

# SOME TAMENESS CONDITIONS INVOLVING SINGULAR DISKS<sup>(1)</sup>

BY  
WARREN WHITE

**Introduction.** A familiar sort of lemma in the study of  $E^3$  is the following:

**LEMMA.** *Let  $D$  and  $F$  be two disks in  $E^3$  with  $\partial D \cap F = D \cap \partial F = \emptyset$ , and let  $U$  be a neighborhood of  $F^\circ$  in  $E^3$ . Then there is a disk  $D'$  in  $E^3$  such that  $\partial D' = \partial D$ ,  $D' \subset D \cup U$ , and  $O(D', F) \subset U$  where  $O(D', F)$  is  $D' \setminus F$  minus the component containing  $\partial D$ . ( $D' \setminus F$  means  $D' \setminus (D' \cap F)$ .)*

Theorem 4 generalizes this lemma, allowing  $E$  to be a singular disk with its "interior" disjoint from its "boundary". It is necessary to redefine  $O(D', E)$ , and this is done in §2; the new definition is motivated by Lemma 5A.

Applications of Theorem 4 to the study of 2-spheres in  $E^3$  are given in §6. Burgess has shown (Theorem 7 in [6]) that a 2-sphere  $S$  in  $E^3$  is tame from the interior (i.e.,  $S \cup \text{int } S$  is a 3-cell) if it is "locally spanned" by disks in the interior; Theorem 6 partially extends this result, letting the spanning disks be singular but imposing a condition on their boundaries. Corollary 6A notes that  $S$  is then tame from the interior if "small loops in  $S$  can be shrunk to points in small subsets of the interior." Corollary 6B answers a question raised by Bing [5, §5].

**1. Notation and terminology.** We use the letter  $d$  to denote the Euclidean metric for Euclidean 3-space  $E^3$ , and let  $\rho(f, g) = \sup_{x \in A} d(f(x), g(x))$  for any two maps  $f$  and  $g$  of a space  $A$  into  $E^3$ . A map  $f$  of a subspace of  $E^3$  into  $E^3$  is a  $\delta$ -map if  $\rho(f, I) < \delta$ , where  $I$  is the identity map.

An  $n$ -manifold  $N$  is a separable metric space such that each point  $p \in N$  has an  $n$ -cell neighborhood in  $N$ .  $N^\circ = \{p \in N : p \text{ has a neighborhood in } N \text{ homeomorphic to } E^n\}$ , and  $\partial N = N \setminus N^\circ$ .  $N$  is an  $n$ -manifold-with-boundary if  $\partial N \neq \emptyset$ . A *Euclidean neighborhood* of a point  $p \in N$  is an  $n$ -cell neighborhood  $U$  together with a linear structure on  $U$ . If  $S$  is a connected  $(n-1)$ -manifold in  $N$  which separates  $N$ , and  $V$  is a component of  $N \setminus S$ , then  $S$  is *tame from  $V$*  if  $S \cup V$  is an  $n$ -manifold. All 2-manifolds and 3-manifolds are assumed to be triangulated [2, Theorem 6], and we use the same symbol for both the manifold and its triangulation.

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Two subsets  $X$  and  $Y$  of an  $n$ -manifold  $N$  are in *relative general position* if, for each point  $p \in N$ , there is a Euclidean neighborhood  $U$  of  $p$  and triangulations  $T_X$  and  $T_Y$  of  $X \cap U$  and  $Y \cap U$  such that

- (i) each simplex of  $T_X$  and  $T_Y$  is a simplex in  $U$ ,
- (ii) dimension  $(|T_X^i| \cap |T_Y^j|) \leq i + j - n$ .

A map  $f: X \rightarrow N$  is in *general position* if, for each point  $p \in N$ , there is a Euclidean neighborhood  $U$  of  $p$  and a triangulation  $T$  of  $f^{-1}(U)$  such that

- (i) for each simplex  $\sigma \in T$ ,  $f(\sigma)$  is a simplex in  $U$ ,
- (ii) for any two distinct simplices  $\sigma_1 \in T^i$  and  $\sigma_2 \in T^j$ ,

$$\text{dimension } (f(\sigma_1) \cap f(\sigma_2)) \leq i + j - n.$$

If  $X$  and  $Y$  are two triangulated spaces, then  $X \oplus Y$  denotes the disjoint union of both the spaces  $X$  and  $Y$  and their triangulations.

A *Dehn disk*  $D$  in  $E^3$  is the image of a real disk  $\Delta$  under a map  $f: \Delta \rightarrow E^3$  such that, for some subdisk  $\Delta_1 \subset \Delta^\circ$ ,  $f(\Delta_1) \cap f(\Delta \setminus \Delta_1) = \emptyset$  and  $f|_{\Delta \setminus \Delta_1}$  is piecewise linear and 1-1. The *singularities of  $f$*  are the points of  $\Delta$  in the closure of  $\{x \in \Delta : f^{-1}f(x) \neq x\}$ , and the *singular points of  $D$*  are the images under  $f$  of these singularities.  $\partial D = f(\partial \Delta)$ .

If  $S$  is a 2-sphere in  $E^3$ , then  $\text{int } S$  and  $\text{ext } S$  are, respectively, the bounded and unbounded components of  $E^3 \setminus S$ . *Sierpinski curve* and *inaccessible point* are as defined in [5].

**2. Algebraic separation.** Let  $N$  be a simply-connected  $n$ -manifold,  $n \leq 3$ . An  $(n-1)$ -polyhedron  $K$  is an *algebraic separator of  $N$*  if  $K \cap N^\circ$  can be given a triangulation in which each  $(n-2)$ -simplex is the face of an even number of  $(n-1)$ -simplices.

