SOME FUNCTION-THEORETIC ASPECTS OF DISCONJUGACY OF LINEAR-DIFFERENTIAL SYSTEMS

BY

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1. **Introduction.** In this paper we consider linear differential systems of the form

$$(1.1) \qquad y'(z) = P(z)y(z),$$

where $y(z)$ is the column vector $[y_1(z), \ldots, y_n(z)]$ and $P(z)$ is the $n \times n$ matrix $[p_{ik}(z)]_1^n$, where the n^2 analytic functions $p_{ik}(z)$ are regular in the bounded simply-connected domain D. Following Schwarz [8], we shall say that (1.1) *is disconjugate in D if for every choice of n (not necessarily distinct) points $z_1, z_2, \ldots, z_n$ in D, the only solution of* (1.1), *which satisfies $y_i(z_i)=0$, $i=1, 2, \ldots, n$, is the trivial one* $y(z)\equiv 0$.

Various aspects and applications of systems disconjugacy were considered by Nehari [6], Schwarz [8], London and Schwarz [3], and Kim [1]. Considering disfocality of second-order differential equations Nehari pointed out the following principle [6, Theorem 1.1], which we state here as a necessary and sufficient condition for disconjugacy of the differential system

$$(1.2) \qquad y_1' = p(z)y_2, \qquad y_2' = q(z)y_1,$$

where $p(z)$ and $q(z)$ are regular functions in the domain D.

Let

$$(1.3) \qquad f(z) = u_1(z)/v_1(z), \qquad g(z) = u_2(z)/v_2(z);$$

where $\boldsymbol{u}=(u_1, u_2)$ and $\boldsymbol{v}=(v_1, v_2)$ are linearly independent solutions of (1.2). *The system* (1.2) *is disconjugate in D if and only if $f(z)$ and $g(z)$ are "relatively schlicht" in D; i.e. if*

$$(1.4) \qquad f(z_1) \neq g(z_2)$$

for every choice of $z_1, z_2 \in D$.

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are replaced by a different set of two linearly independent solutions of (1.2), then, according to (1.3), f and g are replaced by Tf and Tg, where T is given by

$$(1.5) \qquad Tf = (Af+B)/(Cf+D), \qquad AD-BC \neq 0.$$

Received by the editors November 27, 1968.

(1) Research sponsored by Army Research Office Grant No. DA-ARO-D-31-124-G951 and Air Force Office of Scientific Research Grant AF-AFOSR-62-414.

It is therefore necessary, that any relation between the coefficients $p(z)$ and $q(z)$ of (1.2) and the functions $f(z)$ and $g(z)$ will remain invariant under the mapping $f \to Tf$, $g \to Tg$. Two combinations of f and g with this invariance property are

(1.6) $$\Phi[f, g] = f'g'/(f-g)^2,$$

and

(1.7) $$\Psi[f, g] = \frac{f''}{f'} - \frac{g''}{g'} - \frac{2(f'+g')}{f-g}.$$

The relations between the coefficients $p(z)$ and $q(z)$ of (1.2) and the functions $\Phi[f, g]$ and $\Psi[f, g]$ are given by

1.8) $$-p(z)q(z) = \Phi[f, g]$$

and

(1.9) $$p'(z)/p(z) - q'(z)/q(z) = \Psi[f, g].$$

Now, for functions $f(z)$ and $g(z)$ which are "relatively schlicht" in $|z| < 1$ it is known [5, p. 281, 6, Theorem 7.1] that

(1.10) $$|\Phi[f, g]| = \frac{|f'(z)g'(z)|}{|f(z)-g(z)|^2} \leqq \frac{1}{(1-|z|^2)^2}, \qquad |z| < 1.$$

Utilizing this result one obtains the following necessary condition. *If* (1.2) *is disconjugate in* $|z| < 1$, *then*

(1.11) $$|p(z)q(z)| \leqq 1/(1-|z|^2)^2, \qquad |z| < 1.$$

Our principal aim in this paper is to generalize these results of Nehari to differential systems with $n \geqq 3$. The ideas are also related to a recent paper by the author [2], where some function-theoretic aspects of disconjugacy of nth order linear differential equations were considered.

2. **Mappings onto domains with empty intersection.** Let $y_k(z) = [y_{1k}(z), y_{2k}(z), \ldots, y_{nk}(z)]$, $k = 1, 2, \ldots, n$, be n linearly independent solutions of (1.1), then the matrix $Y(z) = [y_{ik}(z)]_1^n$ is a fundamental solution of the matrix differential equation

(2.1) $$Y'(z) = P(z)Y(z)$$

corresponding to (1.1); i.e. the determinant $\det [y_{ik}(z)]_1^n \neq 0$ for all $z \in D$. Without loss of generality we may assume that $y_{in}(z) \not\equiv 0$, $i = 1, 2, \ldots, n$, and define the functions

(2.2) $$f_{ik}(z) = y_{ik}(z)/y_{in}(z), \qquad i, k = 1, 2, \ldots, n,$$

which are meromorphic in D. Furthermore,

(2.3) $$\det [y_{ik}(z)]_1^n = \prod_{i=1}^n y_{in}(z) \det [f_{ik}(z)]_1^n.$$

Hence, $\det [f_{ik}(z)]_1^n \neq 0$ for all $z \in D$.

Let

$$w = H_i(z; a_1, \ldots, a_{n-1}) = \sum_{k=1}^{n-1} a_k f_{ik}(z), \qquad i = 1, 2, \ldots, n, \tag{2.4}$$

and denote by $D_i(a_1, \ldots, a_{n-1})$ the image of D in the w plane given by

$$H_i(z; a_1, \ldots, a_{n-1}).$$

We state now

THEOREM 1. *(1.1) is disconjugate in D if and only if for every choice of finite complex constants $a_1, \ldots, a_{n-1}$, not all zero, the domains $D_i(a_1, \ldots, a_{n-1})$, $i=1, 2, \ldots, n$, have no common point, i.e.*

$$\bigcap_{i=1}^{n} D_i(a_1, \ldots, a_{n-1}) = \varnothing. \tag{2.5}$$

As pointed out by Schwarz [8, Theorem 3], disconjugacy of (1.1) in D is equivalent to the fact that for any fundamental solution $[y_{ik}(z)]_1^n$ of (2.1), we have $\det [y_{ik}(z_i)]_1^n \neq 0$ for every choice of n (not necessarily distinct) points $z_1, z_2, \ldots, z_n \in D$. According to (2.3) it follows now that disconjugacy of (1.1) in D is equivalent to

$$\prod_{i=1}^{n} y_{in}(z_i) \det [f_{ik}(z_i)]_1^n \neq 0$$

for every choice of $z_1, \ldots, z_n \in D$. Thus, if $y_{in}(z) \neq 0$, $i=1, 2, \ldots, n$, for all $z \in D$, the functions $f_{ik}(z)$ defined by (2.2) are regular in D, and Theorem 1 follows from [8, Theorem 3]. But if we do not assume that $y_{in}(z) \neq 0$ the result does not follow immediately, and it is exactly the zeros of $y_{in}(z)$ that cause the difficulty in the proof of Theorem 1. To handle this we shall require the following two lemmas.

LEMMA 1. *Given a set of n points $z_1, z_2, \ldots, z_n$ of D, there always exists a solution $y(z)$ of (1.1) such that $y_i(z_i) \neq 0$, $i=1, 2, \ldots, n$.*

Proof. By the existence theorem there exists a solution $\boldsymbol{u}(z)$ such that $u_1(z_1)=1$. Suppose $u_2(z_2)=0$, then by the same argument there exists a solution $\boldsymbol{v}(z)$ such that $v_2(z_2)=1$. If $v_1(z_1)=0$, then $y(z)=\boldsymbol{u}(z)+t\boldsymbol{v}(z)$, $t \neq 0$, is a solution of (1.1) which satisfies $y_1(z_1) \neq 0$, $y_2(z_2) \neq 0$. Assume now that $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ are solutions of (1.1) which satisfy $u_i(z_i)=\alpha_i \neq 0$, $i=1, 2, \ldots, j<n$, $u_{j+1}(z_{j+1})=0$, $v_1(z_1)=0$, $v_i(z_i)=\beta_i \neq 0$, $i=1, 2, \ldots, j+1$. If $t \neq -\alpha_i\beta_i^{-1}$, $i=2, \ldots, j+1$, then $y(z)=\boldsymbol{u}(z)+t\boldsymbol{v}(z)$ will be a solution of (1.1) which satisfies $y_i(z_i) \neq 0$, $i=1, 2, \ldots, j+1$.

