ZEROS OF ENTIRE FUNCTIONS

BY

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Introduction. This paper is the outgrowth of the author's Ph.D. thesis written under the supervision of Professor A. Peyerimhoff, whose paper on zeros of power series [4] has been the starting point.

§1 contains some lemmas due to A. Peyerimhoff, which are needed in §2.

In all theorems (with the exception of Theorem 4 only) of the remaining part of this paper, real entire functions $f(z)$ of finite order with infinitely many real and finitely many complex(1) zeros are taken as starting point.

Then in §2 functions of the form $g(z)=A(z)f(z)+B(z)f'(z)$ are considered where $A(z)$ and $B(z)$ are real polynomials. In Theorem 1 upper bounds for the number of zeros of $g(z)$ in the complex plane with the exception of certain real intervals are obtained. In certain instances the *exact* number of zeros is obtained. At the end of §2 examples are given which show that the results of Theorem 1 are best possible. Questions of this kind have been considered by Laguerre and Borel especially [1], [2] for the particular case $g(z)=f'(z)$. One of the results of [4] deals with functions $g(z)=\alpha f(z)+zf'(z)$ where α is real and $f(z)$ of order <1. The method of proof of this result has been generalized in the present paper.

In §3 a couple of theorems (2, 3, 5, 6) which are derived from Theorem 1 are partly slight generalizations of known theorems, especially of Laguerre [2], [3], [5], [6]. These theorems deal with questions of the following kind. If one has some information on the zeros of the entire function $f(z)=\sum a_n z^n$, one wants to gain information on the zeros of $F(z)=\sum a_n G(n) z^n$, where $G(z)$ is an entire function of a certain type.

For Theorem 4, which is known already, a short proof is given here which is independent of Theorem 1. From this theorem follows immediately the well-known fact that the Besselfunctions of real order >-1 have real zeros only.

From Theorem 6 Hurwitz's Theorem on the complex zeros of the Besselfunctions of real order <-1 follows as a special case.

In Theorem 9 functions $g(z)=\alpha f(z)+z^l f'(z)$ are considered, where α is real, l odd, and $f(z)$ is a real canonical product of finite genus. Then the *exact* number of complex zeros of $g(z)$ is obtained. Especially we find that for $\alpha>0$ and l odd the functions $\alpha \sin z+z^l \cos z$ and $\alpha \cos z-z^l \sin z$ have *exactly* $l-1$ complex zeros.

1. In the proof of Lemma 1 we will need some elementary properties of difference quotients. In the following we suppose that $f(z)$ is a holomorphic function.

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(1) Here "complex" means "nonreal".

Difference quotients which are undefined shall be defined by the corresponding differential quotients.

Let

$$\Delta_{z_1} f(z) = \frac{f(z)-f(z_1)}{z-z_1}$$

and recursively we define

$$\Delta_{z_l \dots z_1} f(z) = \Delta_{z_l}(\Delta_{z_{l-1} \dots z_1} f(z)).$$

This operation on functions $f(z)$ is homogeneous and linear, and for products $f(z)=g(z)\cdot h(z)$ we have $\Delta_{z_1}(g(z)h(z))=g(z_1)\Delta_{z_1}h(z)+h(z)\Delta_{z_1}g(z)$ and, as can be proved by induction on l:

$$\begin{aligned} \Delta_{z_l \dots z_1}(g(z)h(z)) = g(z_1)\Delta_{z_l \dots z_1}h(z) + h(z)\Delta_{z_l \dots z_1}g(z) \\ + \sum_{n=1}^{l-1} (\Delta_{z_n \dots z_1} g(z))|_{z=z_{n+1}} \Delta_{z_l \dots z_{n+1}} h(z). \end{aligned} \tag{1}$$

Let C_1 denote the exterior of the interval $[1, +\infty)$ with respect to the finite complex plane and C_2 the exterior of the intervals $(-\infty, -1]$ and $[1, +\infty)$ with respect to the finite complex plane.

Then we can prove the following lemma due to A. Peyerimhoff [4] for which a different proof is given here:

LEMMA 1. *Let $f(z)=P_{k-1}(z)+z^k \int_{-1}^{1} (1/(1-zt))\, dg(t)$, where $P_{k-1}(z)$ is a real polynomial of degree $k-1$ at most ($P_{k-1}(z)\equiv 0$ if $k=0$) and $g(t)$ is real and weakly monotone on $-1 \leqq t \leqq 1$ with $g(1)\neq g(-1)$.*

Then $f(z)$ is holomorphic and has at most k zeros in C_2. Similarly if $f(z)=P_{k-1}(z) + z^k \int_0^1 (1/(1-zt))\, dg(t)$, where $g(t)$ is real and weakly monotone on $0\leqq t\leqq 1$, then $f(z)$ is holomorphic and has at most k zeros in C_1.

Proof. We give a proof for the first part, the proof for the second part being similar.

If $k=0$, then $f(z)=\int_{-1}^{1} (1/(1-zt))\, dg(t)$.

If z is real, $z=x, f(x)=\int_{-1}^{1} (1/(1-xt))\, dg(t)\neq 0$ for $-1<x<1$ since $g(1)\neq g(-1)$.