Suppose that  $K$  is an algebraic separator of  $N$ . Any arc  $A \subset N$  in general position relative to  $K$  hits  $K$  at a finite number  $\|A \cap K\|$  of points, and standard counting arguments show that:

**PROPOSITION 2A.** *If  $A \subset N$  and  $B \subset N$  are polygonal arcs in general position relative to  $K$ , and  $A, B$  have the same endpoints, then  $\|A \cap K\| = \|B \cap K\| \pmod{2}$ . In particular, if  $\|A \cap K\|$  is odd then the endpoints of  $A$  are separated in  $N$  by  $K$ .*

Suppose that  $D$  is a disk and  $K \subset D^\circ$  is an algebraic separator of  $D$ . It follows from Proposition 2A that we can define a map  $\phi_{D \setminus K}$  on  $D \setminus K$  by setting  $\phi_{D \setminus K}(x) = \|A \cap K\| \pmod{2}$ , where  $A$  is any arc from  $\partial D$  to  $x$  in general position relative to  $K$ . We let  $O(D, K) = \{x \in D \setminus K : \phi_{D \setminus K}(x) = 1\}$ .

Now suppose that  $\Delta$  is a disk,  $M$  a 3-manifold, and  $f: \Delta \rightarrow M$  a map such that  $f|_{\Delta^\circ}$  is locally piecewise linear and in general position. Let  $D \subset M$  be a polyhedral disk in general position relative to  $f(\Delta^\circ)$ , such that  $\partial D \cap f(\Delta) = D \cap f(\partial \Delta) = \emptyset$ .

**PROPOSITION 2B.**  *$f^{-1}(D) = J_1 \cup \dots \cup J_s$ , where the  $J_i$  are disjoint simple closed curves.  $f(J_i)$  and  $f(\cup J_i) = D \cap f(\Delta)$  are algebraic separators of  $D$ , and  $O(D, f(\cup J_i)) \subset \cup O(D, f(J_i))$ .*

**Proof.** To check that  $O(D, f(\cup J_i)) \subset \cup O(D, f(J_i))$ , just note that, for any polygonal arc  $A$  in general position relative to  $f(\cup J_i)$ ,  $\|A \cap f(\cup J_i)\| = \|A \cap f(J_1)\| + \dots + \|A \cap f(J_s)\|$ . The other statements follow from the general position of  $f|_{\Delta^\circ}$ , and of  $D$  relative to  $f(\Delta^\circ)$ .

**3. Induction lemma.**

LEMMA 3. *Let  $M$  be a 3-manifold-with-boundary,  $D$  and  $\Delta$  disks. Let  $f: \Delta \rightarrow M$  be a simplicial map in general position,  $U$  an open neighborhood of  $f(\Delta)$  in  $M$ .*

*Suppose  $i: D \rightarrow M$  is a simplicial embedding such that  $i(D)$  is in general position relative to  $f(\Delta)$ ,  $i(D) \cap \partial M = i(\partial D)$ , and  $i(\partial D) \cap U = i(D) \cap f(\partial \Delta) = \emptyset$ .*

*If  $O(i(D), f(J)) \not\subset U$  for some simple closed curve  $J \subset f^{-1}i(D)$ , then there is a polyhedral disk  $D'$  in  $M$  such that:*

$$(3.1) \quad \partial D' = i(\partial D),$$

$$(3.2) \quad D' \subset i(D) \cup U,$$

$$(3.3) \quad (i(D) \setminus U) \setminus (D' \setminus U) \neq \emptyset.$$

**Proof.** Our proof will be analogous to those of Papakyriakopoulos [10] and Stallings [11], but where they dealt with maps of disks, we will be working with the map  $i \oplus f: D \oplus \Delta \rightarrow M$  defined by  $i \oplus f|_D = i$ ,  $i \oplus f|_\Delta = f$ . To measure the singularity of this map we use the complex  $S(i \oplus f)$  defined by Stallings in his proof of [11, Lemma 3]; for completeness, we reproduce the definition here.

For any simplicial map  $\gamma$  of a complex  $X$  into a complex  $Y$ , a simplicial map  $\gamma \times \gamma: X \times X \rightarrow Y \times Y$  can be constructed, where  $X \times X$  and  $Y \times Y$  are the cartesian products of complexes as defined in [7, p. 67]. We define  $S(\gamma)$  to be the inverse image under  $\gamma \times \gamma$  of the diagonal of  $Y \times Y$ ; since this diagonal is a subcomplex of  $Y \times Y$ , it follows that  $S(\gamma)$  is a subcomplex of  $X \times X$ . The useful property of  $S(\gamma)$  is that, if  $\Pi: Y \rightarrow Z$  is a simplicial map into some complex  $Z$ , then  $S(\gamma) \subset S(\Pi\gamma)$ , and  $S(\gamma) = S(\Pi\gamma)$  if and only if  $\Pi$  is 1-1.

We will induct on the number  $\mathcal{H}(i, f)$  of simplices in  $S(i \oplus f)$ ; assume that  $O(i(D), f(J)) \not\subset U$  for some simple closed curve  $J \subset f^{-1}i(D)$ .

To simplify notation, we will identify  $D$  with  $i(D)$  from this point on in the proof of Lemma 3.

Through standard combinatorial techniques and the ideas Stallings uses in proving Lemma 3 of [11], one can show:

PROPOSITION 3A. *There is a regular neighborhood  $N$  of  $D \cap f(\Delta)$  in  $M$ , a closed neighborhood  $V$  of  $D \cap f(\Delta)$  in  $D$ , and a piecewise linear embedding  $h: D \times [-1, 1] \rightarrow N$  such that*

- (i)  $h(x \times 0) = x$  and  $h(x \times \pm 1) \in \partial N$  for all  $x \in D \setminus V$ ,
- (ii)  $N \subset U \cup h((D \setminus V) \times [-1, 1])$  and  $f(\Delta) \cap h((D \setminus V) \times [-1, 1]) = \emptyset$ ,
- (iii) the maps  $i: D \rightarrow N$  and  $f: \Delta \rightarrow N$  are simplicial.