LEMMA 2. *If (1.1) is not disconjugate in D, and if $y_{in}(z) \not\equiv 0$, $i=1, 2, \ldots, n$, then there exist n points $z_1^*, z_2^*, \ldots, z_n^*$ of D, and a nontrivial solution $y^*(z)$ of (1.1), such that $y_i^*(z_i^*)=0$ and $y_{in}(z_i^*) \neq 0$, $i=1, 2, \ldots, n$.*

Proof. Since (1.1) is not disconjugate in D, there exists a nontrivial solution $y(z)$, such that $y_i(z_i)=0$ for $z_i \in D$, $i=1, 2, \ldots, n$. If $y_{jn}(z_j)=0$ for some $1 \leqq j \leqq n$, then apply a perturbation $y_\varepsilon(z)=y(z)+\varepsilon u(z)$, where $u(z)$ is a solution of (1.1) which satisfies $u_i(z_i)\neq 0$, $i=1,2,\dots,n$, and ε is a complex parameter. By making a proper choice of ε, say $\varepsilon=\varepsilon^*$, we obtain $y^*(z)=y_{\varepsilon^*}(z)$, and by Rouché's theorem $y_i^*(z_i^*)=0$, where $z_i^*\in D$, $i=1,2,\dots n$. Furthermore, ε^* is chosen in such a way to guarantee that $y_{in}(z_i^*)\neq 0$.

We are ready now to prove Theorem 1.

Proof of Theorem 1.

(i) Suppose $b\in\bigcap_{i=1}^n D_i(a_1,\dots,a_{n-1})$ for some choice of finite constants $a_1,\dots,a_{n-1}$, not all zero. Then there exist n points $z_1,z_2,\dots,z_n\in D$ such that

$$H_i(z_i;a_1,\dots,a_{n-1}) = \sum_{k=1}^{n-1} a_k f_{ik}(z_i) = b, \qquad i = 1,2,\dots,n.$$

If $b=\infty$, then $y_{in}(z_i)=0$, $i=1,\dots,n$, and (1.1) is not disconjugate. If $b\neq\infty$, then

$$y_i(z_i) = \sum_{k=1}^{n-1} a_k y_{ik}(z_i)-by_{in}(z_i) = 0, \qquad i = 1,2,\dots,n.$$

Indeed, if $y_{jn}(z_j)\neq 0$ for $1\leqq j\leqq n$, then clearly $y_j(z_j)=0$, and if $y_{jn}(z_j)=0$, then it follows from $b\neq\infty$ that $\sum_{k=1}^{n-1} a_k y_{jk}(z_j)=0$ and we have again $y_j(z_j)=0$. Hence, disconjugacy of (1.1) in D implies (2.5).

(ii) Assume (1.1) is not disconjugate in D; i.e., there exists a nontrivial solution of (1.1), $y^*(z)=\sum_{k=1}^n a_k y_k(z)$, such that $y_i^*(z_i^*)=0$ for $z_i^*\in D$, $i=1,2,\dots,n$. By Lemma 2 we may assume that $y_{in}(z_i^*)\neq 0$. Hence

$$\frac{y_i^*(z_i^*)}{y_{in}(z_i^*)} = \sum_{k=1}^{n-1} a_k f_{ik}(z_i^*)+a_n = 0, \qquad i = 1,\dots,n,$$

and $-a_n\in\bigcap_{i=1}^n D_i(a_1,\dots,a_{n-1})$. This completes the proof of Theorem 1.

3. **Relations between the coefficients $p_{ik}(z)$ and the functions $f_{ik}(z)$.** Replacement of $y_k(z)$, $k=1,2,\dots,n$, by another set of fundamental solutions $w_k(z)$, $k=1,2,\dots,n$, results in a transformation

$$\text{(3.1)}\quad f_{ik}(z)\to F_{ik}(z) = \frac{w_{ik}(z)}{w_{in}(z)} = \frac{\sum_{j=1}^n \alpha_{jk} f_{ij}(z)}{\sum_{j=1}^n \alpha_{jn} f_{ij}(z)}, \qquad i,k = 1,2,\dots,n,\ \det[\alpha_{st}]_1^n \neq 0$$

applied to the matrix $[f_{ik}(z)]_1^n$. Hence, any relation between the entries of the matrices $[p_{ik}(z)]_1^n$ and $[f_{ik}(z)]_1^n$ must remain invariant under mappings of the type (3.1).

Without loss of generality we may assume that

$$\text{(3.2)}\qquad p_{ii}(z) \equiv 0, \qquad i = 1,2,\dots,n,$$

since this can be achieved by means of a transformation [8, p. 489]

$$\text{(3.3)}\quad y_i(z) = \tau_i(z)u_i(z), \qquad \tau_i(z) = c_i \exp\int_{z_0}^z p_{ii}(\zeta)\,d\zeta, \qquad i = 1,2,\dots,n,$$

which leaves $f_{ik}(z)$ unchanged. Assuming (3.2), it is still possible to apply (3.3) with $\tau_i(z)=c_i\neq 0$, where c_i are arbitrary constants. This results in

$$\mathbf{u}'(z) = R(z)\mathbf{u}(z), \qquad R(z) = [r_{ik}(z)]_1^n, \tag{3.4}$$

where

$$r_{ik}(z) = p_{ik}(z)(c_k/c_i), \qquad i, k = 1, 2, \ldots, n. \tag{3.5}$$

Therefore, the coefficients $p_{ik}(z)$ can be determined by the functions $f_{ik}(z)$ up to a relation of the type (3.5). It is easily verified by (3.5) that

$$\sigma_{ij}(z) = p_{ij}(z)p_{ji}(z), \qquad i \neq j, \quad i, j = 1, 2, \ldots, n \tag{3.6}$$

and

$$\eta_{ij}(z) = p'_{ij}(z)/p_{ij}(z), \qquad i \neq j, \quad i, j = 1, 2, \ldots, n \tag{3.7}$$

are independent of the constants c_i. Next we prove that $\sigma_{ij}(z)$ and $\eta_{ij}(z)$ can be expressed in terms of the functions $f_{ik}(z)$, and therefore remain invariant under the group of transformations of the type (3.1). According to (2.2) we have $y_{ik}(z) = f_{ik}(z)y_{in}(z)$. Differentiating and using (1.1) we obtain

$$\sum_{j=1}^{n} p_{ij}\frac{y_{jn}}{y_{in}}[f_{jk}-f_{ik}] = f'_{ik}, \qquad k = 1, 2, \ldots, n-1. \tag{3.8}$$

Thus for every fixed $1 \leqq i \leqq n$, we have $(n-1)$ linear equations for the $(n-1)$ unknown $p_{ij}(y_{jn}/y_{in})$, $j\neq i$, $j=1, 2, \ldots, n$. The $(n-1)\times(n-1)$ matrix $m_{jk}(i, z) = f_{jk}(z)-f_{ik}(z)$, $j=1, 2, \ldots, i-1, i+1, \ldots, n$, $k=1, 2, \ldots, n-1$, satisfies

$$\det [m_{jk}(i, z)] = (-1)^{n+i} \det [f_{st}(z)]_1^n \neq 0$$

for all $z \in D$. Solving (3.8) we get

$$p_{ij}\frac{y_{jn}}{y_{in}} = \frac{\det [h_{st}(i, j, z)]_1^n}{\det [f_{st}(z)]_1^n}, \qquad i \neq j, \quad i, j = 1, 2, \ldots, n, \tag{3.9}$$

where

$$\left.\begin{aligned} h_{st}(i, j, z) &= f_{st}(z), \quad s \neq j, \\ h_{jt}(i, j, z) &= f'_{it}(z) \end{aligned}\right\} \quad \begin{aligned} &s, t = 1, 2, \ldots, n \\ &j \neq i, \quad i, j = 1, 2, \ldots, n. \end{aligned}$$

