ON A CLASS OF STOCHASTIC PROCESSES WITH TWO STATES AND CONTINUOUS TIME PARAMETER

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0. **Introduction.** In this paper we consider those two state continuous time parameter stochastic processes which have the property that any future and past events are independent given the present, the time elapsed since the last jump and the holding time distribution at the last jump. This is similar to the two state stochastic processes discussed by H. P. McKean, Jr. [6] and D. P. Johnson [4] for which any future and past events are independent given the present and the distribution of the present. The first half of this paper develops the general theory. The second half contains, in the form of our main theorem, a concrete application.

To be more precise, let E be the set of integers $\{+1, -1\}$, Ω the set of all right continuous functions ω mapping the nonnegative real numbers R_+ into E, $\tau_k(\omega)$ the time of the kth jump of ω, $\tau_\infty = \lim_{k\to\infty} \tau_k$, and $m_s(\omega)$ the number of jumps of ω in the interval $[0, s]$. If

$$\begin{aligned} x_t(\omega) &= \omega(t), \qquad t < \tau_\infty(\omega) \\ &= +\infty, \qquad t \geqq \tau_\infty(\omega), \end{aligned}$$

$\mathcal{N}_t^*$ is the σ-field generated by x_s, $s \leqq t$, $\mathcal{N}_k$ is the σ-field consisting of all $\Lambda \in \mathcal{N}_\infty^*$ for which $\Lambda \cap (\tau_k < t) \in \mathcal{N}_t^*$, $t \in R_+$, $\mathcal{N}$ is the smallest σ-field containing $\mathcal{N}_k$, $k=0, 1, 2, \ldots$, and if P is a probability measure on $\mathcal{N}$; then the collection $X=(\Omega, x_t, \mathcal{N}, P)$ will be called a characteristic process.

The main result of this paper is the following theorem:

THEOREM 0.1. *The following two classes of characteristic processes are identical.*

I. *The class ${}_n\mathscr{C}_{\mathrm{I}}$ of characteristic processes $X=(\Omega, x_t, \eta, P)$, where the measure P is defined in the following manner. Let $X^\xi=(\Omega, x_t, \eta, P^\xi)$ be a Markov chain with holding time distribution* $\exp(-\xi t)$ *from both* $+1$ *and* -1. *Let α_e, $e \in E$ be mappings of R_+ into itself which are bounded, nonnegative, monotone increasing and for which $\int_0^\infty d\alpha_e(\xi)=1$, and let δ be the mapping of Ω onto itself defined by*

$$\begin{aligned} \delta(\omega)(t) &= \quad \omega(0) \quad \textit{if } 2kn \leqq m_t(\omega) < (2k+1)n \textit{ for some } k \geqq 0, \\ &= -\omega(0) \quad \textit{otherwise.} \end{aligned}$$

Define for each $B \in \mathcal{N}$

$$P(B \cap (x_0 = e)) = \int_0^\infty P^\xi(\delta^{-1}(B))\, d\alpha_e(\xi).$$

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II. *The class* ${}_n\mathscr{C}_{\mathrm{II}}$ *of all characteristic processes for which* $P(\tau_m=\infty \mid \mathscr{N}_{m-1})=0$, $P(\tau_{m+1}>\tau_m+t \mid \mathscr{N}_{m-1}, \tau_m=s)$ *has derivatives almost everywhere of all orders in* $s, t \in R_+$, *and*

$$(1)\quad P(\tau_{m+1} \leqq \tau_m+s \mid \mathscr{N}_m, \tau_{m+2}) = \frac{\int_{\tau_m}^{\tau_m+s} (\tau_m-\xi)^{n-1}(\tau_{m+2}-\xi)^{n-1}\,d\xi}{\int_{\tau_m}^{\tau_{m+2}} (\tau_m-\xi)^{n-1}(\tau_{m+2}-\xi)^{n-1}\,d\xi}, \quad a.e.$$

for each $m \geqq 0$ *and all* $s \in [0, \tau_{m+2}-\tau_m]$.

(See [2] for a proof, using different methods that ${}_1\mathscr{C}_{\mathrm{I}}$ is identical to the class of all processes for which $P(m_s=m_r+1 \mid m_t=m_r+1)=(s-r)(t-r)^{-1}$ whenever $r<s<t$.)

In proving this theorem we will proceed as follows: in §1 we prove some general theorems on characteristic processes; in §2 we associate families of operators with classes of characteristic processes and study the relationship between these operators and the characteristic processes they represent; and finally, in §§3, 4, and 5 we prove the main theorem.

1. **Characteristic processes.** A function $h=h(\omega; m, t)$ mapping $\Omega \times Z_+ \times R_+$, Z_+ being the set of all nonnegative integers, into $[0, 1]$ will be called a holding functional if there exists a characteristic process $X=(\Omega, x_t, \mathscr{N}, P)$ for which

$$h(\omega; m, t) = P(\tau_{m+1} > \tau_m+t \mid \mathscr{N}_m), \quad \text{a.e.}$$

The following theorem is basic.