The proof of the lemma splits into two parts, depending on whether or not  $N$  is simply connected.

Case I.  $N$  simply connected:

PROPOSITION 3B. *The components  $S_1, \dots, S_r$  of  $\partial N$  are spheres.*

**Proof.** See 7.2 in [10].

PROPOSITION 3C. *If  $p$  is a point of  $O(D, f(J)) \setminus U$ , then  $h(p \times -1)$  and  $h(p \times 1)$  lie in different spheres  $S_i$ .*

**Proof.** If  $E$  is the disk in  $\Delta$  bounded by  $J$ , then it follows from the general position of  $f$ , and of  $D$  relative to  $f(\Delta)$ , that  $O(D, f(J)) \cup f(E) = K$  is an algebraic separator of  $N$ .  $h(p \times [-1, 1])$  is a polyhedral arc in general position relative to  $K$  which hits  $K$  once; by Proposition 2A,  $h(p \times -1)$  and  $h(p \times 1)$  are separated in  $N$  by  $K \subset N^\circ$ , and must therefore lie in different components of  $\partial N$ .

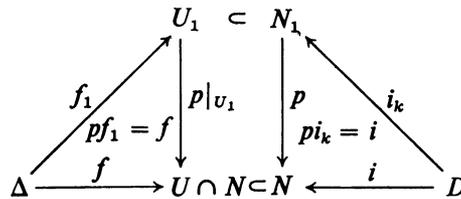
PROPOSITION 3D. *If  $p$  is a point of  $O(D, f(J)) \setminus U$ , then  $\partial D$  bounds a polyhedral disk  $D'$  in  $(D \setminus p) \cup U$ . ( $D'$  satisfies (3.1)–(3.3)).*

**Proof.** Suppose  $\partial D$  lies in  $S_1$ . By Proposition 3C,  $S_1$  does not contain both  $h(p \times -1)$  and  $h(p \times 1)$ .  $\partial D$  misses  $h(p \times \pm 1)$ , so  $\partial D$  bounds a disk  $D_1$  in  $S_1$  missing  $h(p \times \pm 1)$ . For any  $x \in D \setminus V$ ,  $D_1$  contains at most one of the two points  $h(x \times -1)$  and  $h(x \times 1)$ , since  $D$  is an algebraic separator of  $N$  separating them. Using the embedding of  $D \times [-1, 1]$  in  $N$  given by Proposition 3A, we can therefore draw  $D_1$  homeomorphically into  $(D \setminus p) \cup U$ .

Case II.  $N$  not simply connected:

Let  $(N_1, p)$  be the universal (simply connected) covering space for  $N$ .  $N_1$  is a 3-manifold-with-boundary, and we triangulate  $N_1$  so that  $p: N_1 \rightarrow N$  is simplicial. Let  $U_1 = p^{-1}(U \cap N)$ .

Let  $f_1: \Delta \rightarrow N_1$  be a lifting of  $f$ , and let  $i_1, i_2, \dots, i_k, \dots: D \rightarrow N_1$  be the distinct liftings of  $i$ .



It is easy to check that

PROPOSITION 3E. *The hypotheses of Lemma 3 are satisfied by the substitution:*

<i>for</i>	<i>substitute</i>	<i>for</i>	<i>substitute</i>
$M$	$N_1$	$U$	$U_1$
$\Delta$	$\Delta$	$D$	$D$
$f$	$f_1$	$i$	any $i_k$

PROPOSITION 3F. For any  $i_k, \mathcal{H}(i_k, f_1) < \mathcal{H}(i, f)$ .

**Proof.** Consider the commutative diagram:

$$\begin{array}{ccc} \pi_1(i_k(D) \cup f_1(\Delta)) & \xrightarrow{\psi_1} & \pi_1(N_1) = 0 \\ \downarrow (p|_{i_k(D) \cup f_1(\Delta)})_* & & \downarrow p_* \\ \pi_1(D \cup f(\Delta)) & \xrightarrow{\psi} & \pi_1(N) \neq 0, \end{array}$$

where  $\psi_1$  and  $\psi$  are induced by inclusions.  $\psi$  is onto since  $N$  is a regular neighborhood of  $D \cup f(\Delta)$ .

Now,  $S(i_k \oplus f_1) \subset S(p \circ (i_k \oplus f_1)) = S(i \oplus f)$ . If  $S(i_k \oplus f_1) = S(i \oplus f)$ , then  $p|_{i_k(D) \cup f_1(\Delta)}$  is 1-1 and hence a homeomorphism, so  $(p|_{i_k(D) \cup f_1(\Delta)})_*$  is onto. But then

$$0 \neq \pi_1(N) = \psi(p|_{i_k(D) \cup f_1(\Delta)})_* \pi_1(i_k(D) \cup f_1(\Delta)) = p_* \psi_1 \pi_1(i_k(D) \cup f_1(\Delta)) = 0,$$

a contradiction. Thus,  $S(i_k \oplus f_1)$  is properly contained in  $S(i \oplus f)$ , and  $\mathcal{H}(i_k, f_1) < \mathcal{H}(i, f)$ .

PROPOSITION 3G. For some  $K, J \subset f_1^{-1}i_k(D)$  and  $O(i_k(D), f_1(J)) \not\subset U_1$ .

