/2}A(u, v), \qquad u, v\in V_J,$$

is called the Riemann form of the polarized abelian variety (T_J, A).

An R-valued bilinear form B on $V\times V$ assuming integral values on $D\times D$ is called a satellite form of A if $B(u, v)-B(v, u)=A(u, v)$, u, $v\in V$. By means of a satellite form B of A, a *semicharacter* of D attached to A is defined to be a map ψ of D into the multiplicative group of complex numbers with absolute value 1 such that the map $d\to\psi(d)e[B(d, d)/2]$ is a character of D, where we adopt a convention to read $e[*]$ as $\exp 2\pi(-1)^{1/2}*$. A theta function (of reduced type, [3, p. 111]) of type (H_J, ψ) is a holomorphic function on the complex

$$\theta(x+d) = \theta(x)\psi(d)e\left[\frac{H_J(d, x)}{2(-1)^{1/2}}+\frac{H_J(d, d)}{4(-1)^{1/2}}\right], \qquad x\in V,\ d\in D.$$

We choose a base $\{d_1, \dots, d_{n+1}, \dots, d_{2n}\}$ of D such that

$$\text{(i)} \qquad A\left(\sum_{i=1}^{2n}\xi_i d_i, \sum_{i=1}^{2n}\eta_i d_i\right) = \sum_{\alpha=1}^{n} e_\alpha(\xi_\alpha\eta_{n+\alpha}-\xi_{n+\alpha}\eta_\alpha), \qquad \xi_i, \eta_j\in R,$$

$$\text{(ii)} \qquad e_1, \dots, e_n\in Z,\ e_1|e_2|\cdots|e_{n-1}|e_n.$$

The integers $e_1, \dots, e_n$ are the elementary divisors of A with respect to D and the positive integer $e=|e_1\cdots e_n|$ is the Pfaffian of A with respect to D. Both the elementary divisors and the Pfaffian of A are determined by A, independently on the choice of a base $\{d_1, \dots, d_n, d_{n+1}, \dots, d_{2n}\}$ satisfying the above two conditions. Such a base is called a *base of D adapted to A*. Take a base $\{d_1, \dots, d_n, d_{n+1}, \dots, d_{2n}\}$ of D adapted to A. Then, both $\{d_1, \dots, d_n\}$ and $\{d_{n+1}, \dots, d_{2n}\}$ are complex bases of V_J for any admissible complex structure J. Let E and E' be the real vector subspaces of V spanned by $\{d_1, \dots, d_n\}$ and $\{d_{n+1}, \dots, d_{2n}\}$ respectively. The subspaces of E and E' are both maximal isotropy subspaces of V with respect to the alternate bilinear form A. Obviously V is the direct sum of E and E'. We denote by Γ and Γ' the subgroups of D generated by $\{d_1, \dots, d_n\}$ and $\{d_{n+1}, \dots, d_{2n}\}$ respectively; we have $\Gamma=E\cap D$, $\Gamma'=E'\cap D$.

Take $\{d_{n+1}, \dots, d_{2n}\}$ as a C-base of the complex vector space V_J. If

$$H_J\left(\sum_{\alpha-1}^{n} z_\alpha\, d_{n+\alpha}, \sum_{\alpha=1}^{n} w_\alpha\, d_{n+\alpha}\right) = \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta}\bar{z}_\alpha w_\beta,$$

and $h_{\alpha\beta}=h_{\beta\alpha}$ then

$$\Phi_J\left(\sum_{\alpha=1}^{n} z_\alpha\, d_{n+\alpha}, \sum_{\alpha=1}^{n} w_\alpha\, d_{n+\alpha}\right) = \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta} z_\alpha w_\beta \tag{1}$$

defines a symmetric C-bilinear form on $V_J\times V_J$.

1.2. Let (T_J, A) be a polarized abelian variety. It is known that if the elementary divisions $e_1, \dots, e_n$ of A with respect to D are all not less than 3, then there is an imbedding of the abelian variety T_J into the complex projective space of dimension $e-1$ by means of the theta functions such that the image of the cohomology class determined by hyperplane sections of the complex projective space under the dual map of the differential of the imbedding is exactly the cohomology class of T_J which gives rise to the polarization (T_J, A). We describe this imbedding in this number. First, we prepare the following well-known theorem.

THEOREM A. *The complex vector space of theta functions on V_J of type (H_J, ψ) is of dimension equal to the Pfaffian e of A with respect to D.*

Let $\{d_1, \dots, d_n, d_{n+1}, \dots, d_{2n}\}$ be a base of D adapted to A. Consider this base as a base of the real vector space V. We define and $\boldsymbol{R}$-valued bilinear form B_0 on $V\times V$ by

$$B^0\left(\sum_{i=1}^{2n} \xi_i\, d_i, \sum_{i=1}^{2n} \eta_i\, d_i\right) = -\sum_{\alpha}^{n} e_\alpha \xi_{n+\alpha}\eta_\alpha.$$

Then, B_0 is a satellite form of A. Setting

$$\psi_0(d) = e[B_0(d, d)/2] \tag{2}$$

we obtain a semicharacter ψ_0 of D attached to A. Obviously, $\psi_0(\Gamma)=1$ and $\psi_0(\Gamma')=1$.

We denote by $\mathfrak{A}$ the set of row vectors $a=(a_1, \ldots, a_n)$ of integer components such that $0<a_\alpha<|e_\alpha|$, $\alpha=1, \ldots, n$. Let us introduce an order in the set $\mathfrak{A}$ once and for all, and denote by $a(k)$ the kth element in $\mathfrak{A}$. The number of elements in the set $\mathfrak{A}$ is the Pfaffian e.

THEOREM B. *Let H_J be the Riemann form of a polarized abelian variety (T_J, A) and let Φ_J be the symmetric C-bilinear form on V_J defined by* (1). *For each $a(k)=(a(k)_1, \ldots, a(k)_n) \in \mathfrak{A}$, we define an element*

$$d_{a(k)} = \sum_{\alpha=1}^{n} \frac{a(k)_\alpha}{e_\alpha} d_\alpha \in E$$

and put

$$\theta_k(J)(x) = \sum_{d\in\Gamma} \exp\frac{\pi}{2}\{H_J(d_{a(k)}+d, d_{a(k)}+d)+2H_J(d_{a(k)}+d, x) + \Phi_J(d_{a(k)}+d+x, d_{a(k)}+d+x)\}, \qquad x \in V_J.$$

Then θ_k is a theta function of type (H_J, ψ_0), and those e functions $\{\theta_k, 1<k<e\}$ form a $\boldsymbol{C}$-base of the vector space of theta functions of type (H_J, ψ_0).