Setting now

$$B_{ii}(z) = 0, \qquad B_{ij}(z) = \frac{\det [h_{st}(i, j, z)]}{\det [f_{st}(z)]}, \qquad i \neq j, \quad i, j = 1, 2, \ldots, n, \tag{3.10}$$

it follows from (3.9) that

$$\begin{aligned} \sigma_{ij}(z) &= p_{ij}(z)p_{ji}(z) = B_{ij}(z)B_{ji}(z) \\ &= \frac{\det [h_{st}(i, j, z)] \det [h_{st}(j, i, z)]}{(\det [f_{st}(z)])^2}, \qquad i \neq j, \quad i, j = 1, 2, \ldots, n, \end{aligned} \tag{3.11}$$

and

$$\eta_{ij}(z) = \frac{p'_{ij}(z)}{p_{ij}(z)} = \frac{B'_{ij}(z)}{B_{ij}(z)} + \sum_{k=1}^{n} [B_{ik}(z)-B_{jk}(z)], \qquad i \neq j, \quad i, j = 1, \ldots, n. \tag{3.12}$$

By Theorem 1, any condition for the functions $f_{ik}(z)$, $i, k=1, 2, \ldots, n$, to satisfy (2.5), which may be expressed in terms of $\sigma_{ij}(z)$ and $\eta_{ij}(z)$, is equivalent to conditions for disconjugacy of (1.1). For $n=2$, a known result in the theory of functions, namely inequality (1.10), was applied to yield the necessary condition for disconjugacy (1.11). Yet, for $n>2$, we do not know of any necessary condition for the functions $f_{ik}(z)$ to satisfy (2.5). In §7 a condition of this type will be deduced from necessary conditions for disconjugacy obtained in Theorem 5.

4. **A family of "relatively schlicht" functions.** Another way of generalizing Nehari's principle [6, Theorem 1.1] is to generate a family of "relatively schlicht" functions.

Given n points (not necessarily distinct) $z_1, z_2, \ldots, z_n$ in D, let $\boldsymbol{u}=[u_1, \ldots, u_n]$ and $\boldsymbol{v}=[v_1, \ldots, v_n]$ be linearly independent solutions of (1.1), which satisfy

$$(4.1)\quad u_i(z_i) = v_i(z_i) = 0, \qquad i = 1, 2, \ldots, n, \quad i \neq j, k, \quad j \neq k, \quad 1 \leqq j, k \leqq n.$$

Evidently, there always exist at least two linearly independent solutions of (1.1) which satisfy (4.1). (This is an immediate consequence of the existence of a fundamental set of n linearly independent solutions.) Moreover, if $z_i=a \in D$, $i=1, 2, \ldots, n$, $i\neq j, k$, then there exist exactly two linearly independent solutions which satisfy (4.1). But in the general case, where some of the z_i may be distinct, it does not follow from the existence theorem that any three solutions of (1.1) which satisfy $y_i(z_i)=0$, $i=1, \ldots, n$, $i\neq j, k$, are linearly dependent. In Lemma 3 we discuss this situation.

We assume now that $u_t(z)$ and $v_t(z)$, $t=j, k$, (where $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent solutions of (1.1) satisfying (4.1)) are not both identically zero, and we define the functions

$$(4.2)\qquad g_t(z) = u_t(z)/v_t(z), \qquad t = j, k, \quad j \neq k, \quad 1 \leqq j, k \leqq n.$$

In case $u_s(z)\equiv v_s(z)\equiv 0$, $1\leqq s\leqq n$, we do not define $g_s(z)$. We state now

THEOREM 2. *Let $g_j(z)$ and $g_k(z)$ be defined by* (4.2), *where $\boldsymbol{u}$ and $\boldsymbol{v}$ are two linearly independent solutions of* (1.1) *which satisfy* (4.1). *In order that the system* (1.1) *be disconjugate in D, it is necessary and sufficient that for every choice of n points (not necessarily distinct) $z_1, z_2, \ldots, z_n$ in D, and every pair of functions $g_j(z)$ and $g_k(z)$, $j\neq k, j, k=1, 2, \ldots, n$,*

$$(4.3)\qquad g_j(z_j) \neq g_k(z_k), \qquad j \neq k, \quad j, k = 1, 2, \ldots, n$$

holds; i.e. disconjugacy of (1.1) *is equivalent to the "relatively schlichtness" of all pairs of functions $g_j(z)$ and $g_k(z)$, $j\neq k, j, k=1, 2, \ldots, n$.*

For the proof of Theorem 2 we require some preliminary prepositions which we state as a lemma.

LEMMA 3. *Suppose there exist three linearly independent solutions of* (1.1), $\boldsymbol{y}(z)$, $\boldsymbol{v}(z)$, *and* $\boldsymbol{w}(z)$, *which satisfy* $y_i(z_i)=v(z_i)=w_i(z_i)=0$, $z_i \in D$, $i=1, 2, \ldots, n-2$, *then*

(i) (1.1) *is not disconjugate in D.*

(ii) *There exists a pair of functions* $g_j(z)$ *and* $g_k(z)$, $j \neq k$, *which are not relatively schlicht in* D; *i.e.*, $g_j(\zeta_j)=g_k(\zeta_k)$ *for some* ζ_j, $\zeta_k \in D$.

Proof. (i) Let z_{n-1}, $z_n \in D$. There always exists a nontrivial solution $\boldsymbol{u}(z)=\alpha_1\boldsymbol{y}(z)+\alpha_2\boldsymbol{v}(z)+\alpha_3\boldsymbol{w}(z)$ which satisfies $u_{n-1}(z_{n-1})=u_n(z_n)=0$. Hence (1.1) is not disconjugate in D, since $u_i(z_i)=0$, $i=1, 2, \ldots, n$.

(ii) We first make the following remark. Since $\boldsymbol{y}(z)$ and $\boldsymbol{v}(z)$ are linearly independent solutions, then at least one component of each solution, say $y_s(z)$ and $v_m(z)$, $1 \leqq s$, $m \leqq n$, $s \neq m$, is not identically zero. Hence, we may assume that at least two components of $\boldsymbol{v}(z)$ are not identically zero. Suppose now that

$$v_{n-1}(z) \not\equiv 0, \qquad v_n(z) \not\equiv 0, \qquad z \in D, \tag{4.4}$$

and let ζ_{n-1}, $\zeta_n \in D$ be such that $v_{n-1}(\zeta_{n-1}) \neq 0$ and $v_n(\zeta_n) \neq 0$, then the functions $g_{n-1}(z)$ and $g_n(z)$, where $g_t(z)=u_t(z)/v_t(z)$, $t=n-1, n$, are not relatively schlicht in D since $g_{n-1}(\zeta_{n-1})=g_n(\zeta_n)=0$.

In case (4.4) is false and $y_{n-1}(z) \equiv v_{n-1}(z) \equiv w_{n-1}(z) \equiv 0$, we assume that $v_1(z) \not\equiv 0$, $v_n(z) \not\equiv 0$. Let ζ_1, $\zeta_n \in D$ be such that $v_1(\zeta_1) \neq 0$, $v_n(\zeta_n) \neq 0$. Proceeding as before, there exists a nontrivial solution $\boldsymbol{u}(z)=\alpha_1\boldsymbol{y}(z)+\alpha_2\boldsymbol{v}(z)+\alpha_3\boldsymbol{w}(z)$ such that $u_1(\zeta_1)=0$, $u_i(z_i)=0$, $i=2, \ldots, n-2$, $u_{n-1}(z) \equiv 0$, $u_n(\zeta_n)=0$, and $g_1(\zeta_1)=g_n(\zeta_n)=0$. If $y_t(z) \equiv v_t(z) \equiv w_t(z) \equiv 0$ for $t=n-1, n$, we may assume that $v_1(z) \not\equiv 0$, $v_2(z) \not\equiv 0$ and proceed as before.