**Proof.**  $J \subset f^{-1}(D) = (pf_1)^{-1}(D) = f_1^{-1}(p^{-1}(D)) = f_1^{-1}(\cup i_k(D)) = \cup f_1^{-1}i_k(D)$ ; since the disks  $i_k(D)$  are disjoint,  $J \subset f_1^{-1}i_k(D)$  for some  $K$ .  $pi_k = i$  is a homeomorphism, so  $p(O(i_k(D), f_1(J))) = O(D, f(J))$ ; if  $O(i_k(D), f_1(J)) \subset U_1$ , then  $O(D, f(J)) \subset p(U_1) = U \cap N$ , a contradiction to our assumption that  $O(D, f(J)) \not\subset U$ .

PROPOSITION 3H. There is a polyhedral disk  $D'$  in  $M$  satisfying (3.1)–(3.3).

**Proof.** Let  $D_1 = i_k(D)$ , where  $K$  is given by Proposition 3G. By our induction, there is a polyhedral disk  $D'_1$  in  $N_1$  such that:

- (i)  $\partial D'_1 = \partial D_1$ ,
- (ii)  $D'_1 \subset D_1 \cup U_1$ ,
- (iii)  $(D_1 \setminus U_1) \setminus (D'_1 \setminus U_1) \neq \emptyset$ .

Since  $p|_{D_1}$  is a homeomorphism, the singularities of  $p: D'_1 \rightarrow M$  all lie in  $U_1$ .  $p(U_1) \cap p(\partial D'_1) \subset U \cap \partial D = \emptyset$ , so we can apply Dehn's lemma [9, Theorem IV.3] to get a polyhedral disk  $D'$  in  $M$  such that:

- (iv)  $\partial D' = p(\partial D'_1)$ ,
- (v)  $D' \subset p(D'_1) \cup U$ .

It is easy to check that (i)–(v) imply that  $D'$  satisfies (3.1)–(3.3).

#### 4. Using a singular disk to “cut back” a real disk.

THEOREM 4. Let  $U_0$  be an open subset of  $E^3$ ,  $\Delta_0$  a disk, and  $f_0: \Delta_0 \rightarrow E^3$  a map such that  $f_0(\Delta_0) \cap U_0 = f_0(\Delta^\circ_0)$  and  $f_0|_{\Delta^\circ_0}: \Delta^\circ_0 \rightarrow E^3$  is locally piecewise linear and in general position.

Suppose that  $D \subset E^3$  is a polyhedral disk such that  $D \cap f_0(\partial\Delta_0) = \partial D \cap f_0(\Delta_0) = \emptyset$ . Then there is a polyhedral disk  $D'$  in  $E^3$  such that

- (4.1)  $\partial D' = \partial D$ ,
- (4.2)  $D' \subset D \cup U_0$ ,
- (4.3)  $D'$  is in general position relative to  $f_0(\Delta_0^\circ)$ ,
- (4.4)  $O(D', D' \cap f_0(\Delta_0)) \subset U_0$ .

**Proof.** We may assume that  $\bar{U}_0$  is locally polyhedral mod  $f_0(\partial\Delta_0)$ , and that  $U_0 \cap \partial D = \emptyset$ . For any disk  $D'$  satisfying (4.1) and (4.2), let  $\mathcal{H}(D')$  be the number of components of  $D' \setminus U_0$ ;  $\mathcal{H}(D')$  is finite because  $\bar{U}_0$  is polyhedral near  $D'$ .

$D$  satisfies (4.1) and (4.2); we will induct on  $\mathcal{H}(D)$ . By adjusting  $D$  within  $U_0$ , if necessary, we may assume that  $D$  is in general position relative to  $f_0(\Delta_0^\circ)$ ; if  $O(D, D \cap f_0(\Delta_0)) \subset U_0$ , as is the case when  $\mathcal{H}(D) = 1$ , then we have nothing to prove. Suppose that  $O(D, D \cap f_0(\Delta_0)) \not\subset U_0$ .

Since  $(D \cup U_0) \cap f_0(\partial\Delta_0) = \emptyset$ , we can choose a disk  $\Delta \subset \Delta_0^\circ$  such that  $f = f_0|_\Delta: \Delta \rightarrow E^3$  is piecewise linear and in general position, and

$$D \cap f_0(\Delta_0) \subset f(\Delta^\circ) \setminus f(\partial\Delta).$$

Using standard Euclidean-space techniques, together with the fact that

$$f_0(\Delta_0^\circ) \cap f_0(\partial\Delta_0) = \emptyset,$$

one can show:

**PROPOSITION 4A.** *There is a 3-manifold-with-boundary  $M \subset E^3$  such that*

- (i)  $D \cup f(\Delta) \subset M$ ,
- (ii)  $D \cap \partial M = \partial D$ ,
- (iii)  $M \cap f_0(\partial\Delta_0) = \emptyset$ .

Furthermore,  $M$ ,  $D$ , and  $\Delta$  may be triangulated so that the hypotheses of Lemma 3 are satisfied by  $M$ ,  $D$ ,  $\Delta$ ,  $f$ ,  $U = U_0 \cap M$ , and the natural injection  $i: D \rightarrow M$ .

Since  $O(D, D \cap f_0(\Delta_0)) \not\subset U_0$ , we have also  $O(D, D \cap f(\Delta)) \not\subset U$ . By Proposition 2B,  $O(D, D \cap f(J)) \not\subset U$  for some simple closed curve  $J \subset f^{-1}(D)$ . Lemma 3 then gives us a polyhedral disk  $D'$  such that

- (4.1)  $\partial D' = \partial D$ ,
- (4.2)  $D' \subset D \cup U \subset D \cup U_0$ ,
- (3.3)  $(D \setminus U) \setminus (D' \setminus U) \neq \emptyset$ .