Now, consider a holomorphic map of the complex vector space V_J into the complex vector space $\boldsymbol{C}^e$ given by

$$x \to (\theta_1(J)(x), \ldots, \theta_e(J)(x)).$$

It is well known (see [3, Théorème 5, Chapter VI]) that under the assumption that each elementary divisor e_i of A with respect to D is not less than 3, the above holomorphic map of V_J into $\boldsymbol{C}^e$ induces a holomorphic map $\Theta(J)$ of the complex torus T_J into the complex projective space $\boldsymbol{P}^{e-1}$ of dimension $e-1$ which is univalent and everywhere regular. Moreover, the image of the cohomology class of hyperplane cross sections under $\Theta(J)^*$ is the cohomology class determined by A.

1.3. Let $\boldsymbol{S}$ be the group of nonsingular $\boldsymbol{R}$-linear transformations of V, acting on the right-hand side, which leave the alternate bilinear form A invariant, that is, the symplectic group of A, and let $\boldsymbol{S}(D)$ be the subgroup of $\boldsymbol{S}$ consisting of the elements σ such that $D\cdot\sigma=D$. If $\sigma\in\boldsymbol{S}(D)$ and if J is an admissible complex structure of (V, A), then $J^\sigma=\sigma\cdot J\cdot\sigma^{-1}$ is also an admissible complex structure of (V, A), and σ is a complex linear transformation of $V_J\sigma$ onto V_J. Since $D\cdot\sigma=D$, σ induces a holomorphic isomorphism of the abelian variety $T_J\sigma$ onto the abelian variety T_J, which leaves the polarization A invariant.

In terms of a base $\{d_1, \ldots, d_n, d_{n+1}, \ldots, d_{2n}\}$ of D adapted to A, $\sigma\in\boldsymbol{S}(D)$ is represented by a matrix $M(\sigma)$ of integer coefficients;

$$(d_1\cdot\sigma, \ldots, d_{2n}\cdot\sigma) = (d_1, \ldots, d_{2n})\cdot M(\sigma).$$

Let us denote by $\boldsymbol{S}(\nu)$ the normal subgroup of $\boldsymbol{S}(D)$ consisting of those elements σ such that $M(\sigma)\equiv 1_{2n} \bmod \nu$. Now, we are ready to state the main theorem.

THEOREM C [2, Proposition 2.5, p. 308]. *Assume that the elementary divisors* $e_1, \ldots, e_n$ *of* A *with respect to* D *are all not less than* 3. *Let* σ *be an element in the subgroup* $S(2e)$, *where* $2e$ *is twice the Pfaffian of* A, *in* $S(D)$. *Let us denote by the same notation* σ *the holomorphic isomorphism of the abelian variety* $T_J\sigma$ *onto the abelian variety* T_J *induced by* σ. *Then*

$$\Theta(J)\cdot\sigma = \Theta(J^\sigma),$$

namely, the following diagram is commutative:

$$\begin{tikzcd} T_J\sigma \arrow[rr, "\sigma"] \arrow[dr, "\Theta(J^\sigma)"'] & & T_J \arrow[dl, "\Theta(J)"] \\ & P^{e-1} & \end{tikzcd}$$

2. **Schrödinger and Fock representations.**

2.1. Following Weil [4] and Cartier [1], we introduce a nilpotent Lie group G associated to the pair (V, A) of a $2n$ dimensional real vector space V and a non-degenerate alternate bilinear form A on $V\times V$. We first consider a Lie algebra $\mathfrak{G}$ over R whose underlying vector space is the direct sum of $\boldsymbol{R}$ and V and whose bracket multiplication is given by

$$[(s, u), (t, v)] = (A(u, v), 0), \qquad s, t \in \boldsymbol{R},\ u, v \in V.$$

Obviously $\mathfrak{G}$ is a nilpotent Lie algebra and whose center is the subalgebra consisting of elements $(t, 0)$. Let G be the simply connected Lie group corresponding to the Lie algebra $\mathfrak{G}$. The exponential map $e\colon \mathfrak{G} \to G$ is univalent and surjective. The image of the center in G under the exponential map e is the center of the group G. Any element in G is uniquely written as

$$e(t, u) = e^t\cdot e^u, \qquad t \in R,\ v \in V.$$

The law of multiplication is given by

$$(e^s\cdot e^u)\cdot(e^t\cdot e^v) = e^{s+t+\frac{1}{2}A(u,v)}\cdot e^{u+v}, \qquad s, t \in R,\ u, v \in V.$$

Now, we define two unitary representations of the group G which are equivalent to each other. The first one is the Schrödinger representation. We recall that V is the direct sum of two maximal isotropy subspaces E and E' with respect to A. If we express an element $g \in G$ as $g = e^t\cdot e^{v'}\cdot e^v$, $t \in R$, $v \in E$, $v' \in E'$, the law of multiplication reads

$$(e^s\cdot e^{u'}\cdot e^u)\cdot(e^t\cdot e^{v'}\cdot e^v) = e^{s+t+A(u,v')}\cdot e^{u'+v'}\cdot e^{u+v}, \qquad s, t \in R,\ u, v \in E,\ u', v' \in E'.$$

In the Schrödinger representation, the representation module is the Hilbert space $L^2(E)$ of C-valued square-integrable functions on E with respect to a usual Lebesgue measure dv which is kept fixed once and for all on E. The homomorphism τ of G into the group of unitary operators of $L^2(E)$ is given by

$$(\tau(e^t\cdot e^{v'}\cdot e^v)\varphi)(x) = e[t+A(x, v')]\varphi(x+v), \quad \text{for } \varphi \in L^2(E).$$

Another representation is the Fock representation; the representation module F_J is the Hilbert space of holomorphic functions φ on the complex vector space V_J with an admissible complex structure J of the pair (V, A) such that

$$\int_{V_J} e^{-\pi} H_J(x, x)|\varphi(x)|^2\, dx < \infty,$$

where dx denote a usual Lebesgue measure with respect to the hermitian form H_J on V_J. The homomorphism ρ_J of the group G into the unitary operators on F_J is defined by

$$(\rho_J(e^t\cdot e^w)\cdot\varphi)(x) = e\left[\frac{(-1)^{1/2}}{2} H_J(w\cdot w)+(-1)^{1/2}H_J(w, x)\right]\varphi(x+w) \quad \text{for } \varphi\in F_J.$$

These two representations are known to be irreducible [1] and their restrictions to the center of G give rise to the same character of the center: $e^t \to e[t]$. Therefore they are unitary equivalent to each other by the theorem of Stone-Neumann [1].