Proof of Theorem 2. (i) Necessary. Suppose $g_j(z_j)=g_k(z_k)=\beta\alpha^{-1}$, then $\boldsymbol{y}(z)=\alpha\boldsymbol{u}(z)-\beta\boldsymbol{v}(z)$ satisfies $y_i(z_i)=0$, $i=1, 2, \ldots, n$.

(ii) Sufficient. Suppose there exists a solution $\boldsymbol{u}(z)$ such that $u_i(z_i)=0$, $i=1, 2, \ldots, n$, $z_i \in D$. Let $\boldsymbol{v}(z)$ be a solution of (1.1), which is linearly independent of $\boldsymbol{u}(z)$ and satisfies $v_i(z_i)=0$, $i=1, 2, \ldots, n-2$. If

$$v_{n-1}(z_{n-1}) \neq 0, \qquad v_n(z_n) \neq 0, \tag{4.5}$$

then $g_{n-1}(z_{n-1})=g_n(z_n)=0$. So suppose (4.5) is false and $v_{n-1}(z_{n-1})=0$. Assume now that $u_n(z)$ and $v_n(z)$ are not both identically zero. Denote by S_n the set of common zeros in D of $u_n(z)$ and $v_n(z)$, and let $\zeta_n \notin S_n$, $\zeta_n \in D$. There exists a nontrivial solution $\boldsymbol{y}(z)=\alpha_1\boldsymbol{u}(z)+\alpha_2\boldsymbol{v}(z)$, such that $y_n(\zeta_n)=0$ and $y_i(z_i)=0$, $i=1, 2, \ldots, n-1$. Moreover, there exists another solution $\boldsymbol{w}(z)$, which is linearly independent of $\boldsymbol{y}(z)$ and satisfies $w_i(z_i)=0$, $i=3, 4, \ldots, n-1$, $w_n(\zeta_n)=0$. We claim now that $w_t(z_t) \neq 0$, $t=1, 2$. Because if $w_2(z_2)=0$, then $u_i(z_i)=v_i(z_i)=w_i(z_i)=0$, $i=1, 2, \ldots, n-1$, and by Lemma 3 it follows from the relatively schlichtness in D of every pair of functions $g_j(z)$ and $g_k(z)$ that $\boldsymbol{w}(z)=\beta_1\boldsymbol{u}(z)+\beta_2\boldsymbol{v}(z)$. But since $\boldsymbol{w}(z)$ and $\boldsymbol{y}(z)$ are linearly independent, it follows now from $w_n(\zeta_n)=y_n(\zeta_n)=0$ that $u_n(\zeta_n)=v_n(\zeta_n)=0$, which contradicts our assumption that $\zeta_n \notin S_n$. So, $w_2(z_2) \neq 0$, and similarly

$w_1(z_1) \neq 0$. Considering now the functions $g_t(z) = y_t(z)/w_t(z)$, $t = 1, 2$, it follows that $g_1(z_1) = g_2(z_2) = 0$. In case $u_n(z) \equiv v_n(z) \equiv 0$ ($S_n = D$), we may assume that $u_r(z)$ and $v_r(z)$ are not both identically zero for some r, $1 \leqq r \leqq n-1$, and proceed as before.

5. **Quantities invariant under the mapping** $f \to Tf$, $g \to Tg$. Our next goal is to establish relations between the coefficients $p_{ik}(z)$ of the system (1.1) and the functions $g_j(z)$ and $g_k(z)$ defined by (4.2). As has become by now a standard procedure, we have to find out first what kind of transformations may be applied to g_j and g_k without affecting their relations with the coefficients p_{ik}. If $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ are replaced by the linearly independent solutions $A\boldsymbol{u}(z) + B\boldsymbol{v}(z)$ and $C\boldsymbol{u}(z) + D\boldsymbol{v}(z)$ respectively, then according to (4.2), g_j and g_k are replaced by Tg_j and Tg_k, where T is the linear transformation (1.5). Therefore any relation between the coefficients p_{ik} and the functions g_j and g_k should be expressed by quantities which remain invariant under the transformation $g_t \to Tg_t$, $t = j, k$.

This brings up the following question. Given two meromorphic functions, $f(z)$ and $g(z)$, in a domain D, what combinations of $f(z)$ and $g(z)$, and their derivatives remain invariant under the transformation $f \to Tf$, $g \to Tg$. Two combinations of this type were given by Nehari, namely $\Phi[f, g]$ and $\Psi[f, g]$ which are defined by (1.6) and (1.7). By differentiating $\Phi[f, g]$ and $\Psi[f, g]$, it is possible to derive more quantities with this invariance property. One combination of this type which will be of interest later is

$$\Theta[f, g] = \frac{f''}{f'} - \frac{2f'}{f-g} = \frac{1}{2}\left[\frac{\Phi'[f, g]}{\Phi[f, g]} + \Psi[f, g]\right]. \tag{5.1}$$

In the following theorem we shall prove that with some restrictions on the functions $f(z)$ and $g(z)$, every combination of $f(z)$ and $g(z)$ with the desired invariance property can be derived from $\Phi[f, g]$ and $\Theta[f, g]$.

Denote by $RC(D)$ the *restricted class* in D (see [7, p. 159]), namely the class of functions $f(z)$ which are meromorphic in D with simple poles at most and which satisfy $f'(z) \neq 0$ for all $z \in D$. Note that if f belongs to $RC(D)$ so does Tf.

THEOREM 3. *Let $f(z) \in RC(D)$, and let $g(z)$ be a meromorphic function in D such that*

$$f(z) \neq g(z), \qquad z \in D. \tag{5.2}$$

Let $E[f(z), g(z)] = E[f(z), \ldots, f^{(n)}(z), g(z), \ldots, g^{(n)}(z)]$ be a combination of $f(z)$ and $g(z)$ and their derivatives up to order n. If $E[f(z), g(z)]$ remains invariant under the transformation $f \to Tf$, $g \to Tg$, i.e.,

$$E[Tf(z), Tg(z)] = E[f(z), g(z)] = I(z), \tag{5.3}$$

where T is defined by (1.5), *then $E[f(z), g(z)]$ may be derived from $\Phi[f(z), g(z)] = \phi(z)$ and $\Theta[f(z), g(z)] = \theta(z)$, and*

$$I(z) = E[f(z), g(z)] = E^*[\phi(z), \theta(z)] \tag{5.4}$$

where E^* is a combination of $\phi^{(s)}(z)$, $s = 0, 1, \ldots, n-1$, and $\theta^{(r)}(z)$, $r = 0, 1, \ldots, n-2$.

Proof. Let $z_0 \in D$. Without loss of generality we may assume that

$$f(z_0) = 1, \qquad g(z_0) = 0, \qquad f'(z_0) = 1, \tag{5.5}$$

since this situation may be achieved by means of a transformation $f \to Tf$, $g \to Tg$ which, according to (5.3), leaves $I(z)$ unchanged. Setting now $z = z_0$ in (1.6) and (5.1), it follows from (5.5) that

$$g'(z_0) = \phi(z_0), \qquad f''(z_0) = 2 + \theta(z_0). \tag{5.6}$$

Differentiation of (1.6) and (5.1) gives us

$$\phi^{(m)}(z) = \frac{g^{(m+1)}(z)f'(z)}{[f(z)-g(z)]^2} + \frac{M_m[f(z), g(z)]}{[f(z)-g(z)]^{m+2}}, \qquad m = 0, 1, 2, \ldots, \tag{5.7}$$

and

$$\theta^{(m)}(z) = \frac{f^{(m+2)}(z)}{f'(z)} + \frac{N_m[f(z), g(z)]}{[f(z)-g(z)]^{m+1}[f'(z)]^{m+1}}, \qquad m = 0, 1, 2, \ldots, \tag{5.8}$$

where M_m and N_m are polynomials of the arguments $f(z), f'(z), \ldots, f^{m+1}(z)$ and $g(z), g'(z), \ldots, g^{(m)}(z)$. By elimination and induction it follows now from (5.5), (5.6), (5.7) and (5.8) that

$$g^{(m+1)}(z_0) = \phi^{(m)}(z_0) + R_{m-1}[\phi(z), \theta(z)]_{z=z_0}, \qquad m = 1, 2, \ldots, \tag{5.9}$$

and

$$f^{(m+2)}(z_0) = \theta^{(m)}(z_0) + \tilde{R}_{m-1}[\phi(z), \theta(z)]_{z=z_0}, \qquad m = 1, 2, \ldots, \tag{5.10}$$

where R_{m-1} and $\tilde{R}_{m-1}$ are polynomials of the arguments $\phi^{(t)}(z)$ and $\theta^{(t)}(z)$, $t = 0, 1, \ldots, m-1$. Insertion of (5.5), (5.6), (5.9) and (5.10) in $E[f(z), g(z)]$ leads us to

$$\begin{aligned} &E[f(z_0), \ldots, f^{(n)}(z_0), g(z_0), \ldots, g^{(n)}(z_0)] \\ &\quad = E[1, 1, \theta(z_0)+2, \ldots, \theta^{(n-2)}(z_0) + R_{m-3}, 0, \phi(z_0), \ldots, \phi^{(n-1)}(z_0) + \tilde{R}_{m-2}] \\ &\quad = E^*[\phi(z_0), \ldots, \phi^{(n-1)}(z_0), \theta(z_0), \ldots, \theta^{(n-2)}(z_0)]. \end{aligned}$$