THEOREM 1.1. *Let* $X=(\Omega, x_t, \mathscr{N}, P)$ *be a characteristic process and let* h *be its holding functional. If* ψ *is a* $\mathscr{N}_{m+n}$ *measurable function and if* $\omega_e(u_1, \dots, u_k)$ *is a sample function* ω *with* $\omega(0)=e$ *and* $\tau_i(\omega)=u_i$, $i \leqq k$, *then*

$$E(\psi I_{[\tau_{m+n}<\infty]} \mid \mathscr{N}_m) = (-1)^n \int_{\tau_m \leqq u_1 \leqq \dots \leqq u_n} \cdots \int \psi[\omega_{x_0}(\tau_1, \dots, \tau_m, u_1, \dots, u_n)]$$
$$\cdot h(\omega; m, du_1-\tau_m) \prod_{k=1}^{n-1} h[\omega_{x_0}(\tau_1, \dots, \tau_m, u_1, \dots, u_k);$$
$$m+k; du_{k+1}-u_k], \quad a.e.$$

where I_A *is the indicator function of the set* A.

Proof. We first note that if ψ is $\mathscr{N}_m$ measurable, then ψ is a function of $x_0, \tau_1, \dots$ τ_m only; see [3, p. 87] for a proof of this. Thus the theorem holds for $n=1$. Suppose, the theorem holds for n. If ψ is $\mathscr{N}_{m+n+1}$ measurable, then using the fact that

$$E(\psi I_{[\tau_{m+n+1}<\infty]} \mid \mathscr{N}_m) = E[E(\psi I_{[\tau_{m+n+1}<\infty]} \mid \mathscr{N}_{m+n}) \mid \mathscr{N}_m] \quad \text{a.e.},$$

the reader can easily complete the proof by induction on n.

COROLLARY 1.2. *If* $X=(\Omega, x_t, \mathscr{N}, P)$ *is a characteristic process, then the distribution of* X *is uniquely determined by its holding functional and* $P(x_0=e)$.

From Tulcea's Theorem [5], we have

COROLLARY 1.3. *Necessary and sufficient conditions that a function* $h=h(\omega; m, t)$ *mapping* $\Omega \times Z_+ \times R_+ \to [0, 1]$ *be a holding functional are*

(i) *$h(\omega; m, t)$ is a $\mathscr{N}_m$ measurable function of ω for each m, t.*

(ii) *For almost all ω and for each $m \in Z_+$, $h(\omega; m, 0)=1$ and $h(\omega; m, t)$ is a decreasing right continuous function of t.*

2. **Linear families of operators.** Let $X=(\Omega, x_t, \mathscr{N}, P)$ be a characteristic process with associated holding functional h. Suppose that $\mathscr{F}$ is the set of all right continuous functions mapping R_+ into R_+, and that for each $s \in R_+^* = R_+\backslash\{0\}$, V_s is an operator mapping a subset $\mathscr{F}(V)$ of $\mathscr{F}$ into $\mathscr{F}(V)$ such that $h(\omega; i, \cdot) \in \mathscr{F}(V)$, $V_s h(\omega; i, \cdot)(0) \neq 0$, and

$$h(\omega; i, t) = \langle V_{\tau_i - \tau_{i-1}} h(\omega; i-1, \cdot)\rangle(t), \quad \text{a.e.} \tag{2}$$

where $\langle f\rangle(t)=f(t)/f(0)$. The set of all characteristic processes which satisfy (2) will be denoted by $\mathscr{C}(V)$. $\mathscr{D}(V)$ will be the set of all functions $f \in \mathscr{F}(V)$ for which there exists a characteristic process $X=(\Omega, x_t, \mathscr{N}, P) \in \mathscr{C}(V)$ and an $\tilde{\omega} \in \Omega$ with $P[x_0 = x_0(\tilde{\omega})] > 0$, such that $f(t)=h(\tilde{\omega}; 0, t)$. By Corollary 1.2, there corresponds to each function $f \in \mathscr{D}(V)$ and $e \in E$ a unique characteristic process $X=(\Omega, x_t, \mathscr{N}, P)$ contained in $\mathscr{C}(V)$ for which $P(x_0=e)=1$ and $h(\omega; 0, t)=f(t)$, a.e. The process X and the corresponding measure P will be denoted by the symbols $B_e f$ and $A_e f$ respectively. We therefore have

THEOREM 2.1. *Let $V=V_s$, $s \in R_+^*$ be a family of operators mapping a subset $\mathscr{F}(V)$ of $\mathscr{F}$ into $\mathscr{F}(V)$. Then to each function $f \in \mathscr{D}(V)$ and $e \in E$, there corresponds a unique characteristic process $B_e f=(\Omega, x_t, \mathscr{N}, A_e f)$ contained in $\mathscr{C}(V)$ for which $P(x_0=e)=1$ and $f(t)=h(\omega; 0, t)$, a.e. Conversely, if $X=(\Omega, x_t, \mathscr{N}, P) \in \mathscr{C}(V)$ and if $P(x_0=e)=1$, then $X=B_e f$ where $f(t)=P(\tau_1 > t)$.*

We will consider the operators A_e, $e \in E$ of the last theorem as mappings of $\mathscr{D}(V)$ into ca $(\Omega, \mathscr{N})$, the linear space of all measures on the measurable space $(\Omega, \mathscr{N})$. A family of operators $V=V_s$, $s \in R_+^*$ mapping a subset $\mathscr{F}(V)$ of $\mathscr{F}$ into $\mathscr{F}(V)$ will be called linear if:

(i) $\mathscr{F}(V)$ is a linear subspace of $\mathscr{F}$ and $\mathscr{D}(V)$ is convex,

(ii) V_s is a linear operator for each $s \in R_+^*$,

(iii) The corresponding operators A_e, $e \in E$ are affine.