2.2. Let (H, π) be a unitary representation of G on a Hilbert space H and a homomorphism π of G into the group of unitary operators on H. Let H_∞ be the Gårding subspace consisting of C^∞-vectors in H. The subspace H_∞ is a Fréchet space with respect to a certain topology (see [1, §7]) and is $\pi(G)$-invariant. Thus, we have a representation (H_∞, π) on the Fréchet space H_∞. Let $H_{-\infty}$ be the complex vector space of continuous antilinear forms forms of H_∞. The representation (H_∞, π) of G canonically induces a representation $(H_{-\infty}, \pi)$ of G: $(\pi(g)\cdot f)(a) = f(\pi(g^{-1})\cdot a)$, $g\in G$, $a\in H_\infty$ and $f\in H_{-\infty}$. The inner product (a, b) of the Hilbert space H is antilinear with respect to the first entry a and C-linear with respect to the second entry b. Assigning to an element $b\in H$, an antilinear form $f_b(a)=(a, b)$, we identify $b\in H$ with $f_b\in H_{-\infty}$. In this manner, we regard H as a subspace of the complex vector subspace $H_{-\infty}$. Moreover, the restriction of the representation $(H_{-\infty}, \pi)$ to the subspace H is exactly the representation (H, π); this fact justifies our usage of the same notation π for three distinct representations of G.

2.3. Let us apply the above construction of new representations to the Fock representation (F_J, ρ_J). The Lie algebra $\mathfrak{G}$ of G is the direct sum $\boldsymbol{R}+V$. To each $t\in\boldsymbol{R}\subset\mathfrak{G}$, we assign an operator $\rho_J(t)=2\pi it\cdot 1$, where 1 is the identity operator. To each $v\in V\subset\mathfrak{G}$, we assign an operator

$$\rho_J(v)\cdot\varphi = \theta_v\cdot\varphi-\pi H_{J,v}\cdot\varphi \tag{3}$$

acting on a holomorphic function on V_J, where θ_v denotes the direction differentiation along v and $H_{J,v}$ denotes a holomorphic function on V_J given by $H_{J,v}(x) = H_J(v, x)$, $x\in V_J$. Then, the map ρ_J: $(t, v)\to\rho_J(v)|F_{J\infty}$ is nothing but the infinitesimal representation of $\mathfrak{G}$ associated to the group representation $(F_{J\infty}, \rho_J)$.

The Gårding subspace $F_{J-\infty}$ can be identified with the vector space of holomorphic functions φ in F_J such that $\rho_J(v_1)\cdot\ \cdots\ \cdot\rho_J(v_k)\cdot\varphi$ is in F_J for any choice of $v_1, \dots, v_k \in V$. Take a base $\{u_1, \dots, u_{2n}\}$ of V. We define on $F_{J\infty}$ countably many Hilbert norms N_m, $m=1, 2, \dots$, by

$$N_m(\varphi) = \sum_{|\alpha|<m} (2\pi)^{\alpha}\|\rho_J(u_1)^{\alpha_1}, \dots, \rho_J(u_{2n})^{\alpha_{2n}}\cdot\varphi\|F_J$$

with the standard abbreviations $\alpha=(\alpha_0, \alpha_1, \dots, \alpha_{2n})$ and $|\alpha|=\alpha_0+\alpha_1+\cdots+\alpha_{2n}$. With the topology determined by these norms, $F_{J\infty}$ becomes a Fréchet space. The space $F_{J-\infty}$, then, consists of those holomorphic functions on V_J each of which is expressed as a finite sum of the form $\rho_J(v_1)\cdots\rho_J(v_k)\cdot\varphi$ with $v_1, \dots, v_k \in V$ and $\varphi \in F_J$ [1, p. 381].

Now, let V be a $2n$ dimensional real vector space and let D be a discrete subgroup in V of rank $2n$. As in 1.1., we assume that there is a nondegenerate R-valued alternate bilinear form A on $V\times V$ which assumes integral values on $D\times D$. Consider the nilpotent Lie group G associated to the pair (V, A). Cartier shows in his paper [1, §14, Definition of theta functions] that a holomorphic function θ on V_J is a theta function of type (H_J, ψ) with respect to D if and only if θ belongs to $F_{J-\infty}$ and satisfies the equations

$$\rho_J(e^d)\cdot\theta = \psi(d)\cdot\theta \quad \text{for each } d\in D. \tag{4}$$

3. **Theta distributions.**

3.1. In this section we study the solutions of the equations corresponding to (4) in the Schrödinger representation. We start with an observation on semicharacters of D. Let ψ be a semicharacter of D attached to A. If B is a satellite form of A, the map $d\to\psi(d)e[B(d\cdot d)/2]$ is a character of G. Hence

$$\psi(d)\psi(d')e[B(d, d)+B(d', d')/2] = \psi(d+d')e[B(d+d', d+d')/2], \qquad d, d' \in D.$$

Consider the direct sum decomposition $D=\Gamma+\Gamma'$, $\Gamma=D\cap E$, $\Gamma'=D\cap E'$. If particularly d_1, d_2 are in Γ, then $\psi(d_1)\psi(d_2)=\psi(d_1+d_2)$, and similarly if d_1', d_2' are in Γ', then $\psi(d_1')\psi(d_2')=\psi(d_1'+d_2')$. Therefore there is an $\boldsymbol{R}$-linear form l on V such that

$$\text{(i)} \qquad \psi(d) = e[l(d)], \qquad d\in\Gamma$$

$$\text{(ii)} \qquad \psi(d') = e[l(d')], \qquad d'\in\Gamma$$

and that

$$\psi(d+d') = e\left[l(d+d')+\frac{A(d, d')}{2}\right], \qquad d\in\Gamma,\ d'\in\Gamma'. \tag{5}$$

For a given ψ, the choice of l is not unique. Conversely, if l is an $\boldsymbol{R}$-linear form on V, the right-hand side of the above formula (5) defines a semicharacter of D attached to A. If especially we choose l to be identically zero, we obtain the semicharacter ψ_0 defined by (2).

3.2. Let $(L^2(E), \tau)$ be the Schrödinger representation defined in 2.1. In this case, the Gårding subspace is the Fréchet space (S) of rapidly decreasing C^∞-functions on E and its antidual space is nothing but the space (S') of tempered distributions [1, p. 374].

The equations in the Schrödinger representation corresponding to (4) is of the form

$$\tau(e^d)\cdot t = \psi(d)\cdot t, \qquad d\in D,\ t\in(S'). \tag{6}$$

A distribution $t\in(S')$ is a solution of the system (6) if and only if t satisfies the following two systems of equations:

$$\tau_{d'}t = \psi(d)\cdot t, \qquad d\in\Gamma, \tag{7}$$

where τ_d denotes the translation in the vector space E along d, and

$$e[A(x, d')]\cdot t = \psi(d')\cdot t, \qquad d\in\Gamma', \tag{8}$$

where the left-hand side of the equality is the multiplication of a function $e[A(x, d')]$ of $x\in E$ and a distribution t.