To show that (3.3) implies  $\mathcal{H}(D') < \mathcal{H}(D)$ , we note

**PROPOSITION 4B.**  $D' \setminus U = D' \setminus U_0$ ,  $D \setminus U = D \setminus U_0$ , and each component of  $D' \setminus U$  is a component of  $D \setminus U$ .

**Proof.**  $D^* \setminus U = D^* \setminus (U_0 \cap M) = (D^* \setminus U_0) \cup (D^* \setminus M) = D^* \setminus U_0$ , where  $D^*$  is either  $D'$  or  $D$ . That the components of  $D' \setminus U$  are components of  $D \setminus U$  follows from (4.1) and (4.2) above.

**REMARK.** The proof of Theorem 4 shows that we can actually have  $D'$  satisfy

$$(4.4') \quad O(D', D' \cap f_0(J)) \subset U_0, \text{ for each simple closed curve } J \subset f_0^{-1}(D').$$

**5. Applying Theorem 4.** Throughout the remainder of the paper,  $\Delta$  will represent a standard disk. Let  $M$  be a 3-manifold,  $S$  a 2-manifold in  $M$ , and  $F \subset S$  a disk.

**PROPOSITION 5A.** *Let  $\mathcal{G}$  be the class of all maps  $g: \partial\Delta \rightarrow F^\circ$  which are piecewise linear into  $F$  and in general position. Then, for an arbitrary map  $f: \partial\Delta \rightarrow F$ ,  $\phi_{F \setminus f(\partial\Delta)} = \lim_{g \in \mathcal{G}; \rho(f, g) \rightarrow 0} \phi_{F \setminus g(\partial\Delta)}$  exists on  $F \setminus f(\partial\Delta)$ , and  $\phi_{F \setminus f(\partial\Delta)}(x) = \phi_{F \setminus g(\partial\Delta)}(x)$  for any map  $g \in \mathcal{G}$  which is homotopic to  $f$  in  $F \setminus x$ .*

**Proof.** Both assertions follow from the easily demonstrated fact that if two maps  $g_1$  and  $g_2: \partial\Delta \rightarrow F$  are piecewise linear into  $F$ , in general position, and homotopic in  $F \setminus x$ , then  $\phi_{F \setminus g_1(\partial\Delta)}(x) = \phi_{F \setminus g_2(\partial\Delta)}(x)$ .

If  $V$  is a component of  $M \setminus S$ , then a *blister of  $F$  in  $V$*  is a map  $f: \Delta \rightarrow F \cup V$  such that  $f(\Delta) \cap S = f(\partial\Delta)$ . We let  $O(F, f) = \{x \in F \setminus f(\partial\Delta) : \phi_{F \setminus f(\partial\Delta)}(x) = 1\}$ , and denote  $f(\Delta) \cup O(F, f)$  by  $(f)_F$ .

**LEMMA 5A.** *Let  $S$  be a 2-sphere in  $E^3$ ,  $F \subset S$  a disk. Let  $f$  be a blister of  $F$  in  $\text{int } S$ , and  $B$  a 3-cell in  $E^3$ , such that  $(f)_F \subset B^\circ$  and  $f|_{\Delta^\circ}: \Delta^\circ \rightarrow E^3$  is locally piecewise linear and in general position.*

*Suppose  $p$  is a point of  $O(F, f)$ ,  $q$  is a point of  $\text{int } S \setminus B$ , and  $qp \subset E^3$  is a polygonal arc in general position relative to  $f(\Delta^\circ)$  such that  $qp \setminus p \subset \text{int } S$ . Then  $\|qp \cap f(\Delta)\|$  is odd.*

**Proof.** Pick a point  $r$  in  $\text{ext } S \setminus B$ . We can use Theorem 5.37 of [12] to extend  $qp$  to a polygonal arc  $qpr \subset E^3$  such that  $pr$  misses  $S \setminus O(F, f)$  and  $f(\Delta)$ .

Let  $\delta = d(f(\partial\Delta), (E^3 \setminus B) \cup qpr)$ , and triangulate  $S$  so that  $F$  is a polyhedron in  $S$ . Using Bing's approximation theorem [1, Theorem 1], we can find a piecewise linear  $\delta/2$ -homeomorphism  $h: S \rightarrow E^3$  such that

$$(1) \quad qpr \cap h(S) \subset h(O(F, f)) \\ (= \{x \in h(F) \setminus hf(\partial\Delta) : \phi_{h(F) \setminus hf(\partial\Delta)}(x) = 1\}),$$

- (2)  $q \in \text{int } h(S)$ ,  $r \in \text{ext } h(S)$ ,
- (3)  $h(S)$  is in general position relative to  $f(\Delta^\circ) \cup qpr$ .

**PROPOSITION 5B.** *There is a polyhedron  $K \subset B \subset E^3$  such that:*

- (i)  $K$  is an algebraic separator of  $E^3$  in general position relative to  $qpr$ .
- (ii)  $qpr \cap (f(\Delta) \cup h(S)) = qpr \cap K$ .

**Proof.** Let  $g: \partial\Delta \rightarrow E^3$  be a piecewise linear map in general position, such that:

- (i)  $g(\partial\Delta) \subset h(F^\circ)$ ,
- (ii)  $g$  and  $hf|_{\partial\Delta}$  are homotopic in  $h(F) \setminus qpr$ ,
- (iii)  $\rho(g, hf|_{\partial\Delta}) < \delta/2$ .

Let  $\gamma: \Delta \rightarrow \Delta^\circ$  be a homeomorphism such that

- (iv)  $\rho(f\gamma, f) < \delta$ ,
- (v)  $f\gamma: \Delta \rightarrow E^3$  is piecewise linear and in general position, and  $f\gamma(\Delta)$  is in general position relative to  $h(S)$ .