A family of operators V_s, $s \in R_+^*$ will be called singular if for some $s \in R_+^*$, $\langle V_s f\rangle = \langle V_s g\rangle$ for all $f, g \in \mathscr{D}(V)$.

THEOREM 2.2. *Let $V=V_s$, $s \in R_+^*$ be a nonsingular family of linear operators mapping a linear subspace $\mathscr{F}(V)$ of $\mathscr{F} \cap C^\infty$, C^∞ being the set of all infinitely differentiable functions on R_+, into $\mathscr{F}(V)$. Suppose that $\mathscr{D}(V)$ is convex and that for each $f \in \mathscr{F}(V)$ and $t \in R_+$, $V_s f(t)$ is continuous in s. Then a necessary and sufficient condition that the operators A_e, $e \in E$ be affine, is that there exist a function $a(\cdot)$ such that for all $f \in \mathscr{D}(V)$, $D_s f(s)=a(s)V_s f(0)$.*

Proof. To prove the necessity, assume that the operators A_e are affine. If $f, g \in \mathscr{D}(V)$, $\alpha \in [0, 1]$ and $e \in E$, and if h_f, h_g and $h_{\alpha f+(1-\alpha)g}$ are the holding functions associated with $B_e f$, $B_e g$ and $B_e(\alpha f+(1-\alpha)g)$ respectively, then by Theorem 1.1 and our assumptions of the continuity of $V_s f(t)$ in s and t, we must have

$$[D_s h_{\alpha f+(1-\alpha)g}(\omega_e; 0, s)][D_t h_{\alpha f+(1-\alpha)g}(\omega_e(s); 1, t)]$$
$$= \alpha[D_s h_f(\omega_e; 0, s)][D_t h_f(\omega_e(s); 1, t)]+(1-\alpha)[D_s h_g(\omega_e; 0, s)][D_t h_g(\omega_e(s); 1, t)]$$

or, in terms of the operator V_s,

$$[\alpha D_s f(s)+(1-\alpha)D_s g(s)][\alpha D_t V_s f(t)+(1-\alpha)D_t V_s g(t)]/\alpha V_s f(0)+(1-\alpha)V_s g(0)$$
$$= \alpha[D_s f(s)]D_t V_s f(t)/V_s f(0)+(1-\alpha)[D_s g(s)]D_t V_s g(t)/V_s g(0).$$

This last equation can be rewritten as

$$(-\alpha^2+\alpha)[V_s f(0)D_t V_s g(t)-V_s g(0)D_t V_s f(t)][V_s g(0)D_s f(s)-V_s f(0)D_s g(s)] = 0.$$

Thus for each pair of functions $f, g \in \mathscr{D}(V)$ and $s \in R_+^*$, either

$$[D_s f(s)]/V_s f(0) = [D_s g(s)]/V_s g(0)$$

or

$$D_t V_s f(t)/V_s f(0) = D_t V_s g(t)/V_s g(0), \quad \text{all } t \in R_+.$$

This implies that for each $s \in R_+^*$, either

$$[D_s f(s)]/V_s f(0) = [D_s g(s)]/V_s g(0), \quad \text{all } f, g \in \mathscr{D}(V)$$

or

$$D_t V_s f(t)/V_s f(0) = D_t V_s g(t)/V_s g(0), \quad \text{all } f, g \in \mathscr{D}(V) \quad \text{and} \quad t \in R_+.$$

But the family of operators V is nonsingular and so the first equality must hold. Letting $a(s)$ equal $[D_s f(s)]/V_s f(0)$, the necessity is proven. To prove sufficiency, we need only show that the functional

$$\Gamma(f) = [D_{u_1} f(u_1)] \prod_{k=1}^{m-1} [D_{u_{k+1}} h_f[\omega_e(u_1, \ldots, u_k); k, u_{k+1}-u_k]]$$

satisfies

$$\Gamma(\alpha f+(1-\alpha)g) = \alpha\Gamma(f)+(1-\alpha)\Gamma(g), \quad f, g \in \mathscr{D}(V) \text{ and } \alpha \in [0, 1].$$

The reader can easily verify that this is the case.

If V_s, $s \in R_+^*$ is a nonsingular linear family of operators with $\mathscr{F}(V) \subseteq C^\infty$ and $V_s f(t)$ continuous in s, then since we are interested in the value of $\langle V_s f\rangle$ and not $V_s f$, we can and will assume that $D_s f(s) = V_s f(0)$, all $f \in \mathscr{F}(V)$.

THEOREM 2.3. *Let $V = V_s$, $s \in R_+^*$ be a nonsingular family of linear operators mapping a linear subspace $\mathscr{F}(V)$ of $\mathscr{F} \cap C^\infty$ into $\mathscr{F}(V)$. Suppose that $D_s f(s) = V_s f(0)$ for all $f \in \mathscr{F}(V)$ and that for each $f \in \mathscr{F}(V)$ and $t \in R_+$, $V_s f(t)$ is continuous in s. Then a necessary and sufficient condition that $f \in \mathscr{D}(V)$ is that $f(0)=1$ and*

$$(3) \qquad (-1)^n V_{s_n} \cdots V_{s_1} f(t) \geqq 0$$

for all choices of $n, s_1, \ldots, s_n, t$.