Let us denote by Γ_0 the subgroup in E consisting of those elements v each of which satisfies $A(v, d')\equiv 0 \bmod 1$ for all $d'\in\Gamma'$. Clearly $\Gamma_0\supset\Gamma$. The subgroups Γ and Γ' are generated by $\{d_1,\dots,d_n\}$ and $\{d_{n+1},\dots,d_{2n}\}$ respectively, where these $2n$ elements form a base of G adapted to A. Hence, Γ_0 is generated by $d_1^0=(1/e_1)d_1,\dots,d_n^0=(1/e_n)d_n$. A complete system of representatives of the coset group Γ_0/Γ is the set of elements $d_a=\sum_{i=1}^n a_1 d_1^0$, where $a=(a_1,\dots,a_n)\in\mathfrak{A}$ (cf. 1.2).

By selecting a suitable $\boldsymbol{R}$-linear form l on V, let us express the semicharacter ψ as in (5). Let b be a unique solution vector in E of the equations $A(d, a')=l(d')$ for each $d'\in\Gamma'$. We denote by X the set of vectors $v\in E$ such that $e[A(v, d')]=\psi(d')$ for each $d'\in\Gamma'$. Then the set X is written as $b+\Gamma_0$. Now, suppose that t is a solution of (8), then t must be a distribution of the form $\sum_{v\in X}\alpha(v)\delta_v$ is the Dirac distribution at the point $v\in E$, and $\alpha(v)\in\boldsymbol{C}$. If, furthermore, t satisfies (7), then the coefficients $\alpha(v)$'s satisfy the relation

$$\alpha(v-d) = \psi(d)\alpha(v) \quad \text{for } d\in\Gamma.$$

Therefore a solution t of (6) is of the form

$$\sum_{a\in A}\alpha(d_a)\sum\psi(-d)\delta_{d_a+d+b}, \tag{9}$$

where the $\alpha(d_a)$'s are arbitrary constants in $\boldsymbol{C}$. Conversely, a distribution of this form is a solution of the system of equations (7). We may call a distribution of the form (9) a *theta distribution* associated to a semicharacter ψ of D.

As a result of the above observation, we obtain the following:

PROPOSITION 1. *The complex vector space of theta distribution associated to a semicharacter ψ of D attached to A is of dimension e, the Pfaffian of A with respect to D.*

Let i_J be an isomorphism of the Hilbert space $L^2(E)$ onto the Hilbert space F_J through which the Schrödinger representation and the Fock representation are unitary equivalent. Then, i_J canonically extends to a G-isomorphism of S' onto $F_{J-\infty}$. Obviously, $\theta \in F_{J-\infty}$ satisfies (4) if and only if $(i_J)^{-1}\cdot\theta$ satisfies (7). Namely, θ is a theta function of type (H_J, ψ) if and only if $(i_J)^{-1}$. θ is a theta distribution associated to ψ. Thus, Proposition 1 immediately implies Theorem A. In his paper [1], Cartier proves this theorem using the lattice representation [1, **12**]. The idea employed in our proof is an analogue of his in the Schrödinger representation.

4. **Symplectic group.**

4.1. If we express the vector space V as the direct sum $E+E'$, a linear transformation σ of V into itself is written as

$$(u, u')\cdot\sigma = (u, u')\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}, \qquad u \in E,\ u' \in E,$$

where α, β, γ and δ are $\boldsymbol{R}$-linear transformations such that $\alpha\colon E\to E$, $\beta\colon E\to E'$, $\gamma\colon E'\to E$ and $\delta\colon E'\to E'$. It is seen that σ belongs to the symplectic group $\boldsymbol{S}$ with respect to A if and only if

$$A(u\cdot\alpha, u'\cdot\delta) + A(u\cdot\beta, u'\cdot\gamma) = A(u, u') \quad \text{for all } u \in E,\ u' \in E,$$
$$A(u\cdot\alpha, v\cdot\beta)+A(u\cdot\beta, v\cdot\alpha) = 0 \quad \text{for all } u,\ v \in E,$$

and

$$A(u'\cdot\gamma, v'\cdot\delta)+A(u'\cdot\delta, v'\cdot\gamma) = 0 \quad \text{for all } u',\ v' \in E'.$$

We denote by $A(G)$ the group of automorphisms of the Lie group G which leave each element in the center $\{e^t, t \in \boldsymbol{R}\}$ of G fixed. An automorphism $s \in A(G)$ determines a pair of an element $\sigma \in \boldsymbol{S}$ and and $\boldsymbol{R}$-linear form η on V by the equality

$$(e^t\cdot e^v)\cdot s = e^{t+\eta(v)}\cdot e^{v\cdot\sigma}, \qquad t \in R,\ v \in V.$$

Conversely, if $\sigma \in S$ and η is an $\boldsymbol{R}$-linear form on V, then the pair (σ, η) gives rise to an automorphism of G belonging to $A(G)$ in this manner. The subgroup in $A(G)$ consisting of the pairs (σ, η) with $\eta=0$ is canonically isomorphic to the sympletic group $\boldsymbol{S}$. Thus, we realize $\boldsymbol{S}$ as a subgroup in $A(G)$.

Another way to describe automorphisms in $A(G)$ is employed by Weil [4, p. 150], which utilizes the expression of an element in G as $e^t\cdot e^{u'}\cdot e^u$, $t \in \boldsymbol{R}$, $u \in E$, $u' \in E'$. Take particularly an automorphism σ belonging to $\boldsymbol{S}$; $(u, u')\cdot\sigma=(u\cdot\alpha+u'\cdot\gamma, u\cdot\beta+u'\cdot\delta)$. Then

$$(e^{u'}\cdot e^u)\cdot\sigma = e^{g_\sigma(u,u')}\cdot e^{u\cdot\beta+u'\cdot\delta}\cdot e^{u\cdot\alpha+u'\cdot\gamma}, \qquad u \in E,\ u' \in E', \tag{10}$$

where $g_\sigma(u, u')=\frac{1}{2}(A(u, u')-A(u\cdot\alpha+u'\cdot\gamma, u\cdot\beta+u'\cdot\delta))$. The $\boldsymbol{R}$-valued function g_σ on $E\times E'$ satisfies the equality

$$g_\sigma(u+v, u'+v')-g_\sigma(u, u')-g_\sigma(v, v') = A(u, v')-A(u\cdot\alpha+u'\cdot\gamma, v\cdot\beta+v'\cdot\delta),$$
$$u,\ v \in E,\ u',\ v' \in E'.$$

Let $(L^2(E), \tau)$ be the Schrödinger representation of the group G. Each unitary operator U in the normalizer of $\tau(G)$ induces an automorphism of $\tau(G)$: $\tau(g) \to U^{-1} \cdot \tau(g) \cdot U$, $g \in G$, which obviously leaves each element $\tau(e^t)$, $t \in \boldsymbol{R}$, fixed. Since G is simply connected and since the homomorphism is locally isomorphic, the automorphism associated to U determines uniquely an automorphism $s(U) \in \grave{A}(G)$; $U^{-1} \cdot \tau(g) \cdot U = \tau(g) \cdot s(U)$, $g \in G$. A theorem of Weil [4, Théorème 1] states that the homomorphism $U \to s(U)$ of the normalizer of $\tau(G)$ in the group of unitary operators into $A(G)$ is surjective and its kernel is the group of scalar multiplications $\{e[t]1, t \in R\}$.