As a result of our care with  $\delta$ , we can get a piecewise linear homotopy  $G: \partial\Delta \times [0, 1] \rightarrow E^3$  such that

- (vi)  $G_0 = g, G_1 = f\gamma|_{\partial\Delta}$ ,
- (vii)  $G(\partial\Delta \times [0, 1]) \subset B \setminus qpr$ ,
- (viii)  $G(\partial\Delta \times (0, 1))$  is in general position relative to  $h(S)$  and  $f\gamma(\Delta)$ .

It is simple to check that  $K = O(h(F), g(\partial\Delta)) \cup G(\partial\Delta \times [0, 1]) \cup f\gamma(\Delta)$  satisfies the requirements.

PROPOSITION 5C.  $\|qp \cap f(\Delta)\|$  is odd.

**Proof.**  $K$  is contained in  $B$ , which does not separate  $q$  and  $r$  in  $E^3$ , so by Proposition 2A  $\|qpr \cap K\|$  is even.  $\|qpr \cap K\| = \|qpr \cap (f(\Delta) \cup h(S))\| = \|qpr \cap f(\Delta)\| + \|qpr \cap h(S)\|$ , by condition (3) on  $h$ .  $\|qpr \cap h(S)\|$  is odd since  $h(S)$  is a manifold separating  $q$  and  $r$  in  $E^3$ , so  $\|qpr \cap f(\Delta)\|$  is also odd.

LEMMA 5B. Let  $S$  be a 2-sphere in  $E^3$ ,  $F \subset S$  a disk. Let  $f_1, \dots, f_s$  be blisters of  $F$  in  $\text{int } S$ , and  $B_1, \dots, B_s$  3-cells in  $E^3$  such that  $(f_i)_F \subset B_i^2$  for each  $i$ .

Suppose  $D$  is a polyhedral disk in  $E^3$  such that  $\partial D \subset \text{int } S \cup B_i, \text{int } S \cup (F \setminus \bigcup f_i(\partial\Delta)) \cup D$  retracts to  $\text{int } S \cup (F \setminus \bigcup f_i(\partial\Delta))$ , and  $D \cap S \subset \bigcup O(F, f_i)$ .

Then there is a disk  $D'$  in  $E^3$  such that

- (5.1)  $\partial D' = \partial D$ ,
- (5.2)  $D' \subset D \cup (\bigcup B_i)$ ,
- (5.3)  $D' \subset \text{int } S$ .

**Proof.** Suppose that we have a polyhedral disk  $D_j$  in  $E^3$  which satisfies the following conditions:

- (1)  $\partial D_j = \partial D$ ,
- (2)  $D_j \subset D \cup (\bigcup B_i)$ ,
- (3)  $\text{int } S \cup (F \setminus \bigcup f_i(\partial\Delta)) \cup D_j$  retracts to  $\text{int } S \cup (F \setminus \bigcup f_i(\partial\Delta))$ ,
- (4)  $D_j \cap S \subset \bigcup_{j < i \leq s} O(F, f_i)$ .

We can choose  $D_0 = D$ , for example, and if we had  $D_s$  we could choose  $D' = D_s$ . For the proof of Lemma 5B it is, therefore, sufficient to produce  $D_{j+1}$ .

We may assume that, for each  $i, f_i|_{\Delta^0}$  is locally piecewise linear and in general position. The hypotheses of Theorem 4 are then satisfied by the following substitutions:

for	substitute	for	substitute
$\Delta_0$	$\Delta$	$U_0$	$B_{j+1} \cap \text{int } S$
$f_0$	$f_{j+1}$	$D$	$D_j$

There is, therefore, a polyhedral disk, which we shall call  $D_{j+1}$ , satisfying:

- (4.1)  $\partial D_{j+1} = \partial D_j$ ,
- (4.2)  $D_{j+1} \subset D_j \cup (B_{j+1} \cap \text{int } S)$ ,
- (4.3)  $D_{j+1}$  is in general position relative to  $f_{j+1}(\Delta^0)$ ,
- (4.4)  $O(D_{j+1}, D_{j+1} \cap f_{j+1}(\Delta)) \subset B_{j+1} \cap \text{int } S$ .

From (4.1) and (4.2) it follows that  $D_{j+1}$  satisfies conditions (1)–(3); it remains to check (4).

**PROPOSITION 5D.**  $D_{j+1} \cap S \subset \bigcup_{j+1 < i \leq s} O(F, f_i)$ .

**Proof.** (4.2) implies that  $D_{j+1} \cap S \subset \bigcup_{j < i \leq s} O(F, f_i)$ , so all we need check is  $O(F, f_{j+1})$ . Suppose that  $D_{j+1} \cap O(F, f_{j+1}) \neq \emptyset$ , and use (4.3) to choose a polygonal arc  $A$  with endpoints  $p$  and  $q$ , such that

- (i)  $A \subset D_{j+1}$ ,
- (ii)  $q \in \partial D_{j+1}, p \in O(F, f_{j+1})$ ,
- (iii)  $A$  is in general position relative to  $f_{j+1}(\Delta^\circ)$ .

Using the facts that  $D_{j+1}$  satisfies (3) and  $\text{int } S$  is locally 0-connected [12, Theorem 5.35], we can get a polygonal arc  $A'$  with endpoints  $p'$  and  $q$ , such that

- (iv)  $A' \setminus p' \subset \text{int } S$ ,
- (v)  $p' \in O(F, f_{j+1})$ ,
- (vi)  $A' \cap W = A \cap W$ , for some neighborhood  $W$  of  $f_{j+1}(\Delta^\circ)$  in  $\text{int } S$ .