Proof. If $f \in \mathscr{D}(V)$, then, as the reader can easily check,

$$h[\omega(s_1, s_1+s_2, \ldots, s_1+\cdots+s_n); n, \cdot] = \langle V_{s_n} \cdots V_{s_1} f\rangle$$

and hence:

(i) $V_{s_n} \cdots V_{s_1} f$ is either positive and monotone decreasing or negative and monotone increasing.

(ii) $D_s V_{s_n} \cdots V_{s_1} f(s) = V_s V_{s_n} \cdots V_{s_1} f(0)$.

Clearly $D_s f(s) \leqq 0$ and hence $V_s f(0) = D_s f(s) \leqq 0$ which implies by (i) above that $V_s f(t) \leqq 0$. Thus (3) is proved for $n=1$. Suppose (3) holds for n. Then (i) implies that $(-1)^n V_{s_n} \cdots V_{s_1} f$ is monotone decreasing and hence $(-1)^n D_s V_{s_n} \cdots V_{s_1} f \leqq 0$. Thus by (ii), $(-1)^{n+1} V_s V_{s_n} \cdots V_{s_1} f(0) \geqq 0$. Using (i) once again we see that $(-1)^{n+1} V_s V_{s_n} \cdots V_{s_1} f(t) \geqq 0$ which proves the necessity of (3). To prove sufficiency, suppose that f is in the domain of V and satisfies (3). Then $(-1)^n D_s V_{s_n} \cdots V_{s_1} f(s) \leqq 0$ and thus the functions $\langle V_{s_n} \cdots V_{s_1} f\rangle$ are nonnegative and decreasing. It follows from Corollary 1.3 that the functionals

$$h(\omega; m, t) = \langle V_{\tau_m - \tau_{m-1}} \cdots V_{\tau_1} f\rangle(t)$$

are holding functionals and thus $f \in \mathscr{D}(V)$.

THEOREM 2.4. *Suppose that $V = V_s$, $s \in R_+^*$ is a nonsingular linear family of operators whose domain $\mathscr{F}(V)$ is contained in C^∞. If $X = (\Omega, x_t, \mathscr{N}, P) \in \mathscr{C}(V)$, $f(\cdot) = h(\omega; 0, \cdot)$, a.e., and if ψ is a $\mathscr{N}_{m+n}$ measurable function, then*

$$E(\psi I_{[\tau_{m+n} < \infty]} \mid \mathscr{N}_m) = (-1)^n \int_{\tau_m}^\infty du_1 \int_{u_1}^\infty du_2 \cdots \int_{u_{n-1}}^\infty du_n \psi[\omega_{x_0}(\tau_1, \ldots, \tau_m, u_1, \cdots, u_n)]$$
$$\cdot V_{u_n - u_{n-1}} \cdots V_{u_1 - \tau_m} \langle V_{\tau_m - \tau_{m-1}} \cdots V_{\tau_1} f\rangle(0), \quad a.e.$$

Proof. Let h be the holding functional of X. Then

$$[D_{u_1} h(\omega; m, u_1 - \tau_m)] \prod_{k=1}^{n-1} [D_{u_{k+1}} h[\omega_{x_0}(\tau_1, \ldots, \tau_m, u_1, \ldots, u_k); m+k, u_{k+1} - u_k]]$$
$$= [D_{u_1} \langle V_{\tau_m - \tau_{m-1}} \cdots V_{\tau_1} f\rangle(u_1 - \tau_m)]$$
$$\cdot \prod_{k=1}^{n-1} [D_{u_{k+1}} \langle V_{u_k - u_{k+1}} \cdots V_{u_2 - u_1} V_{u_1 - \tau_m} \cdots V_{\tau_1} f\rangle(u_{k+1} - u_k)]$$
$$= V_{u_1 - \tau_m} \langle V_{\tau_m - \tau_{m-1}} \cdots V_{\tau_1} f\rangle(0)$$
$$\cdot \prod_{k=1}^{n-1} V_{u_{k+1} - u_k} \langle V_{u_k - u_{k-1}} \cdots V_{u_2 - u_1} V_{u_1 - \tau_m} \cdots V_{\tau_1} f\rangle(0)$$
$$= \frac{V_{u_1 - \tau_m} V_{\tau_m - \tau_{m-1}} \cdots V_{\tau_1} f(0)}{V_{\tau_m - \tau_{m-1}} \cdots V_{\tau_1} f(0)} \prod_{k=1}^{n-1} \frac{V_{u_{k+1} - u_k} \cdots V_{u_2 - u_1} V_{u_1 - \tau_m} \cdots V_{\tau_1} f(0)}{V_{u_k - u_{k-1}} \cdots V_{u_2 - u_1} V_{u_1 - \tau_m} \cdots V_{\tau_1} f(0)}$$
$$= \frac{V_{u_n - u_{n-1}} \cdots V_{u_2 - u_1} V_{u_1 - \tau_m} \cdots V_{\tau_1} f(0)}{V_{\tau_m - \tau_{m-1}} \cdots V_{\tau_1} f(0)}$$
$$= V_{u_n - u_{n-1}} \cdots V_{u_1 - \tau_m} \langle V_{\tau_m - \tau_{m-1}} \cdots V_{\tau_1} f\rangle(0), \quad \text{a.e.}$$

Now apply Theorem 1.1.