4.2. Let J be an admissible complex structure of (V, A). If $\sigma \in S$, then $J^\sigma = \sigma \cdot J \cdot \sigma^{-1}$ is also an admissible complex structure of (V, A), and the map σ: $V_{J^\sigma} \to V_J$ is a $\boldsymbol{C}$-isomorphism. If, furthermore, $\sigma \in \boldsymbol{S}(D)$, σ induces a holomorphic isomorphism of the complex torus T_{J^σ} onto T_J. The Riemann forms H_J and H_{J^σ} of the polarized abelian varieties (T_J, A) and (T_{J^σ}, A) are related in the following fashion:

$$H_{J^\sigma}(u, v) = H_J(u \cdot \sigma, v \cdot \sigma), \qquad u, v \in V. \tag{11}$$

Indeed,

$$\begin{aligned} H_{J^\sigma}(u, v) &= A(u, v \cdot \sigma \cdot J \cdot \sigma^{-1}) + (-1)^{1/2} A(u \cdot v) \\ &= A(u \cdot \sigma, v \cdot \sigma \cdot J) + (-1)^{1/2} A(u \cdot \sigma, v \cdot \sigma) \\ &= H_J(u \cdot \sigma, v \cdot \sigma). \end{aligned}$$

Given an element $\sigma \in \boldsymbol{S}(D)$, we denote by σ^* a correspondence which assigns to a function φ on V_J a function $\varphi \circ \sigma$ on V_{J^σ}. Then σ^* gives rise to an isomorphism of the Hilbert space F_J onto the Hilbert space F_{J^σ}. Indeed, for φ and $\psi \in F_{J^\sigma}$,

$$\begin{aligned} (\sigma^* \cdot \varphi, \sigma^* \cdot \psi) &= \int_V e^{-\pi H_{J^\sigma}(v,v)} |\varphi(v \cdot \sigma) \psi(v \cdot \sigma)| \, dv \\ &= \int_V e^{-\pi H_J(v \cdot \sigma, v \cdot \sigma)} |\varphi(v \cdot \sigma) \psi(v \cdot \sigma)| \, dv \\ &= \int_V e^{-\pi H_J(v \cdot v)} |\varphi(v) \psi(v)| \, dv = (\varphi, \psi). \end{aligned}$$

In the above computation, we make use of (11) and the fact that the real linear transformation $\sigma \in \boldsymbol{S}$ of V leaves the measure dv invariant.

PROPOSITION 2. *Let i_J be an isomorphism of the Hilbert space $L^2(E)$ onto the Hilbert space F_J under which the two unitary representations $(L^2(E), \tau)$ and (F_J, ρ_J) of the group G are unitary equivalent. Let σ be an element in the sympletic group $\boldsymbol{S}$ with respect to A. Put $U_J(\sigma) = (i_{J^\sigma})^{-1} \cdot \sigma^* \cdot i_J$. Then the unitary operator $U_J(\sigma)$ of $L^2(E)$ is in the normalizer of $\tau(G)$ and induces the automorphisms σ of G;*

$$U_J(\sigma)^{-1} \cdot \tau(e^t \cdot e^u) \cdot U_J(\sigma) = \tau(e^t \cdot e^{u \cdot \sigma}), \qquad t \in \boldsymbol{R}, u \in V.$$

Proof. First, we show

$$\sigma^{*-1}\cdot\rho_J\sigma(e^t\cdot e^u)\cdot\sigma^* = \rho_J(e^t\cdot e^{u-\sigma}), \qquad t\in \boldsymbol{R},\ u\in V. \tag{12}$$

Indeed, if $\varphi\in F_J$,

$$\begin{aligned}(\rho_J\sigma(e^t\cdot e^u)\cdot\sigma^*\cdot\varphi)(x) &= e[t]\cdot\exp-\pi(H_J\sigma(u,u)/2+H_J\sigma(u,x))\varphi((x+u)\cdot\sigma)\\ &= e[t]\cdot\exp-\pi(H_J(u\cdot\sigma,u\cdot\sigma)/2+H_J(u\cdot\sigma,x\cdot\sigma))\varphi((x+u)\cdot\sigma)\\ &= (\sigma^*\cdot\rho_J(e^t\cdot e^{u\cdot\sigma})\cdot\varphi)(x), \qquad x\in V,\ t\in\boldsymbol{R},\ u\in V.\end{aligned}$$

Since $\rho_J\sigma(e^t\cdot e^u)=i_J\sigma\cdot\tau(e^t\cdot e^u)\cdot(i_J\sigma)^{-1}$ and $\rho_J(e^t\cdot e^u)=i_J\cdot\tau(e^t\cdot e^{u\sigma})\cdot(i_J)^{-1}$,

$$(\sigma^*)^{-1}\cdot i_J\sigma\cdot\tau(e^t\cdot e^u)\cdot(i_J\sigma)^{-1}\cdot\sigma^* = i_J\cdot\tau(e^t\cdot e^{u\cdot\sigma})\cdot(i_J)^{-1}.$$

Thus, $U_J(\sigma)^{-1}\cdot\tau(e^t\cdot e^u)\cdot U_J(\sigma)=\tau(e^t\cdot e^{u\cdot\sigma})$, $t\in\boldsymbol{R}$, $u\in V$.

4.3. PROPOSITION 3. *Let $(L^2(E),\tau)$ be the Schrödinger representation. If U is a unitary operator on $L^2(E)$ which is in the normalizer of $\tau(G)$ and induces an automorphism σ of G belonging to the symplectic group $\boldsymbol{S}$, then U leaves the subspace (S) of rapidly decreasing C^∞-functions on E invariant, defining an isomorphism of the Fréchet space (S) onto itself. Therefore U canonically extends to an isomorphism of the space (S') of tempered distribution onto itself.*

Proof. By assumption the unitary operation U induces an automorphism σ of G belonging to $\boldsymbol{S}$. In virtue of a theorem of Weil mentioned in 4.1., U is equal to the unitary operator $U_J(\sigma)=(i_J\sigma)^{-1}\cdot\sigma^*\cdot i_J$ in Proposition 2 times a scalar with absolute value 1. Therefore, it is enough to show that the proposition is valid for $U_J(\sigma)$. The isomorphism i_J of $L^2(E)$ onto F_J maps naturally the Gårding space (S) of the representation $(L^2(E),\tau)$ onto the Gårding space $F_{J\infty}$ of the representation (F_J,ρ_J). Similarly, $i_{J\sigma}((S))=F_J\sigma_\infty$. Therefore, it suffices to prove that the unitary operator σ^* of F_J onto $F_J\sigma$ induces an isomorphism of $F_{J\infty}$ onto $F_J\sigma_\infty$.