If $z=x+iy$ with $y\neq 0$, consider

$$zf(z) = \int_{-1}^{1} \frac{z(1-\bar{z}t)}{|1-zt|^2}\, dg(t) = \int_{-1}^{1} \frac{x-|z|^2 t}{|1-zt|^2}\, dg(t) + iy \int_{-1}^{1} \frac{1}{|1-zt|^2}\, dg(t).$$

Then

$$\operatorname{Im}(zf(z)) = y \int_{-1}^{1} \frac{1}{|1-zt|^2}\, dg(t) \neq 0,$$

since $y\neq 0$. Therefore $zf(z)\neq 0$ for $z\neq 0$ and $f(0)=g(1)-g(-1)\neq 0$. Therefore, if $k=0, f(z)$ has no zeros in C_2.

Assume now $k \geqq 1$. Then $f(z) = P_{k-1}(z) + z^k \int_{-1}^{1} (1/(1-zt))\, dg(t)$ and we prove by induction on k that for $z, z_1, z_2, \ldots, z_k \in C_2$ we have

$$\Delta_{z_k \ldots z_1} f(z) = \int_{-1}^{1} \frac{1}{(1-zt)(1-z_1 t)\cdots(1-z_k t)}\, dg(t). \tag{2}$$

This is trivial for $k=1$ and $\Delta_{z_k \ldots z_1}(P_{k-1}(z)) = 0$ is trivial for any k.

Assume now that (2) holds for $k-1$ instead of k. Then

$$\Delta_{z_{k-1} \ldots z_1}\left(\frac{z^{k-1}}{1-zt}\right) = \frac{1}{(1-zt)(1-z_1 t)\cdots(1-z_{k-1} t)}.$$

Now apply (1) with $h(z) = z$ and $g(z) = z^{k-1}/(1-zt)$. Then

$$\begin{aligned}
\Delta_{z_k \ldots z_1}\left(\frac{z^{k-1}}{1-zt} z\right) &= \left(\Delta_{z_{k-1} \ldots z_1}\left(\frac{z^{k-1}}{1-zt}\right)\right)\Bigg|_{z=z_k} \Delta_{z_k}(z) + z \Delta_{z_k \ldots z_1}\left(\frac{z^{k-1}}{1-zt}\right) \\
&= \frac{1}{(1-z_k t)(1-z_1 t)\cdots(1-z_{k-1} t)} + \frac{zt}{(1-zt)(1-z_1 t)\cdots(1-z_k t)} \\
&= \frac{1}{(1-zt)(1-z_1 t)\cdots(1-z_k t)} \quad \text{which proves (2).}
\end{aligned}$$

Now we assume that $f(z) = P_{k-1}(z) + z^k \int_{-1}^{1} (1/(1-zt))\, dg(t)$ has at least k zeros $z_1, \ldots, z_k$ in C_2. Then

$$\Delta_{z_k \ldots z_1} f(z) = \frac{f(z)}{(z-z_1)\cdots(z-z_k)} = \int_{-1}^{1} \frac{1}{(1-zt)(1-z_1 t)\cdots(1-z_k t)}\, dg(t). \tag{3}$$

If k is even we may assume that $z_1, \ldots, z_k$ consists of pairs of conjugate complex zeros and of real zeros of $f(z)$ in C_2. Then (3) gives:

$$\frac{f(z)}{(z-z_1)\cdots(z-z_k)} = \int_{-1}^{1} \frac{1}{1-zt}\, d\gamma(t), \quad \text{where } \gamma(t) = \int_{-1}^{t} \frac{dg(\tau)}{(1-z_1\tau)\cdots(1-z_k\tau)}$$

is a real monotone function for $-1 \leqq t \leqq 1$. According to what has been proved already $\int_{-1}^{1} (1/(1-zt))\, d\gamma(t)$ has no zeros in C_2 and so $f(z)$ does not have more zeros than $z_1, \ldots, z_k$.

If k is odd and $f(z)$ has at least k zeros in C_2, then $f(z)$ has at least 1 real zero in C_2. Otherwise $f(z)$ would have at least $k+1$ complex zeros $z_1, \ldots, z_k, z_{k+1}$ in C_2 which we can assume to consist of pairs of conjugate complex zeros. Especially we assume that $z_{k+1} = \bar{z}_k$. Then (3) gives .

$$\frac{f(z)}{(z-z_1)\cdots(z-z_k)} = \int_{-1}^{1} \frac{1}{(1-zt)(1-z_k t)}\, d\gamma(t) \tag{4}$$

where

$$\gamma(t) = \int_{-1}^{t} \frac{dg(\tau)}{(1-z_1\tau)\cdots(1-z_{k-1}\tau)}$$

is a real monotone function for $-1 \leqq t \leqq 1$.