$A'$  is in general position relative to  $f_{j+1}(\Delta^\circ)$  since  $A$  is, so Lemma 5A tells us that  $\|A' \cap f_{j+1}(\Delta)\|$  is odd. Since  $\|A' \cap f_{j+1}(\Delta)\| = \|A \cap f_{j+1}(\Delta)\|$ , this means that  $p \in O(D_{j+1}, D_{j+1} \cap f_{j+1}(\Delta))$ . According to (4.4),  $p$  then lies in  $\text{int } S$ ; but we assumed that  $p \in S$ , which is a contradiction. Therefore,  $D_{j+1} \cap O(F, f_{j+1}) = \emptyset$ .

**6. 2-spheres in  $E^3$ .** Let  $M$  be a 3-manifold,  $S$  a 2-manifold in  $M$ , and  $V$  a component of  $M \setminus S$ .  $S$  satisfies condition (1) toward  $V$  at a point  $p \in S$  if, for any neighborhood  $B$  of  $p$  in  $M$ , and any Cantor set  $C$  in  $S$ , there is a disk  $F \subset S \cap B$  and a blister  $f$  of  $F$  in  $V$  such that  $p \in O(F, f) \subset (f)_F \subset B$  and  $f(\partial\Delta) \cap C = \emptyset$ .

**THEOREM 6.** *Let  $S$  be a 2-sphere in  $E^3$  which satisfies condition (1) toward its interior at every point. Then  $S$  is tame from the interior.*

**Proof.** Let  $F \subset S$  be a disk,  $U$  a neighborhood of  $F$  in  $E^3$ , and  $D \subset E^3$  a polyhedral disk with  $\partial D \subset \text{int } S$  and  $D \cap S \subset F^\circ$ .

**PROPOSITION 6A.** *To prove Theorem 6, it is sufficient to show that there is a disk  $D'$  in  $E^3$  such that*

- (i)  $\partial D' = \partial D$ ,
- (ii)  $D' \subset D \cup U$ ,
- (iii)  $D' \subset \text{int } S$ .

**Proof.** As Hempel has noted (in the proof of [8, Theorem 1]), this is a consequence of Bing's proof of Theorem 1 in [3].

**PROPOSITION 6B.** *There is a Dehn disk  $D_0$  in  $E^3$  and a Cantor set  $C$  in  $S$  such that*

- (i)  $\partial D_0 = \partial D$ ,
- (ii)  $D_0 \subset D \cup U$ , with the singular points of  $D_0$  contained in  $U$ ,
- (iii)  $D_0 \subset \text{int } S \cap (F^\circ \cap C)$ .

**Proof.** We can use the Tietze extension theorem to get a Dehn disk  $D'_0$  such that  $\partial D'_0 = \partial D$  and  $D'_0 \subset (D \cap \text{int } S) \cup F^\circ$ , with the singular points of  $D'_0$  contained in  $F$ . Theorem 2.1 of [5] then gives us  $D_0$ .

**PROPOSITION 6C.** *There are blisters  $f_1, \dots, f_s$  of  $F$  in  $\text{int } S$ , 3-cells  $B_1, \dots, B_s$  in  $U \setminus \partial D$ , and disjoint disks  $G_1, \dots, G_r$  in  $S$ , such that  $(f_i)_F \subset B_i^\circ$  for each  $i$ , and  $D_0 \cap S \subset \bigcup G_j \subset \bigcup G_j \subset (\bigcup O(F, f_i)) \setminus \bigcup f_i(\partial \Delta)$ .*

**Proof.** For any point  $p$  in  $F^\circ$ , we can choose a 3-cell neighborhood  $B$  in  $U \setminus (\partial D \cup (S \setminus F))$ . Let  $C$  be the Cantor set described in Proposition 6B; since  $S$  satisfies condition (1) toward  $\text{int } S$  at  $p$ , there is a disk  $F_p \subset S \cap B_p \subset F$  and a blister  $f_p$  of  $F_p$  in  $\text{int } S$  such that  $p \in O(F_p, f_p) = O(F, f_p) \subset (f_p)_F \subset B_p^\circ$  and  $f_p(\partial \Delta) \cap C = \emptyset$ .  $D_0 \cap S$  is compact, so we can pick  $p_1, \dots, p_s \in F^\circ$  such that  $D_0 \cap S \subset (\bigcup O(F, f_{p_i})) \setminus \bigcup f_{p_i}(\partial \Delta)$ . We let  $\{f_1, \dots, f_s\} = \{f_{p_1}, \dots, f_{p_s}\}$ ,  $\{B_1, \dots, B_s\} = \{B_{p_1}, \dots, B_{p_s}\}$ , and use the fact that  $D_0 \cap S \subset C$  is 0-dimensional to choose the disks  $G_1, \dots, G_r$ .

We can use the Tietze extension theorem to show:

**PROPOSITION 6D.**  *$\text{int } S \cup (\bigcup G_j)$  is a retract of some neighborhood  $V$  of  $\text{int } S \cup (\bigcup G_j)$  in  $E^3$ .*

**PROPOSITION 6E.** *There is a polyhedral disk  $D_1$  in  $E^3$  such that*

- (i)  $\partial D_1 = \partial D$ ,
- (ii)  $D_1 \subset D \cup U$ ,
- (iii)  $S, F, \{f_1, \dots, f_s\}, \{B_1, \dots, B_s\}$ , and  $D = D_1$  satisfy the hypotheses of Lemma 5B.

**Proof.** The singular points of  $D_0$  are contained in  $U \cap V$ , where  $V$  is as in Proposition 6D, so Dehn's Lemma gives us a polyhedral disk  $D_1 \subset D_0 \cup (U \cap V)$  with  $\partial D_1 = \partial D_0$ . It is simple to check that  $D_1$  meets the requirements.