Let φ be an arbitrary holomorphic function on V_J. Since $\sigma^*\cdot\varphi$ is the composition $\varphi\circ\sigma$ of $\sigma\colon V_J\sigma\to V_J$ and the function φ on V_J, $\sigma^*(\theta_v\cdot\varphi)=\theta_{v\cdot\sigma^{-1}}\cdot(\sigma^*\cdot\varphi)$ for $v\in V$. On account of (11), we have $\sigma^*(H_{J,v}\cdot\varphi)=H_J\sigma_{v\cdot\sigma^{-1}}\cdot(\sigma^*\cdot\varphi)$, $v\in V$. Thus,

$$\begin{aligned}\sigma^*\cdot(\rho_J(v)\cdot\varphi) &= \sigma^*(\theta_t\cdot\varphi-H_{J\cdot v}\cdot\varphi)\\ &= \theta_{v\cdot\sigma^{-1}}(\sigma^*\cdot\varphi)-H_J\sigma_{v\cdot\sigma^{-1}}\cdot(\sigma^*\cdot\varphi)\\ &= \rho_J\sigma(v\cdot\sigma^{-1})\cdot(\sigma^*\cdot\varphi),\end{aligned}$$

which implies

$$\sigma^*(\rho_J(v_1)\cdots\rho_J(v_k)\cdot\varphi) = \rho_J\sigma(v_1\cdot\sigma^{-1})\cdots\rho_J\sigma(v_k\cdot\sigma^{-1})(\sigma^*\cdot\varphi),$$

for any choice of $v_1,\ldots,v_k\in V$.

As is observed in 2.3., the space $F_{J\infty}$ consists of holomorphic function φ on V_J such that $\rho_J(v_1)\cdots\rho_J(v_1)\cdot\varphi\in F_J$ for any choice of $v_1,\ldots,v_k\in V$. Now, it is obvious that $\varphi\in F_{J\infty}$ if and only if $\sigma^*\cdot\varphi\in F_{J^\sigma\infty}$. Since σ^* is a unitary operator of F_J onto $F_J\sigma$, $\|\rho_J(v_1)\cdots\rho_J(v_k)\cdot\varphi\|=\|\rho_J\sigma(v_1\cdot\sigma^{-1})\cdots\rho_J\sigma(v_k\cdot\sigma^{-1})(\sigma^*\cdot\varphi)\|$ for $v_1,\ldots,v_k\in V$.

It follows that the restriction of σ^* to $F_J\infty$ is an isomorphism of the Fréchet space $F_{J\infty}$ onto the Fréchet space $F_J\sigma_\infty$, completing the proof.

5. A congruence subgroup of $S(D)$.

5.1. We denote by Γ^* the subgroup in E' consisting of those elements d' such that $A(d, d') \equiv \text{mod } 1$ for any $d \in \Gamma$. Obviously, $\Gamma^* \supset \Gamma$. In terms of the base $\{d_1, \ldots, d_n, d_{n+1}, \ldots, d_{2n}\}$ of D adapted to A in 1.1, $\{d_1, \ldots, d_n\}$ is a base of Γ^* and $\{d_{n+1}/e_1, \ldots, d_{2n}/e_n\}$ is a base of Γ^*. Let us write an automorphism σ of the group G belonging to the sympletic group S as in (10); $(e^{u'} \cdot e^{u}) = e^{g_\sigma(u,u')} \cdot e^{u\cdot\beta+u'\cdot\delta} \cdot e^{u\cdot\alpha+u'\cdot\gamma}$, where $(u, u')\cdot\sigma = (u\cdot\alpha + u'\cdot\gamma, u\cdot\beta + u'\cdot\delta)$. In this section, we are concerned with an automorphism $\sigma \in S$ satisfying the following conditions:

$$
\text{(13)} \qquad \begin{aligned}
&\text{(i)} && D\cdot\sigma = D,\\
&\text{(ii)} && (\Gamma+\Gamma^*)\cdot\sigma = \Gamma+\Gamma^*,\\
&\text{(iii)} && g_\sigma(d, d') \equiv 0 \mod 1 \quad \text{if } d \in \Gamma,\ d' \in \Gamma^*.
\end{aligned}
$$

LEMMA. *Suppose that σ is in the congruence subgroup $S(2e)$ of $S(D)$ and is expressed as $(u, u')\sigma = (u\cdot\alpha + u'\cdot\gamma, u\cdot\beta + u'\cdot\delta)$. Then σ satisfies the above three conditions* (13). *Moreover, if $d_a = \sum_{\alpha=1}^n a_\alpha\, (d_\alpha/e_\alpha)$, $a_\alpha \in Z$, then*

$$
\text{(14)} \qquad d_\alpha\cdot\sigma - d_\alpha \in 2(\Gamma+\Gamma^*),
$$

and

$$
\text{(15)} \qquad \tfrac{1}{2}A(d_\alpha\cdot\alpha, d_\alpha\cdot\beta) \equiv 0 \mod 1.
$$

Proof. If we write

$$
d_i\cdot\sigma - d_i = \sum_{j=1}^{2n} m_{j,i}\, d_j, \qquad i = 1, \ldots, n,
$$

then $m_{ji} \equiv 0 \text{ mod } 2e$ by our assumption. It follows that

$$
\text{(16)} \qquad \begin{aligned}
d_\alpha\cdot\sigma - d_\alpha &= \sum_{\beta=1}^{n} m_{\beta,\alpha}\, d_\beta + \sum_{\beta=1}^{n} m_{n+\beta,\alpha} e_\beta \frac{d_{n+\beta}}{e_\beta} \in 2(\Gamma+\Gamma^*),\\
\frac{d_{n+\alpha}}{e_\alpha}\cdot\sigma - \frac{d_{n+\alpha}}{e_\alpha} &= \sum_{\beta=1}^{n} \frac{m_{\beta,n+\alpha}}{e_\alpha}\, d_\beta + \sum_{\beta=1}^{n} m_{n+\beta,n+\alpha} \frac{e_\beta}{e_\alpha} \frac{d_{n+\beta}}{e_\beta} \in 2(\Gamma+\Gamma^*).
\end{aligned}
$$

Thus, σ satisfies (i) and (ii) in (13). Since $d\cdot\alpha + d'\cdot\gamma - d \in 2\cdot\Gamma$ and $d\cdot\beta + d'\cdot\delta - d' \in 2\cdot\Gamma^*$,

$$
g_\sigma(d, d') = \tfrac{1}{2}A(d, d') - \tfrac{1}{2}A(d\cdot\alpha + d'\cdot\gamma, d\cdot\beta + d'\cdot\delta) \equiv 0 \mod 1,
$$

which shows that σ satisfies (iii) in (13). From (16), it follows that

$$
\frac{d_\alpha}{e_\alpha}\cdot\sigma - \frac{d_\alpha}{e_\alpha} = \sum_{\beta=1}^{n} \frac{m_{\beta,\alpha}}{e_\alpha}\, d_\beta + \sum_{\beta=1}^{n} \frac{m_{n+\beta,\alpha} e_\beta}{e_\alpha} \frac{d_{n+\beta}}{e_\beta}.
$$