In the following corollary, integrals of functions with values in C^∞ are to be thought of as integrals of functions with values in the Banach space $C(R_+)$ of continuous functions mapping R_+ into R (see [1]).

COROLLARY 2.5. *Suppose that $V=V_s$, $s\in R_+^*$ is a nonsingular linear family of operators whose domain $\mathscr{F}(V)$ is contained in C^∞. Let $\mathscr{E}(V)$ be the set of all extremal points of $\mathscr{D}(V)$ and define the extremal processes of $\mathscr{C}(V)$ to be the processes B_ef, $f\in\mathscr{E}(V)$. If there exists a σ-field Σ for which every function $f\in\mathscr{D}(V)$ is the barycenter of some probability measure ν on $(\mathscr{E}(V),\Sigma)$, and if for each probability measure ν on $(\mathscr{E}(V),\Sigma)$ and $s_1,\dots,s_n\in R_+^*$,*

$$V_{s_n}\cdots V_{s_1}\int_{\mathscr{E}(V)} g\nu(dg) = \int_{\mathscr{E}(V)} (V_{s_n}\cdots V_{s_1}g)\nu(dg), \quad s\in R_+^*,$$

then every characteristic process $X=(\Omega, x_t, \mathscr{N}, P)\in\mathscr{C}(V)$ is a mixture of extremal processes. That is, if $X=(\Omega, x_t, \mathscr{N}, P)\in\mathscr{C}(V)$, then there exist measures ν_e, $e\in E$ on $(\mathscr{E}(V),\Sigma)$ and a measure μ on E, such that for all $\Lambda\in\mathscr{N}$,

$$P(\Lambda) = \int_E \mu(de)\int_{\mathscr{E}(V)} A_ef(\Lambda)\nu_e(df).$$

Proof. The corollary follows readily from the observation that if the operators $V_{s_n}\cdots V_{s_1}$, $s_1,\dots,s_n\in R_+^*$ commute with the integral, then so do the operators A_e, $e\in E$.

Proof of Theorem 0.1. Theorem 0.1 will be proved by showing that ${}_n\mathscr{C}_{\mathrm{I}}={}_n\mathscr{C}_{\mathrm{II}}=\mathscr{C}({}_nV)$ where ${}_nV={}_nV_s$, $s\in R_+^*$ is the nonsingular linear family of operators whose domain $\mathscr{F}(V)$ consists of the set of all infinitely differentiable functions on R_+ and for which

$${}_nV_sf(t) = (-s)^{n-1}\sum_{k=0}^{n-1}(-t)^kD_t^k[(-s-t)^{1-n}D_tf(s+t)]/k!.$$

For the duration of the paper, n will be fixed and we will write ${}_nV$ as V, ${}_n\mathscr{C}_{\mathrm{I}}$ as $\mathscr{C}_{\mathrm{I}}$ and ${}_n\mathscr{C}_{\mathrm{II}}$ as $\mathscr{C}_{\mathrm{II}}$. In §3 we calculate $\mathscr{D}(V)$. In §4 we show that $\mathscr{C}(V)=\mathscr{C}_{\mathrm{I}}$ and finally, in §5 we show that $\mathscr{C}(V)=\mathscr{C}_{\mathrm{II}}$.

3. Calculation of $\mathscr{D}(V)$.

THEOREM 3.1. *A necessary and sufficient condition that $f\in\mathscr{D}(V)$ is that*

$$f(t) = \int_0^\infty f_\lambda(t)\alpha(d\lambda), \tag{4}$$

where $\alpha(\lambda)$ is a nonnegative, bounded and nondecreasing function with $\int_0^\infty \alpha(d\lambda)=1$ and

$$f_\lambda(t) = e^{-\lambda t}\sum_{k=0}^{n-1}(\lambda t)^k/k!.$$

Proof. According to Theorem 2.3, a necessary and sufficient condition that $f \in \mathscr{D}(V)$ is that $f \in \mathscr{F}(V) = C^\infty(R_+)$, $f(0)=1$ and

$$(-1)^p V_{s_p} \cdots V_{s_1} f(t) \geqq 0$$

for all choices of $p, s_1, \ldots, s_p$ and t. But for each $f \in C^\infty(R_+)$

$$\begin{aligned}(5)\qquad D_t V_{s_p} \cdots V_{s_1} f(t) &= (-s_p)^{n-1} \cdots (-s_1)^{n-1}[(n-1)!]^{-p} t^{n-1} \\ &\quad \cdot D_t^{pn}[(t+s_1+\cdots+s_p)^{1-n} D_t f(t+s_1+\cdots+s_p)]\end{aligned}$$

for all $t \in R_+$, $s_1, \ldots, s_p \in R_+^*$ and $p \geqq 1$. This is clearly true for $p=1$ since