Hence

$$I(z_0) = E[f(z), g(z)]_{z=z_0} = E^*[\phi(z), \theta(z)]_{z=z_0}. \tag{5.11}$$

Since (5.11) holds for every $z_0 \in D$, it implies the identity (5.4).

REMARK. It is easily confirmed that for $f(z)$ and $g(z)$ satisfying the assumptions of Theorem 3, $\phi(z) = \Phi[f(z), g(z)]$ and $\theta(z) = \Theta[f(z), g(z)]$ are regular functions in D. Moreover, $\phi(z) \neq 0$ for $z \in D$, if and only if in addition to the assumptions of the theorem we have $g(z) \in RC(D)$.

For functions $f(z)$ and $g(z)$ which belong to $RC(D)$ and satisfy (5.2), the function $\psi(z) = \Psi[f(z), g(z)]$, defined by (1.7), is also regular in D, and by (5.1)

$$\theta(z) = \tfrac{1}{2}[\phi'(z)/\phi(z) + \psi(z)] \tag{5.12}$$

holds. By inserting (5.12) in $E^*[\phi(z), \theta(z)]$ we obtain

$$E[f(z), g(z)] = E^*[\phi(z), \theta(z)] = E^{**}[\phi(z), \psi(z)],$$

where E^{**} is a combination of $\phi^{(s)}(z)$, $s=0, 1, \ldots, n-1$, and $\psi^{(r)}(z)$, $r=0, 1, \ldots, n-2$. Hence, for $f(z), g(z) \in RC(D)$ satisfying (5.2), we can replace $\Theta[f(z), g(z)]$ in Theorem 3 by $\Psi[f(z), g(z)]$.

6. **A subfamily of relatively schlicht functions.** For the applications it is useful to consider only a subfamily of functions of the type (4.2). Let $a \in D$, and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be linearly independent solutions of (1.1) satisfying

$$(6.1) \quad u_i(a) = v_i(a) = 0, \qquad i = 1, 2, \ldots, n, \quad i \neq j, k, \quad j \neq k, \quad 1 \leqq j, k \leqq n.$$

Similarly to (4.2), we set

$$(6.2) \qquad g_t(z, a) = u_t(z)/v_t(z), \qquad t = j, k, \quad j \neq k, \quad 1 \leqq j, k \leqq n.$$

Before we consider the problem of establishing relations between the functions (6.2) and the coefficients $p_{ik}(z)$ of (1.1), we first make the following remarks.

(i) As already discussed in §4, there exist exactly two linearly independent solutions which satisfy (6.1). Therefore, any solution of (1.1) satisfying $y_i(a)=0$, $i=1, 2, \ldots, n, i\neq j, k$, is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$. It follows that replacement of $\boldsymbol{u}$ and $\boldsymbol{v}$ in (6.2), by a different set of two linearly independent solutions $\boldsymbol{y}$ and $\boldsymbol{w}$ satisfying $y_i(a)=w_i(a)=0$, $i=1, 2, \ldots, n$, $i\neq j, k$, results in a transformation $g_t(z, a) \to Tg_t(z, a)$, $t=j, k$, where T is defined by (1.5). Hence, the relations between the functions (6.2) and the coefficients $p_{ik}(z)$ must stay invariant under the transformation $g_t \to Tg_t$, $t=j, k$.

(ii) Since the transformation (3.3) leaves the functions (6.2) unchanged, we may assume that $p_{ii}(z)\equiv 0$, $i=1, 2, \ldots, n$. In this case the coefficients $p_{ik}(z)$ can be determined by the functions (6.2) only up to a relation of the type (3.5).

THEOREM 4. *Let $p_{ik}(z)$, $i, k=1, 2, \ldots, n$ be regular functions in D and assume*

$$(3.2) \qquad p_{ii}(z) \equiv 0, \qquad i = 1, 2, \ldots, n.$$

Let the functions $g_j(z, a)$ and $g_k(z, a)$ be defined by (6.2), *where $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent solutions of* (1.1) *satisfying* (6.1). *If*

$$(6.3) \qquad \phi_{jk}(z, a) = \Phi[g_j(z, a), g_k(z, a)] = g_j' g_k'/(g_j - g_k)^2$$

and

$$(6.4) \qquad \theta_{jk}(z, a) = \Theta[g_j(z, a), g_k(z, a)] = g_j''/g_j' - 2g_j'/(g_j - g_k)$$

where $g_t' = d[g_t(z, a)]/dz$, $t=j, k$, then

$$(6.5) \quad \phi_{jk}(a, a) = -p_{jk}(a)p_{kj}(a), \qquad j \neq k, \quad j, k = 1, 2, \ldots, n, \quad a \in D,$$

and if $p_{jk}(a)\neq 0$, then

$$(6.6) \quad \theta_{jk}(a, a) = \frac{p_{jk}'(a)}{p_{jk}(a)} + \frac{\sum_{i=1}^{n} p_{ji}(a)p_{ik}(a)}{p_{jk}(a)}, \qquad j \neq k, \quad j, k = 1, 2, \ldots, n, \quad a \in D.$$

Proof. Let $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ satisfy

$$(6.7)\quad u_i(a) = \delta_{ik}, \qquad v_i(a) = \delta_{ij}, \qquad j \neq k, \quad i = 1, 2, \ldots, n, \quad 1 \leqq j, k \leqq n.$$

According to (1.1) and (6.2) we have

$$(6.8)\qquad g_t'(z, a) = \frac{\sum_{i=1}^{n} p_{ti}(z)[u_i(z)v_t(z) - u_t(z)v_i(z)]}{v_t^2(z)}.$$

Therefore,

$$(6.9)\qquad \phi_{jk}(z, a) = \frac{\sum_{i=1}^{n} p_{ji}[u_i v_j - u_j v_i] \sum_{s=1}^{n} p_{ks}[u_s v_k - u_k v_s]}{[u_j v_k - u_k v_j]^2},$$

and (6.5) follows now from (6.9) and (6.7). By setting $t=j$ and $z=a$ in (6.8) we obtain $g_j'(a, a) = p_{jk}(a)$. Hence if $p_{jk}(a) \neq 0$ for $a \in D$, $g_j(z, a)$ belongs to the restricted class of functions in some neighborhood $N(a) \subset D$ of the point a. Clearly, both $g_j(z, a)$ and $g_k(z, a)$ are meromorphic functions in D. So, we conclude now that $\theta_{jk}(z, a)$ is regular in $N(a)$. By differentiating (6.8) and using (6.7) we obtain (6.6).

Since any solution of (1.1) which satisfies $y_i(a) = 0$, $i \neq j, k$, $i = 1, 2, \ldots, n$, is a linear combination of the normalized solutions $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ defined by (6.7), a different choice of the two solutions would replace g_t by Tg_t, $(t = j, k)$ where T is of the form (1.5). But $\phi(z, a)$ and $\theta(z, a)$ are not affected by this transformation, hence (6.5) and (6.6) hold for any choice of the solutions $\boldsymbol{u}(z)$ and $\boldsymbol{v}(z)$ regardless of the normalization (6.7).