Thus,

$$
\frac{d_\alpha}{e_\alpha}\cdot\sigma - \frac{d_\alpha}{e_\alpha} \in 2(\Gamma+\Gamma^*).
$$

Finally,

$$\tfrac{1}{2}A(d_\alpha\cdot\alpha,\ d_\alpha\cdot\beta) = \tfrac{1}{2}A(d_a\cdot\sigma,\ d_\alpha\cdot\beta) \equiv \tfrac{1}{2}A(d_a,\ d_a\cdot\beta) \quad \text{mod } 1.$$

Applying the fact that $d_\alpha/e_\alpha = \sum_{\beta=1}^{n} (m_{n+\beta,\alpha}/e_\alpha)\cdot d_{n+\beta}$, we see that $\tfrac{1}{2}A(d_a\cdot\alpha,\ d_a\cdot\beta)\equiv 0$ mod 1, completing the proof.

5.2. PROPOSITION 4. *Let us denote by U a unitary operator on $L^2(E)$ which induces an automorphism σ of G belonging to $\boldsymbol{S}(2e)$ and also the isomorphism of the space (S') of tempered distributions onto itself determined canonically by U. If t is a theta distribution associated to the semicharacter ψ_0 defined in* (2), *then $Ut=\gamma\cdot t$ with a scalar λ of absolute value* 1 *which is independent on t.*

This proposition generalizes slightly a theorem of Weil [4, Théorème 4], a generalized Poisson summation formula, in the case where the base field is $\boldsymbol{R}$. Indeed, if the elementary divisors $e_1, \ldots, e_n$ of A relative to D are all 1, this proposition reduces to the result by Weil for the case where the base field is $\boldsymbol{R}$. We modify his proof using some of his results [4, §§16–19].

Proof. We start with a resume of Weil's results. Let E and E' be the maximal isotropy subspaces of V with respect to A defined in 1.1; $V=E+E'$, $E\cap E'=(0)$. We regard E and E' the dual groups of each other with respect to a coupling $\langle u, u'\rangle = e[A(u, u')]$, $u\in E$, $u'\in E'$. The dual group of Γ is E'/Γ^* and that of Γ^* is E/Γ. Let du be a usual Lebesgue measure on E and let du' be a Lebesgue measure on E' such that we have Plancherel formula

$$\int |\varphi(u)|^2\, du = \int |F\cdot\varphi(u')|^2\, du',$$

where $F\cdot\varphi$ is the Fourier transform of φ; $F\cdot\varphi(u') = \int \varphi(u)\langle u, u'\rangle\, du$.

Let K be a Hilbert space consisting of $\boldsymbol{C}$-valued everywhere locally integrable function Φ on $E\times E'$ satisfying the functional equality

$$\Phi(u+d,\ u'+d') = \langle u, u'\rangle\Phi(du') \tag{17}$$

almost everywhere on $E\times E'$ for each $(d, d')\in\Gamma\times\Gamma^*$, and having a finite norm

$$\|\Phi\| = \int_{E/\Gamma\times E'/\Gamma^*} |\Phi|^2\, du\cdot du'.$$

Since $|\Phi(u+d,\ u'+d')| = |\Phi(u, u')|$, $(d, d')\in\Gamma\times\Gamma^*$, $|\Phi|$ induces a locally integrable function on $E/\Gamma\times E'/\Gamma^*$ with respect to the induced measure $du\cdot du'$ on the space. To each $\Phi\in L^2(E)$, we assign a function Φ in K given by

$$\Phi(u, u') = \sum_{d\in\Gamma} \varphi(u+d)\langle d, u'\rangle. \tag{18}$$

Then, the correspondence $Z\colon \varphi\to\Phi$ is an isomorphism of the Hilbert space $L^2(E)$ onto the Hilbert space K. Take an element $e^{v'}\in G$. Then

$$((Z\cdot\tau(e^{v'}\cdot e^{v})\cdot Z^{-1})\Phi)(u, u') = \langle u, v'\rangle\Phi(u+v,\ u'+v'), \qquad u\in E,\ u'\in E'. \tag{19}$$

Consider the subgroup $\boldsymbol{S}_0$ of automorphisms σ of G belonging to $\boldsymbol{S}$ and satisfying the conditions (i) and (iii) in (13). Weil defines a unitary representation of $\boldsymbol{S}_0$ on the Hilbert space K [4, §19] as follows: To each $\sigma \in \boldsymbol{S}_0$, we assign a unitary operator $\gamma(\sigma)$ given by

$$(\gamma(\sigma)\Phi)(u, u') = e[g_\sigma(u, u')]\Phi((u, u')\cdot\sigma), \qquad (u, u') \in E \times E'. \tag{20}$$

We see easily that the unitary operator $V(\sigma)=Z^{-1}\cdot\gamma(\sigma)\cdot Z$ of $L^2(E)$ induces the automorphism σ of G.

Now, let σ be an element in the group $\boldsymbol{S}(2e)$, and let ξ be an element in E which is a linear combination of $d_1/e_1, \ldots, d_n/e_n$ with integral coefficients. Since $V(\sigma)$ induces the automorphism σ of the group G,

$$V(\sigma)\cdot\tau(e^\xi)\cdot V(\sigma)^{-1} = e[\tfrac{1}{2}A(\xi\cdot\alpha_1, \xi\cdot\beta_1)]\tau(e^{\xi\cdot\beta_1}\cdot e^{\alpha_1}),$$

where $(u, u')\sigma^{-1}=(u\cdot\alpha_1+u'\cdot\alpha_1, u\cdot\beta_1+u'\cdot\delta_1)$. By the equality (15) in Lemma, we have $V(\sigma)\cdot\tau(e^\xi)\cdot V(\sigma)^{-1}=\tau(e^{\xi\cdot\beta_1}\cdot e^{\xi\cdot\alpha_1})$. Take an arbitrary rapidly decreasing C^∞-function φ on E. Then

$$(Z\cdot V(\sigma)\cdot\tau(e^\xi))\cdot\varphi = (Z\cdot\tau(e^{\xi\cdot\beta_1}\cdot e^{\xi\cdot\alpha_1})\cdot V(\sigma))\varphi. \tag{21}$$