$$D_t V_s f(t) = (-s)^{n-1}(-t)^{n-1} D_t^n[(-s-t)^{1-n} D_t f(s+t)]/(n-1)!.$$

Assuming equation (5) to hold for p, we have

$$\begin{aligned}
&D_t V_{s_{p+1}} \cdots V_{s_1} f(t) \\
&= (-s_{p+1})^{n-1} D_t \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} D_t^k[(-t-s_{p+1})^{1-n} D_t V_{s_p} \cdots V_{s_1} f(t+s_{p+1})] \\
&= (-s_{p+1})^{n-1} \cdots (-s_1)^{n-1} D_t \sum_{k=0}^{n-1} \frac{(-t)^k}{k!\,[(n-1)!]^p} \\
&\qquad \cdot D_t^k[(-t-s_{p+1})^{1-n}(t+s_{p+1})^{n-1} D_t^{pn}[(t+s_1+\cdots+s_{p+1})^{1-n} \\
&\qquad\qquad \cdot D_t f(t+s_1+\cdots+s_{p+1})]] \\
&= (-s_{p+1})^{n-1} \cdots (-s_1)^{n-1} D_t \sum_{k=0}^{n-1} \frac{(-t)^k(-1)^{n-1}}{k!\,[(n-1)!]^p} \\
&\qquad \cdot D_t^{k+pn}[(t+s_1+\cdots+s_{p+1})^{1-n} D_t f(t+s_1+\cdots+s_{p+1})] \\
&= -(-s_{p+1})^{n-1} \cdots (-s_1)^{n-1} \sum_{k=1}^{n-1} \frac{(-t)^{k-1}(-1)^{n-1}}{(k-1)!\,[(n-1)!]^p} \\
&\qquad \cdot D_t^{k+pn}[(t+s_1+\cdots+s_{p+1})^{1-n} D_t f(t+s_1+\cdots+s_{p+1})] \\
&\quad + (-s_{p+1})^{n-1} \cdots (-s_1)^{n-1} \sum_{k=0}^{n-1} \frac{(-t)^k(-1)^{n-1}}{k!\,[(n-1)!]^p} \\
&\qquad \cdot D_t^{k+1+pn}[(t+s_1+\cdots+s_{p+1})^{1-n} D_t f(t+s_1+\cdots+s_{p+1})] \\
&= (-s_{p+1})^{n-1} \cdots (-s_1)^{n-1} t^{n-1} \\
&\qquad \cdot D_t^{(p+1)n}[(t+s_1+\cdots+s_{p+1})^{1-n} D_t f(t+s_1+\cdots+s_{p+1})]/[(n-1)!]^{p+1},
\end{aligned}$$

and so equation (5) holds for all $p \in Z_+$.

From the proof of Theorem 2.3 we see that if $f \in \mathscr{D}(V)$, then for each $p \in Z_+$, $t \in R_+$, $s_1, \ldots, s_p \in R_+^*$ we have

$$(-1)^p D_t V_{s_p} \cdots V_{s_1} f(t) \leqq 0$$

and

$$-\int_0^\infty (-1)^p D_t V_{s_p} \cdots V_{s_1} f(t)\, dt < \infty.$$

Thus letting

$$F_s(t) = -(n-1)!(s+t)^{1-n}D_t f(s+t), \qquad s \in R_+^*, \quad t \in R_+,$$

we see from equation (5) that if $f \in \mathscr{D}(V)$, then F_s is nonnegative and has the properties:

$$(6) \qquad (-D_t)^{pn}F_s(t) \geqq 0, \qquad t \in R_+, \quad p \in Z_+$$

and

$$(7) \qquad \int_0^\infty \xi^{n-1}(-D_\xi)^{pn}F_s(\xi)\, d\xi < \infty, \qquad p \in Z_+.$$

We wish to conclude that F_s is a completely monotone function for each $s \in R_+^*$. Let $p \in Z_+$ be fixed. Then from inequality (6) we see that there exists a $T \in R_+^*$ such that for each $0 \leqq k \leqq n$, $(-D_t)^{pn+k}F_s(t)$ has the same sign for all $t \geqq T$. But it follows from inequality (7) that $(-D_t)^{pn}F_s(t) \to 0$ as $t \to \infty$. Thus for $t \geqq T$, $(-D_t)^{pn+1}F_s(t) \geqq 0$. But

$$\begin{aligned} 0 \leqq t^n(-D_t)^{pn}F_s(t) &= \int_0^t D_\xi \xi^n(-D_\xi)^{pn}F_s(\xi)\, d\xi \\ &= n\int_0^t \xi^{n-1}(-D_\xi)^{pn}F_s(\xi)\, d\xi + \int_0^t \xi^n(-D_\xi)^{pn}D_\xi F_s(\xi)\, d\xi \end{aligned}$$

and so

$$\int_0^t \xi^n(-D_\xi)^{pn+1}F_s(\xi)\, d\xi \leqq n\int_0^t \xi^{n-1}(-D_\xi)^{pn}F_s(\xi)\, d\xi.$$

Thus

$$\int_0^\infty \xi^n(-D_\xi)^{pn+1}F_s(\xi)\, d\xi < \infty$$

and hence $(-D_t)^{pn+1}F_s(t) \to 0$ as $t \to \infty$. This, in turn, implies that for all $t \geqq T$,

$$(-D_t)^{pn+2}F_s(t) \geqq 0.$$

Proceeding in this manner we can conclude that $(-D_t)^{pn+k}F_s(t) \geqq 0$ for $0 \leqq k \leqq n$, $t \geqq T$, $s \in R_+^*$. But $(-D_t)^{pn+n}F_s(t) \geqq 0$ for all $t \in R_+$ and so $(-D_t)^{pn+n-1}F_s(t) \geqq 0$ for all $t \in R_+$. Proceeding back in this manner we finally have $(-D_t)^{pn+k}F_s(t) \geqq 0$ for all $t \in R_+$, $s \in R_+^*$, $0 \leqq k \leqq n$. Since p is arbitrary, we conclude that F_s is a completely monotone function. In particular, $F_0(t)$ is a completely monotone function for $t \in R_+^*$. It follows from Bernstein's Theorem [7], that there exists a positive, bounded and nondecreasing function β for which