Remarks. (i) Note that (6.5) holds even without the assumption (3.3), but in this case $p_{ii}(z)$, $i = 1, 2, \ldots, n$, are not determined by the functions (6.2).

(ii) If $p_{jk}(z) \neq 0$ for all $z \in D$, $j \neq k$, $j, k = 1, 2, \ldots, n$, then (6.5) and (6.6) are the "fundamental relations" between the functions $g_j(z, a)$ and $g_k(z, a)$ and the coefficients $p_{jk}(z)$ of (1.1).

7. Necessary conditions for disconjugacy in the unit disk.

Theorem 5. *Let $p_{jk}(z)$, $j, k = 1, 2, \ldots, n$ be regular for $|z| < 1$. If the system* (1.1) *is disconjugate in $|z| < 1$, then*

$$(7.1)\qquad |p_{jk}(z)p_{kj}(z)| \leqq 1/(1-|z|^2)^2, \qquad |z| < 1, \quad j \neq k.$$

Proof. By Theorem 2 disconjugacy of (1.1) in $|z| < 1$ implies the "relatively schlichtness" in $|z| < 1$ of every pair of functions $g_j(z)$ and $g_k(z)$ defined by (4.2). In particular $g_j(z, a)$ and $g_k(z, a)$ defined by (6.2) are relatively schlicht. Applying (1.10), it follows that

$$(7.2)\qquad |\phi_{jk}(z, a)| = |\Phi[g_j(z, a), g_k(z, a)]| \leqq \frac{1}{(1-|z|^2)^2}, \qquad |z| < 1$$

holds for every $j, k = 1, 2, \ldots, n$, $j \neq k$, and any $|a| < 1$. Setting $z = a$ in (7.2), we obtain by (6.5) and by remark (i) of the last section

$$|p_{jk}(a)p_{kj}(a)| = |\phi_{jk}(a, a)| \leqq \frac{1}{(1-|a|^2)^2}, \qquad |a| < 1, \quad j \neq k.$$

We add the following remarks.

(i) As regards the function $\theta_{jk}(z, a)$ defined by (6.4), it is clear that there cannot exist an upper bound unless an assumption is made which bounds $|g_j'(z, a)|$ away from zero. Therefore, (6.6) does not yield a necessary condition for disconjugacy of (1.1). Furthermore, in order to utilize Nehari's result [6, Theorem 7.2] and obtain an upper bound for $\Psi[g_j(z, a), g_k(z, a)]$, one has to assume that both $g_j(z, a)$ and $g_k(z, a)$ are univalent in $|z|<1$, besides being relatively schlicht there.

(ii) Let $\pi_{ik}(\zeta)$, $i, k=1, 2, \dots, n$, be regular in the domain Δ, and consider the differential system

$$\boldsymbol{\omega}'(\zeta) = \Pi(\zeta)\boldsymbol{\omega}(\zeta) \tag{7.3}$$

where $\boldsymbol{\omega}(\zeta)=[\omega_1(\zeta), \omega_2(\zeta), \dots, \omega_n(\zeta)]$ and $\Pi(\zeta)=[\pi_{ik}(\zeta)]_1^n$. If Δ is conformally equivalent to D, i.e., if there exists a one-to-one regular function $\zeta(z)$ which maps D onto Δ, then (7.3) may be transformed by $y_j(z)=\omega_j[\zeta(z)]$, $j=1, \dots, n$, into the system (1.1), and

$$p_{jk}(z)p_{kj}(z) = \pi_{jk}[\zeta(z)]\pi_{kj}[\zeta(z)](d\zeta/dz)^2 \tag{7.4}$$

holds. Furthermore, (7.3) is disconjugate in Δ if and only if the transformed system (1.1) is disconjugate in D. Thus, in view of (7.4), Theorem 5 yields a necessary condition for disconjugacy in any simply-connected domain Δ having more than one boundary point and such that $\infty \notin \Delta$.

We conclude this section with the following corollary.

Let $f_{ik}(z)$, $i, k=1, 2, \dots, n$, be regular functions in the unit disk D, such that $f_{in}(z)\equiv 1$, $i=1, 2, \dots, n$, and $\det [f_{ik}(z)]_1^n\neq 0$ *for $z \in D$. Let $w=H_i(z; a_1, \dots, a_{n-1})$ be defined as in* (2.4), *and denote by $D_i(a_1, \dots, a_{n-1})$ the image of D in the w plane given by $H_i(z; a_1, \dots, a_{n-1})$. If for every choice of finite complex constants, $a_1, \dots, a_{n-1}$, not all zero, the domains $D_i(a_1, \dots, a_{n-1})$, $i=1, 2, \dots, n$, have no common point, i.e.*

$$\bigcap_{i=1}^{n} D_i(a_1, \dots, a_{n-1}) = \varnothing, \tag{2.5}$$

then

$$|B_{ij}(z)B_{ji}(z)| \leq \frac{1}{(1-|z|^2)^2}, \qquad i \neq j, \quad i,j = 1, \dots, n, \quad |z| < 1, \tag{7.5}$$

where $B_{ij}(z)$ are defined by (3.10).

Proof. Since the functions $f_{ik}(z)$ are regular in $|z|<1$ and $\det [f_{ik}(z)]_1^n\neq 0$, we may set

$$Y(z) = [f_{ik}(z)]_1^n, \tag{7.6}$$

where $Y(z)$ is a fundamental solution of the matrix equation (2.1), and $P(z)$ is given by $P(z)= Y'(z)Y^{-1}(z)$. By Theorem 1, (2.5) implies the disconjugacy of the corresponding system (1.1). In view of (3.11) and (7.1) the result follows.

REMARKS. (i) (7.5) is a generalization of (1.10) for the case $n>2$.

(ii) The last result can be extended to functions $f_{ik}(z)$, $i, k=1, 2, \ldots, n$, which are meromorphic in the unit disk D, provided there exist n polynomials $s_i(z)$, $i=1, 2, \ldots, n$, such that $s_i(z)f_{ik}(z)$, $i, k=1, 2, \ldots, n$, are regular for $|z|<1$, and

$$\det [s_i(z)f_{ik}(z)]_1^n = \prod_{i=1}^{n} s_i(z) \det [f_{ik}(z)] \neq 0, \qquad |z| < 1.$$

In this case we replace (7.6) by

$$Y(z) = [s_i(z)f_{ik}(z)]_1^n \tag{7.6'}$$

and proceed as before.

8. **Disfocality of nth order differential equations.** In the special case where

$$P(z) = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ -q_n & -q_{n-1} & \cdot & \cdot & \cdot & \cdot & \cdot & -q_2 & -q_1 \end{bmatrix} \tag{8.1}$$

the column vector $y(z)=[y_1(z), \ldots, y_n(z)]$ of (1.1) becomes $[w(z), w'(z), \ldots, w^{(n-1)}(z)]$, and (1.1) is equivalent to the differential equation

$$w^{(n)}(z)+q_1(z)w^{(n-1)}(z)+\cdots+q_n(z)w(z) = 0. \tag{8.2}$$

In this case disconjugacy of (1.1) in D is equivalent to disfocality of (8.2) in the same domain D. (8.2) *is called disfocal in D if for every choice of n (not necessarily distinct) points $z_1, \ldots, z_n$ of D, the only solution of* (8.2) *satisfying* $w(z_1)=w'(z_2) =\cdots=w^{(n-1)}(z_n)=0$, *is the trivial one* $w(z)\equiv 0$ (see [6]).

Let $q_k(z)$, $k=1, 2, \ldots, n$, be regular functions in $|z|<1$. If (8.2) is disfocal in $|z|<1$, it follows from (7.1) and (8.1) that

$$|q_2(z)| \leqq 1/(1-|z|^2)^2, \qquad |z| < 1. \tag{8.3}$$

But (7.1) does not yield bounds for the other coefficients of (8.2), since by (8.1) $p_{in}(z)\equiv 0$ for $i=1, 2, \ldots, n-2$. Yet, such bounds may be obtained by slight modifications of Theorems 4 and 5.