The left-hand side of the above equality is $(\gamma(\sigma)\cdot Z\cdot\tau(e^\xi))\varphi$, which turns out to be

$$e[g_\sigma(u, u')]\sum_{d\in\Gamma}\varphi(\xi+d+u\cdot\alpha+u'\cdot\gamma)\langle d, u\cdot\beta+u'\cdot\delta\rangle$$

by the definition of $\tau(e^\xi)$, (18) and (20). The right-hand side of the equality (21) is

$$\sum_{d\in\Gamma} e[A(u+d, \xi\cdot\beta_1)](V(\sigma)\cdot\varphi)(u+d+\xi\cdot d_1)\langle d, u'\rangle. \tag{22}$$

Applying (14) in Lemma, we see that $A(d\cdot\xi\cdot\beta_1)=A(d, \xi\cdot\sigma^{-1})\equiv 0 \bmod 1$ for $d\in\Gamma$, and that $\xi_{\alpha_1}-\xi=\eta\in\Gamma$. Thus (22) becomes

$$\sum_{d\in\Gamma} e[A(u, \xi\cdot\beta_1)](V(\sigma)\cdot 0)(u+d+\eta+\xi)\langle d, u'\rangle.$$

Finally, evaluating the values at $(0.0)\in E\times E'$ of both sides of the equality (21), we see that

$$\sum_{d\in\Gamma}\varphi(\xi+d) = \sum_{d\in\Gamma}(V(\sigma)\varphi)(\xi+d). \tag{23}$$

In 3.2 we have observed that an arbitrary theta distribution associated to the semicharacter ψ_0 is a linear combination of

$$t_k = \sum_{d\in\Gamma}\delta_{d_{a(k)}+d}, \qquad k = 1, \ldots, e,$$

where $d_{a(k)}=\sum_{\alpha=1}^n (a_\alpha/e_\alpha)d_\alpha$, $a_\alpha\in Z$. The above equality (23) implies $t_k\cdot\varphi=t_k\cdot(U(\sigma)\cdot\varphi)$ for any $\varphi\in(S)$, $k=1,\ldots,e$. This means that if t is a theta distribution, then $V(\sigma)\cdot t=t$. If a unitary operator U includes an automorphism σ of G, then

$U=\lambda\cdot V(\sigma)$, $|\lambda|=1$. Thus, we have proved that $U\cdot t=\lambda\cdot t$ for an arbitrary theta distribution associated to ψ_0, completing the proof.

5.3. On account of Proposition 1 and 2, the proof of Theorems B and C is to be completed if we determine the theta functions of type (H_J, ψ_0) corresponding to the theta distributions $t_1, \ldots, t_k$ under the isomorphism $i_J\colon (S') \to F_{J-\infty}$.

We state the following two facts: The first one is that t_k is written as

$$\sum_{d\in\Gamma} \tau(e^{d+d}a(k))\delta_0$$

with the Dirac distribution δ_0 at the origin $0 \in E$; and the second one is that a theorem of Cartier [1, Theorem 1] asserts that the one-dimensional subspace in (S') spanned by δ_0 is characterized as the set of solutions t in (S') of the system of equations

$$\mathring{\tau}(u')\cdot t = 0, \qquad u' \in E',$$

where $\mathring{\tau}$ is the respresentation of the Lie algebra of G corresponding to the group representation $((S'), \tau)$. Let us determine the one-dimensional subspace in $F_{J-\infty}$ consisting of the solutions φ of the system of equations

$$\mathring{\rho}(u')\cdot\varphi = \varphi, \qquad u' \in E'. \tag{24}$$

As is shown in (3), 2.3,

$$\mathring{\rho}(u')\varphi = \theta_{u'}\cdot\varphi - \pi\cdot H_{J,u'}\cdot\varphi.$$

If $\{d_1, \ldots, d_n, d_{n+1}, \ldots, d_{2n}\}$ is the base of G adapted to A in 1.1, $\{d_{n+1}, \ldots, d_{2n}\}$ is a $\boldsymbol{C}$-base of the complex vector space V_J. Let $\{z_1, \ldots, z_n\}$ be the complex cartesian coordinates of V_J; $v=\sum_{\alpha=1}^n z_\alpha(v)\cdot d_{n+\alpha}$. Put

$$H_J(d_{n+\alpha}, d_{n+\beta}) = h_{\alpha\beta}, \qquad \alpha, \beta = 1, \ldots, n.$$

Then

$$\begin{aligned} h_{\alpha\beta} &= A(d_{n+\alpha}, d_{n+\beta}\cdot J) + (-1)^{1/2}A(d_{n+\alpha}, d_{n+\beta}) \\ &= A(d_{n+\alpha}, d_{n+\beta}\cdot J) \in \boldsymbol{R}, \quad \text{and} \quad h_{\alpha\beta} = h_{\beta\alpha}. \end{aligned}$$

Clearly, $H_{J,\, d_{n+\alpha}}(\sum_{\beta=1}^n z_\beta\cdot d_{n+\beta}) = \sum_{\beta=1}^n h_{\alpha\beta}\cdot z_\beta$. In terms of the cartesian coordinates, the system of equations (24) is written as

$$\frac{\partial\varphi}{\partial z_\alpha} = \pi\cdot\left(\sum_{\beta=1}^n h_{\alpha\beta}\cdot z_\beta\right)\cdot\varphi, \qquad \alpha = 1, \ldots, n. \tag{25}$$

We can easily see that a holomorphic function on V_J

$$\varphi = \lambda\cdot\exp\left(1/2\pi \sum_{\alpha,\beta=1}^n h_{\alpha\beta}z_\alpha z_\beta\right), \qquad \lambda \in \boldsymbol{C},$$

satisfies the system (25). The uniqueness of a holomorphic solution, up to a multiplicative constant, is easily proved by making use of the power series expansion of a holomorphic solution. From the theorem of Cartier mentioned above, we see that the one-dimensional subspace of solutions of the system (25) in $F_{J-\infty}$ is spanned by the holomorphic function exp $(1/2\pi\cdot\Phi_J(xx))$ where

$$\Phi_J\left(\sum_{\alpha=1}^{n} z_\alpha\, d_{n+\alpha},\ \sum_{\alpha=1}^{n} z_\alpha\, d_{n+\alpha}\right) = \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta} z_\alpha z_\beta.$$

Now, it is obvious that the theta function $\varphi_k(J)$ which is the image of the theta distribution t_k under i_J is given by

$$\begin{aligned}\sum_{d\in\Gamma} \rho_J(e^{d+d}a(k))\cdot\exp \pi/2\Phi_J(x, x)\\ = \sum_{d\in\Gamma} \exp \pi/2\{H_J(d_{a(k)}+d,\ d_{a(k)}+d)+2H_J(d_{a(k)}+d,\ x)\\ +\Phi_J(d_{a(k)}+d+x,\ d_{a(k)}+d+x)\},\end{aligned}$$

as is shown in Theorem B. Thus, we have finished the proof of Theorems B and C.

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