$$-(n-1)!\, t^{1-n}D_t f(t) = \int_{0+}^\infty e^{-\lambda t}\beta(d\lambda), \qquad t > 0,$$

and so,

$$f(t) = 1-\frac{1}{(n-1)!}\int_{0+}^{\infty}\left[\int_0^t s^{n-1}e^{-\lambda s}\,ds\right]\beta(d\lambda)$$
$$= 1-\frac{1}{(n-1)!}\int_{0+}^{\infty}[1-f_\lambda(t)]\frac{(n-1)!}{\lambda^n}\beta(d\lambda)$$
$$= 1-\int_{0+}^{\infty}[1-f_\lambda(t)]\alpha(d\lambda), \qquad \alpha(d\lambda) = \lambda^{-n}\beta(d\lambda).$$

Since $0\leqq f(t)\leqq 1$ and $0\leqq 1-f_\lambda(t)\leqq 1$, it follows that for each $\varepsilon>0$ and $t\in R_+$,

$$\int_{\varepsilon}^{\infty}[1-f_\lambda(t)]\alpha(d\lambda) \leqq 1.$$

Thus

$$\int_{\varepsilon}^{\infty}\alpha(d\lambda) \leqq \lim_{t\to\infty}\int_{\varepsilon}^{\infty}[1-f_\lambda(t)]\alpha(d\lambda) \leqq 1.$$

We may therefore choose $\alpha(0)$ in such a way that $\int_0^\infty \alpha(d\lambda)=1$ giving us

$$f(t) = \int_0^{\infty} f_\lambda(t)\alpha(d\lambda).$$

To prove sufficiency we need only note that if f has the form (4), then

$$D_t f(t) = (-t)^{n-1}g^{(n)}(t)/(n-1)!$$

where

$$g(t) = \int_0^{\infty} e^{-\lambda t}\,\alpha(d\lambda)$$

is a completely monotone function. The reader can easily check that $\langle V_s f\rangle$ is again of the form (4) so that $f\in\mathscr{D}(V)$.

4. **Proof that $\mathscr{C}(V)=\mathscr{C}_{\mathrm{I}}$.** We can prove now that $\mathscr{C}(V)=\mathscr{C}_{\mathrm{I}}$. A simple calculation verifies that

$$P^\lambda(m_t = m_s+k \mid \mathscr{N}_s) = e^{-\lambda(t-s)}[\lambda(t-s)]^k/k!, \qquad k \geqq 0.$$

Thus if

$$Q_\lambda(B) = P^\lambda(\delta^{-1}(B)),$$

and if $h_\lambda(\omega; m, t)$ is the holding functional for the characteristic process $X^\lambda=(\Omega, x_t, \mathscr{N}, Q_\lambda)$, then

$$h_\lambda(\omega; m, t) = Q_\lambda(\tau_{m+1} > \tau_m+t \mid \mathscr{N}_m)$$
$$= P^\lambda[\delta^{-1}(\tau_{m+1} > \tau_m+t) \mid \delta^{-1}(\mathscr{N}_m)]$$
$$= \sum_{k=0}^{n-1} P^\lambda(\tau_{nm+k+1} > \tau_{nm}+t \geqq \tau_{nm+k} \mid \mathscr{N}_{nm})$$
$$= e^{-\lambda t}\sum_{k=0}^{n-1}(\lambda t)^k/k! = f_\lambda(t).$$

As one can easily check,

$$h_\lambda(\omega; m, t) = \langle V_{\tau_m - \tau_{m-1}} h_\lambda(\omega; m-1, \cdot)\rangle(t)$$

and so $X^\lambda \in \mathscr{C}(V)$. It therefore follows from Theorem 3.1 and Corollary 2.5 that $\mathscr{C}_{\mathrm{I}} = \mathscr{C}(V)$.

5. **Proof that** $\mathscr{C}(V) = \mathscr{C}_{\mathrm{II}}$. Suppose that $X \in \mathscr{C}_{\mathrm{II}}$ is a characteristic process associated with the holding functional h. Define

$$J(\omega; m, s, t) = [D_s h(\omega; m, s-\tau_m)] D_t h[\omega_{x_0}(\tau_1, \ldots, \tau_m, s); m+1, t-s].$$

Then we have, using Theorem 1.1,

$$\begin{aligned} P(\tau_m \leqq \tau_{m-1}+s \mid \mathscr{N}_{m-1}, \tau_{m+1}) &= \frac{\int_{\tau_{m-1}}^{\tau_{m-1}+s} J(\omega; m-1, u, \tau_{m+1})\, du}{\int_{\tau_{m-1}}^{\tau_{m+1}} J(\omega; m-1, u, \tau_{m+1})\, du} \\ &= \frac{\int_{\tau_{m-1}}^{\tau_{m-1}+s} (\tau_{m-1}-\xi)^{n-1}(\tau_{m+1}-\xi)^{n-1}\, d\xi}{\int_{\tau_{m-1}}^{\tau_{m+1}} (\tau_{m-1}-\xi)^{n-1}(\tau_{m+1}-\xi)^{n-1}\, d\xi}, \quad \text{a.e.} \end{aligned} \tag{8}$$