THEOREM 6. *Let $q_k(z)$, $k=1, 2, \ldots, n$, be regular in the domain D, and let $u(z)$ and $v(z)$ be linearly independent solutions of* (8.2) *which satisfy*

$$u^{(s)}(a) = v^{(s)}(a) = 0, \qquad s = 0, 1, \ldots, n-1, \quad s \neq j-1, j, \quad 1 \leqq j \leqq n-1, \quad a \in D. \tag{8.4}$$

Let

$$g_j(z, a) = \frac{u^{(j-1)}(z)}{v^{(j-1)}(z)}, \qquad g_{j+1}(z, a) = \frac{u^{(j)}(z)}{v^{(j)}(z)}, \qquad 1 \leqq j \leqq n-1. \tag{8.5}$$

If

$$\phi_{j,j+1}(z,a) = \Phi[g_j(z,a), g_{j+1}(z,a)] = \frac{g_j' g_{j+1}'}{(g_j - g_{j+1})^2}, \qquad j = 1, 2, \ldots, n-1, \tag{8.6}$$

and

$$\theta_{n-1,n}(z,a) = \Theta[g_{n-1}(z,a), g_n(z,a)] = \frac{g_{n-1}''}{g_{n-1}'} - \frac{2g_{n-1}'}{g_{n-1} - g_n}, \tag{8.7}$$

then

$$\phi_{j,j+1}(a,a) = \phi_{j,j+1}'(a,a) = \cdots = \phi_{j,j+1}^{(n-j-2)}(a,a) = 0, \tag{8.8}$$

$$\phi_{j,j+1}^{(n-j-1)}(a,a) = q_{n-j+1}(a), \qquad j = 1, 2, \ldots, n-1 \tag{8.9}$$

and

$$\theta_{n-1,n}(a,a) = -q_1(a). \tag{8.10}$$

All derivatives are with respect to z.

Proof. Since (8.6) and (8.7) remain invariant under the transformation $g_t \to Tg_t$, $t = j, j+1$, where T is given by (1.5), we may assume that

$$u(z) = w_j(z), \qquad v(z) = w_{j+1}(z), \qquad 1 \leqq j \leqq n-1, \tag{8.11}$$

where $w_t(z)$, $t = 1, 2, \ldots, n$, is a fundamental set of solutions of (8.2) which satisfy

$$w_t^{(s-1)}(a) = \delta_{st}, \qquad s, t = 1, 2, \ldots, n. \tag{8.12}$$

This assumption results in simplification of the calculations. According to (8.5) and (8.11) we obtain now

$$g_j'(z,a) = \frac{w_j^{(j)}(z) w_{j+1}^{(j-1)}(z) - w_j^{(j-1)}(z) w_{j+1}^{(j)}(z)}{[w_{j+1}^{(j-1)}(z)]^2} = \frac{L_j(z)}{[w_{j+1}^{(j-1)}(z)]^2},$$

$$g_{j+1}'(z,a) = \frac{w_j^{(j+1)}(z) w_{j+1}^{(j)}(z) - w_j^{(j)}(z) w_{j+1}^{(j+1)}(z)}{[w_{j+1}^{(j)}(z)]^2} = \frac{K_j(z)}{[w_{j+1}^{(j)}(z)]^2}.$$

Hence

$$\phi_{j,j+1}(z,a) = K_j(z)/L_j(z). \tag{8.13}$$

By (8.12) we obtain for $z = a$

$$L_j(a) = -1, \qquad K_j(a) = K_j'(a) = \cdots = K_j^{(n-j-2)}(a) = 0, \qquad j = 1, 2, \ldots, n-1, \tag{8.14}$$

and

$$K_j^{(n-j-1)}(a) = w_j^{(n)}(a) = -q_{n-j+1}(a), \qquad j = 1, 2, \ldots, n-1. \tag{8.15}$$

(8.8) and (8.9) follow now from (8.13), (8.14) and (8.15).

In a similar way, it is easily verified that

$$\theta_{n-1,n}(z,a) = L_{n-1}'(z)/L_{n-1}(z).$$

Setting $z = a$, (8.10) follows.

We apply now Theorem 6 in order to obtain necessary conditions for disfocality of (8.2) in the unit disk.

THEOREM 7. *Let $q_k(z)$, $k=1, 2, \ldots, n$, be regular functions in the unit disk. If equation* (8.2) *is disfocal in* $|z|<1$, *then*

$$|q_k(z)| \leqq A_k/(1-|z|^2)^k, \qquad k = 2, 3, \ldots, n, \quad |z| < 1, \tag{8.16}$$

where

$$A_2 = 1, \quad A_k = (k-2)!\left(\frac{k+2}{4}\right)^2\left(\frac{k+2}{k-2}\right)^{(k-2)/2}, \qquad k = 3, 4, \ldots, n. \tag{8.17}$$

We require the following elementary result for the proof of Theorem 7.

LEMMA 4. *Let $h_k(z)$, $k=1, 2, \ldots$, be a regular function in* $|z|<1$. *If*

$$|h_k(z)| \leqq 1/(1-|z|^2)^k, \qquad |z| < 1, \tag{8.18}$$

then

$$|h_k^{(s)}(z)| \leqq C(s, k)/(1-|z|^2)^{s+k}, \qquad |z| < 1, \quad s = 1, 2, \ldots, \tag{8.19}$$

where $C(s, k)$ are constants depending only on s and k.

Proof. Let $h_k(z)=\sum_{j=0}^{\infty} b_j z^j$, then by the Cauchy inequality

$$|b_j| \leqq r^{-j}M(r), \qquad M(r) = \max_{|z|=r<1} |h_k(z)|.$$

By (8.18), $M(r)\leqq(1-r^2)^{-k}$. Therefore,

$$|b_j| \leqq \min_{0<r<1} r^{-j}(1-r^2)^{-k} = m(j, k) = \left(\frac{2k+j}{2k}\right)^k\left(\frac{2k+j}{j}\right)^{j/2}, \qquad j = 1, 2, \ldots. \tag{8.20}$$

Set

$$\eta_k(\zeta) = h_k[z(\zeta)]\left(\frac{dz}{d\zeta}\right)^k, \qquad z(\zeta) = \frac{\zeta+a}{1+\bar{a}\zeta}, \quad |a| < 1. \tag{8.21}$$

$z(\zeta)$ is a mapping of $|\zeta|<1$ onto $|z|<1$, and therefore $\eta_k(\zeta)=\sum_{j=0}^{\infty} \beta_j\zeta^j$ is regular in $|\zeta|<1$. Moreover, since

$$|dz/d\zeta| = (1-|z|^2)/(1-|\zeta|^2),$$

it follows from (8.18) that

$$|\eta_k(\zeta)| \leqq 1/(1-|\zeta|^2)^k, \qquad |\zeta| < 1. \tag{8.22}$$

Consequently,

$$|\beta_j| \leqq m(j, k), \qquad j = 1, 2, \ldots. \tag{8.23}$$

Differentiation of (8.21) leads us to

$$h_k'(z) = \eta_k'(\zeta)(d\zeta/dz)^{k+1}+k\eta_k(\zeta)(d\zeta/dz)^{k-1}(d^2\zeta/dz^2). \tag{8.24}$$

It is easily confirmed that

$$\zeta''(a) = 2\bar{a}/(1-|a|^2)^2, \qquad |z| < 1,$$

and by setting now $z=a$ in (8.24) we obtain

$$\text{(8.25)} \qquad |h_k'(a)| \leqq \frac{|\eta_k'(0)|+2k|a|\,|\eta_k(0)|}{(1-|a|^2)^{k+1}} \leqq \frac{m(1,k)+2k}{(1-|a|^2)^{k+1}} = \frac{C(1,k)}{(1-|a|^2)^{k+1}}.$$

To obtain a bound for $|h_k''(z)|$, one can either apply (8.19) to $h_k'(z)$ or differentiate (8.21) twice. Higher derivatives may be obtained in a similar way.