If we apply Lemma 5B to  $D_1$ , we obtain a disk  $D'$  in  $E^3$  such that

- (5.1)  $\partial D' = \partial D_1 = \partial D$ ,
- (5.2)  $D' \subset D_1 \cup (\bigcup B_i) \subset D \cup U$ ,
- (5.3)  $D' \subset \text{int } S$ .

$D'$  satisfies conditions (i)–(iii) of Proposition 6A, and the proof is therefore complete.

Let  $M$  be a 3-manifold,  $S$  a 2-manifold in  $M$ ,  $V$  a component of  $M \setminus S$ , and  $B$  a subset of  $M$ . A loop  $f: \partial \Delta \rightarrow S$  can be shrunk to a point through  $B \cap V$  if there is a homotopy  $H_t: \partial \Delta \rightarrow M$  such that  $H_0 = f$ ,  $H_1$  is constant, and  $H_t(\partial \Delta) \subset B \cap V$  for all  $t > 0$ .  $S$  is 1-LC through  $V$  at a point  $p \in S$  if, for any neighborhood  $B$  of  $p$  in  $M$ , there is a neighborhood  $B_1$  of  $p$  in  $B \cap S$  such that any loop  $f: \partial \Delta \rightarrow B_1$  can be shrunk to a point through  $B \cap V$ .

**COROLLARY 6A.** *Let  $S$  be a 2-sphere in  $E^3$  which is 1-LC through its interior at every point. Then  $S$  is tame from the interior.*

**Proof.** This follows from the observation:

**PROPOSITION 6F.** *Let  $B$  be a subset of  $E^3$ , and suppose that  $i: \Delta \rightarrow S$  is an embedding such that  $i(\partial\Delta)$  can be shrunk to a point through  $B \cap \text{int } S$ . Then there is a blister  $f$  of  $i(\Delta)$  in  $\text{int } S$  such that  $f(\partial\Delta) = i(\partial\Delta)$  and  $f(\Delta) \subset B$ , and we have  $O(F, f) = i(\Delta^\circ)$ , for any disk  $F \subset S$  containing  $i(\Delta)$ .*

Let  $M$  be a 3-manifold,  $S$  a 2-manifold in  $M$ , and  $V$  a component of  $M \setminus S$ . A set  $X$  in  $S$  can be deformed into  $V$  if there is a homotopy  $H_t: X \rightarrow M$  such that  $H_0 = I$  and  $H_t(X) \subset V$  for  $t > 0$ .

**COROLLARY 6B.** *Let  $S$  be a 2-sphere in  $E^3$  such that every Sierpinski curve in  $S$  can be deformed into  $\text{int } S$ . Then  $S$  is tame from the interior.*

**Proof.** The following proposition is an adaptation of Theorem 14 in [6].

**PROPOSITION 6G.** *Let  $E_1, E_2, \dots, E_k, \dots$  be a decreasing sequence of disks in  $S$  whose intersection is a point  $p \in \bigcap E_k^\circ$ , and suppose that  $p \cup (\bigcup \partial E_k)$  can be deformed into  $\text{int } S$ .*

*Then for any open set  $B \subset E^3$  containing  $E_1$ , there is a blister  $f$  of  $E_1$  in  $\text{int } S$  such that  $p \in O(E_1, f) \subset (f)_{E_1} \subset B$  and  $f(\partial\Delta) = \partial E_K$  for some  $K$ .*

**Proof.** Let  $H_t: p \cup (\bigcup \partial E_k) \rightarrow E^3$  be a homotopy such that  $H_0 = I$  and  $H_t(p \cup (\bigcup \partial E_k)) \subset \text{int } S$  for  $t > 0$ . We may assume that  $H_t(p \cup (\bigcup \partial E_k)) \subset B$  for each  $t$ ; let  $B'$  be an open 3-cell in  $B \setminus S$  containing  $H_1(p)$ .  $H_1(\partial E_k)$  lies in  $B'$  for large enough  $K$ , and can be shrunk to a point in  $B'$ . By Proposition 6F, there is then a blister  $f$  of  $E_K$  (and hence of  $E_1$ ) in  $\text{int } S$  such that  $p \in E_K^\circ = O(E_1, f) \subset (f)_{E_1} \subset B$  and  $f(\partial\Delta) = \partial E_K$ .

**PROPOSITION 6H.** *Let  $p$  be a point of  $S$ ,  $C$  a Cantor set in  $S$ . Then  $p$  is an inaccessible point of some Sierpinski curve in  $S$  which misses  $C$ .*

**Proof.** We just construct such a Sierpinski curve, using the fact that  $C$  is 0-dimensional.

If  $p$  is an inaccessible point of a Sierpinski curve  $X$  in  $S$ , then there is a decreasing sequence of disks  $E_1, E_2, \dots, E_k, \dots$  in  $S$  such that  $\partial E_k \subset X$  and  $\bigcap E_k = \bigcap E_k^\circ = p$ . Therefore, Propositions 6G and 6H together imply that  $S$  satisfies condition (1) toward its interior at every point.

**REMARKS.** (1) The hypothesis of Corollary 6B requires that any Sierpinski curve in  $S$  can be *continuously* approximated from  $\text{int } S$ . For any 2-sphere  $S$  in  $E^3$ , any Sierpinski curve  $X$  in  $S$ , and any  $\delta > 0$ , it follows from Bing's side approximation theorem [4, Theorem 16] that there is a  $\delta$ -homeomorphism  $h: X \rightarrow \text{int } S$ .

(2) Theorem 6 is stated for 2-spheres in  $E^3$ , but its proof is based on a local criterion for tameness [8, Condition A], so Theorem 1 of [6] can be used with Lemma 5B to extend our results to two-sided 2-manifolds in 3-manifolds.

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ARIZONA STATE UNIVERSITY,  
TEMPE, ARIZONA