Differentiating both sides of this last equality with respect to s and letting

$$K(\omega; m-1, t) = \frac{\int_{\tau_{m-1}}^{t} J(\omega; m-1, u, t)\, du}{\int_{\tau_{m-1}}^{t} (\tau_{m-1}-\xi)^{n-1}(t-\xi)^{n-1}\, d\xi},$$

we have

$$J(\omega; m-1, \tau_{m-1}+s, \tau_{m+1}) = (-s)^{n-1}(\tau_{m+1}-\tau_{m-1}-s)^{n-1} K(\omega; m-1, \tau_{m+1}), \quad \text{a.e.},$$

which becomes upon integrating,

$$\begin{aligned} -[D_s h(\omega; m-1, s)] h[\omega_{x_0}(\tau_1, \ldots, \tau_{m-1}, \tau_{m-1}+s); m, \tau_{m+1}-\tau_{m-1}-s] \\ = \int_{\tau_{m+1}}^{\infty} (-s)^{n-1}(\xi-\tau_{m-1}-s)^{n-1} K(\omega; m-1, \xi)\, d\xi, \quad \text{a.e.} \end{aligned} \tag{9}$$

We can now calculate $K(\omega; m-1, \eta)$. If $\tau_{m+1} = \eta = \tau_{m-1}+s$ in equation (9), then

$$-D_\eta h(\omega; m-1, \eta-\tau_{m-1}) = (\tau_{m-1}-\eta)^{n-1} \int_{\eta}^{\infty} (\xi-\eta)^{n-1} K(\omega; m-1, \xi)\, d\xi, \quad \text{a.e.}$$

Multiplying both sides of this equation by $(\tau_{m-1}-\eta)^{1-n}$ and differentiating n times with respect to η gives

$$-D_\eta^n[(\tau_{m-1}-\eta)^{1-n} D_\eta h(\omega; m-1, \eta-\tau_{m-1})] = (-1)^n (n-1)!\, K(\omega; m-1, \eta), \quad \text{a.e.}$$

or

$$K(\omega; m-1, \eta) = (-1)^{n-1} D_\eta^n[(\tau_{m-1}-\eta)^{1-n} D_\eta h(\omega; m-1, \eta-\tau_{m-1})]/(n-1)!, \quad \text{a.e.}$$

Substituting this last expression for K into equation (9) now gives us

$$\begin{aligned} &h[\omega_{x_0}(\tau_1, \ldots, \tau_{m-1}, \tau_{m-1}+s); m, \tau_{m+1}-\tau_{m-1}-s] \\ &\quad = -\frac{\int_{\tau_{m+1}-\tau_{m-1}-s}^{\infty} \xi^{n-1} D_\xi^n[(\xi+s)^{1-n} D_\xi h(\omega; m-1, \xi+s)]\, d\xi}{(n-1)!\,(-s)^{1-n} D_s h(\omega; m-1, s)}, \quad \text{a.e.} \end{aligned}$$

If we let $\tau_m = s + \tau_{m-1}$ and $t = \tau_{m+1} - \tau_m$, this last equation becomes

$$h[\omega_{x_0}(\tau_1, \ldots, \tau_{m-1}, \tau_m); m, t]$$
$$= -\frac{\int_t^\infty \xi^{n-1} D_\xi^n[(\xi + \tau_m - \tau_{m-1})^{1-n} D_\xi h(\omega; m-1, \xi + \tau_m - \tau_{m-1})]\, d\xi}{(n-1)!\,(\tau_{m-1} - \tau_m)^{1-n} D_{\tau_m} h(\omega; m-1, \tau_m - \tau_{m-1})}$$
$$= \langle V_{\tau_m - \tau_{m-1}} h(\omega; m-1, \cdot)\rangle(t), \quad \text{a.e.},$$

which implies that $X \in \mathscr{C}(V)$.

On the other hand, suppose that $X \in \mathscr{C}(V)$. If h is the holding functional for X, then

$$D_t\langle V_{\tau_m - \tau_{m-1}} h(\omega; m-1, \cdot)\rangle(t)$$
$$= \frac{t^{n-1} D_t^n[(\tau_m + t - \tau_{m-1})^{1-n} D_t h(\omega; m-1, \tau_m + t - \tau_{m-1})]}{(n-1)!\,(\tau_{m-1} - \tau_m)^{1-n} D_{\tau_m} h(\omega; m-1, \tau_m - \tau_{m-1})}, \quad \text{a.e.}$$

Thus

$$J(\omega; m, u, \tau_{m+2}) = (\tau_m - u)^{n-1}(\tau_{m+2} - u)^{n-1} H, \quad \text{a.e.},$$

where H does not depend on u. Substituting this expression for J into equation (8) gives us

$$P(\tau_{m+1} \leqq \tau_m + s \mid \mathscr{N}_m, \tau_{m+2}) = \frac{\int_{\tau_m}^{\tau_m + s} (\tau_m - u)^{n-1}(\tau_{m+2} - u)^{n-1}\, du}{\int_{\tau_m}^{\tau_{m+2}} (\tau_m - u)^{n-1}(\tau_{m+2} - u)^{n-1}\, du}, \quad \text{a.e.}$$

This completes the proof.

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