REMARK. If

$$\text{(8.26)} \qquad h_k(a) = h_k'(a) = \cdots = h_k^{(s-1)}(a) = 0, \qquad s = 1, 2, \ldots,$$

then for $z=a$ we have

$$\text{(8.27)} \qquad |h_k^{(s)}(a)| = |\eta_k^{(s)}(0)|/(1-|a|^2)^{s+k} \leqq s!\,m(s,k)/(1-|a|^2)^{s+k}.$$

Proof of Theorem 7. Since (8.2) is disfocal in $|z|<1$, it follows from Theorem 2 (and may easily be verified directly) that for every $1 \leqq j \leqq n-1$ and any $|a|<1$, the functions $g_j(z,a)$ and $g_{j+1}(z,a)$, defined by (8.5), are "relatively schlicht" in $|z|<1$. Consequently,

$$\text{(8.28)} \quad |\phi_{j,j+1}(z,a)| = |\Phi[g_j(z,a), g_{j+1}(z,a)]| \leqq \frac{1}{(1-|z|^2)^2}, \qquad |z| < 1.$$

We utilize now the relations between the functions $\phi_{j,j+1}$ and the coefficients q_{n-j+1}, established in Theorem 6. For $j=n-1$, it follows immediately from (8.9) and (8.28) that

$$|q_2(a)| = |\phi_{n-1,n}(a,a)| \leqq 1/(1-|a|^2)^2, \qquad |a| < 1.$$

For $1 \leqq j \leqq n-2$ we apply Lemma 4 to $\phi_{j,j+1}(z,a)$ with $k=2$ and $s=n-j-1$. By (8.9) and (8.19) we conclude that

$$|q_{n-j+1}(a)| = |\phi_{j,j+1}^{(n-j-1)}(a,a)| \leqq \frac{A_{n-j+1}}{(1-|z|^2)^{n-j+1}}, \qquad j = 1, 2, \ldots, n-2.$$

Moreover, according to (8.8) and to the remark following Lemma 4,

$$A_{n-j+1} \leqq (n-j-1)!\,m(n-j-1,2) = (n-j-1)!\left(\frac{n-j+3}{4}\right)^2\left(\frac{n-j+3}{n-j-1}\right)^{(n-j-1)/2},$$

which completes the proof of the theorem.

We add the following remarks:

(i) (8.10) cannot be utilized to yield a bound for $|q_1(z)|$, since a bound for $|\theta_{n-1,n}(z,a)|$ may be obtained only if $g_{n-1}(z,a)$ is univalent in $|z|<1$, which is more than we can conclude from our assumptions.

(ii) The technique of differentiating the functions ϕ, may also be applied in the general case when the matrix $P(z)$ does not take the special form (8.1). Assume now that (1.1) is disconjugate in $|z|<1$ and that (3.3) holds. By differentiating (6.9) once and setting $z=a$, we obtain

$$\begin{aligned}\phi'_{jk}(a, a) = & -p'_{jk}(a)p_{kj}(a)-p_{jk}(a)p'_{kj}(a) \\ & -\sum_{i=1}^{n} [p_{ji}(a)p_{ik}(a)p_{kj}(a)+p_{ki}(a)p_{ij}(a)p_{jk}(a)].\end{aligned} \tag{8.29}$$

According to (7.1) and (7.2) we may apply Lemma 4 to $p_{jk}(z)p_{kj}(z)$ as well as to $\phi_{jk}(z, a)$. It follows now from (8.19) that

$$|\phi'_{jk}(a, a)| \leqq \frac{C(1, 2)}{(1-|a|^2)^3}, \qquad |a| < 1$$

$$|p'_{jk}(a)p_{kj}(a)+p_{jk}(a)p'_{kj}(a)| \leqq \frac{C(1, 2)}{(1-|a|^2)^3}, \qquad |a| < 1$$

which by (8.29) yields

$$\left|\sum_{i=1}^{n} [p_{ji}(a)p_{ik}(a)p_{kj}(a)+p_{ki}(a)p_{ij}(a)p_{jk}(a)]\right| \leqq \frac{2C(1, 2)}{(1-|a|^2)^3}, \qquad |a| < 1. \tag{8.30}$$

For $n=3, j=1, k=2$ (8.30) reduces to

$$|\det [P(a)]| \leqq 2C(1, 2)/(1-|a|^2)^3, \qquad |a| < 1.$$

By taking the second derivative of (6.9) at the point $z=a$, it is possible to obtain sums of products of 4 coefficients of the matrix $P(z)$ ($n \geqq 4$), and similar results for higher derivatives. The actual calculation is somewhat cumbersome.

We end with the following corollary for second-order equations.

If $q_2(z)$ is regular in $|z|<1$ and if the differential equation

$$w''(z)+q_2(z)w(z) = 0 \tag{8.31}$$

is disfocal in $|z|<1$, then it is also disconjugate in $|z|<1$. We recall that a second-order differential equation is called disconjugate in a domain D, if the only solution that vanishes twice in D is the trivial one. As for the proof of the corollary, since (8.31) is disfocal in $|z|<1$, it follows from (8.16) that

$$|q_2(z)| \leqq 1/(1-|z|^2)^2, \qquad |z| < 1$$

which is sufficient to guarantee the disconjugacy of (8.31) in $|z|<1$ (see [4]).

We note that this result holds only if $q_1(z)\equiv 0$ and is not true in the general case of second-order differential equations of the type (8.2). Considering the differential equation

$$y''(z)-(m+1)y'(z)+my(z) = 0, \qquad m > 1$$

London and Schwarz [3] showed that, in general, disfocality neither implies disconjugacy nor is implied by it.

In view of the fact that disconjugacy of (8.31) is equivalent to univalence of $f(z)=w_1(z)/w_2(z)$, where $w_1(z)$ and $w_2(z)$ are linearly independent solutions of (8.31), our last corollary may be stated as a univalence criterion.

THEOREM 8. *Denote by D the disk $|z-b|<R$, $0<R<\infty$, and let $f(z)$ be a meromorphic function in D. If*

$$f(z_1)-2[f'(z_1)]^2/f''(z_1) \neq f(z_2) \tag{8.32}$$

for every pair of points (not necessarily distinct) $z_1, z_2 \in D$, then $f(z)$ is univalent in D and

$$|\{f(z), z\}| \leqq 2/(R^2-|z-b|^2)^2, \qquad z \in D,$$

where

$$\{f(z), z\} = f'''(z)/f'(z)-(3/2)[f''(z)/f'(z)]^2$$

is the Schwarzian derivative.

Proof. Without loss of generality we may assume that D is the unit disk, since this situation may be achieved by means of a transformation $\zeta(z)=(z-b)/R$, which does not violate (8.32).

Consider now the second-order differential equation

$$w''(z)+q_1(z)w'(z)+q_2(z)w(z) = 0. \tag{8.33}$$

According to (8.9) and (8.10) we have

$$-q_1(z) = \Theta[f(z), g(z)], \qquad q_2(z) = \Phi[f(z), g(z)],$$

where

$$f(z) = w_1(z)/w_2(z), \qquad g(z) = w_1'(z)/w_2'(z), \tag{8.34}$$

and $w_1(z)$ and $w_2(z)$ are linearly independent solutions of (8.33). If $q_1(z)\equiv 0$, it follows from (5.1) that

$$g(z) = f(z)-2[f'(z)]^2/f''(z) \tag{8.35}$$

and

$$\Phi[f(z), g(z)] = \tfrac{1}{2}\{f(z), z\}.$$

In view of (8.35), formula (8.32) takes the form $g(z_1)\neq f(z_2)$, which by (8.34) is equivalent to the disfocality of the differential equation

$$w''(z)+\tfrac{1}{2}\{f(z), z\}w(z) = 0. \tag{8.36}$$

By Theorem 6, disfocality of (8.36) in the unit disk implies

$$|\{f(z), z\}| \leqq 2/(1-|z|^2)^2, \qquad |z| < 1, \tag{8.37}$$

which is a sufficient condition for disconjugacy of (8.36) in $|z|<1$. Since disconjugacy of (8.36) is equivalent to the univalence of $f(z)$ [4], this completes the proof.

ACKNOWLEDGEMENT. I am grateful to Professor Z. Nehari for his valuable advice offered during many discussions.

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