In (4) the left side is holomorphic in C_2 and vanishes at $z=z_{k+1}=\bar{z}_k$ whereas the right side is $\int_{-1}^{1} (1/|1-z_k t|^2)\, d\gamma(t)\neq 0$. Therefore, if $f(z)$ has at least k zeros in C_2, where k is odd, $f(z)$ has at least one real zero in C_2. Now we apply the same argument as in the case where k was even to the k zeros $z_1,\dots,z_k$ in C_2, where we now can assume that $z_1,\dots,z_k$ (k odd) consists of pairs of conjugate complex and of real zeros (at least one) of $f(z)$ in C_2. This again shows that $f(z)$ cannot have more than k zeros in C_2.

Lemma 2. *If $P_l(z)$ is a real polynomial of degree l and $g(t)$ is real and weakly monotone for $-1\leqq t\leqq 1$, then for $z\in C_2$*

$$P_l(z)\int_{-1}^{1}\frac{1}{1-zt}\,dg(t) = P_{l-1}(z)+z^l\int_{-1}^{1}\frac{t^l P_l(1/t)}{1-zt}\,dg(t)$$

where $P_{l-1}(z)$ is a real polynomial of degree $l-1$ at most.

A similar formula holds if the integrals from -1 to 1 are replaced by integrals from 0 to 1, and if C_2 is replaced by C_1.

Proof(2). $h(z)=P_l(z)-z^l t^l P_l(1/t)$ is a polynomial in z of degree l at most which vanishes at $z=1/t$. Therefore $h(z)/(1-zt)$ is a real polynomial in z of degree $l-1$ at most.

The same is true of $\int_{-1}^{1} (h(z)/(1-zt))\, dg(t)$.

Lemma 3. *If $l<k$ and $g(t)$ is real and weakly monotone for $-1\leqq t\leqq 1$ then*

$$z^l\int_{-1}^{1}\frac{1}{1-zt}\,dg(t) = P_{k-1}(z)+z^k\int_{-1}^{1}\frac{t^{k-l}}{1-zt}\,dg(t)\quad \text{for } z\in C_2.$$

A similar formula holds for $z\in C_1$ if the integrals from -1 to 1 are replaced by integrals from 0 to 1.

Proof(2). $h(z)=z^l-z^k t^{k-l}$ is a real polynomial in z of degree k at most which vanishes at $z=1/t$.

Therefore $h(z)/(1-zt)$ is a real polynomial in z of degree $k-1$ at most. The same is true of $\int_{-1}^{1} (h(z)/(1-zt))\, dg(t)$.

2. We assume that $f(z)$ is an entire function of finite order ρ which is real for real z and which has finitely many complex zeros and infinitely many real zeros.

If r_n ($n=1,2,\dots$) are the absolute values in increasing order of the zeros $z_n\neq 0$ of $f(z)$ then there exists a smallest nonnegative integer p (the genus of the sequence r_n) such that $\sum_{n=1}^{\infty} 1/r_n^p=\infty$ and $\sum_{n=1}^{\infty} 1/r_n^{p+1}<\infty$. Then always $p\leqq\rho$ and $p=[\rho]$ if ρ is not an integer [6]. The convergence exponent σ of r_n is the greatest lower bound of positive numbers α such that $\sum_{n=1}^{\infty} 1/r_n^{\alpha}<\infty$. Then always $p\leqq\sigma\leqq p+1$.

(2) I am indebted to the referee for simplifying the proofs of Lemmas 2 and 3.

From Hadamard's factorization theorem [6] follows that $f(z)$ can be written as

$$(1) \qquad f(z) = z^k e^{Q(z)} \prod_{n=1}^{\infty} \left(1-\frac{z}{z_n}\right) \exp\,[z/z_n+(z/z_n)^2/2+\cdots+(z/z_n)^p/p]$$

where $Q(z)$ is a real polynomial of degree q with $q \leqq [\rho]$ and

$$\prod_{n=1}^{\infty} \left(1-\frac{z}{z_n}\right) \exp\,[z/z_n+\cdots+(z/z_n)^p/p]$$

is the canonical product of genus p associated with the zeros z_n which, as an entire function is of order σ [6]. Then always $\rho=\max(q, \sigma)$ [6].

Now we consider the entire function (of order $\leqq \rho$).

$g(z)=A(z)f(z)+B(z)f'(z)$ where $A(z)$ and $B(z)$ are real polynomials of degrees a and b respectively, where we agree that always $b \geqq 0$ and $A(z) \equiv 0$ if $a=-1$.

With this notation we have the following

THEOREM 1.

Case I. *Let $f(z)$ have infinitely many positive, finitely many negative and finitely many complex zeros ($f(z)$ may have a zero at $z=0$).*

Let a_1 denote a real zero of $f(z)$ with multiplicity α_1 such that all real zeros of $B(z)$ are less than a_1. All real zeros of $f(z)$ which are bigger than a_1 are denoted by a_n in increasing order with multiplicities α_n ($n=2, 3, \ldots$). If $B(z)$ does not have any real zeros, a_1 can be any real zero of $f(z)$.

Then at each point a_n ($n=1, 2, \ldots$), $g(z)$ has a zero of multiplicity α_n-1 exactly and between each pair a_n, a_{n+1} ($n=1, 2, \ldots$) $g(z)$ has an odd number of zeros.

Let k be the number of zeros of $f(z)$ which are different from a_n ($n=1, 2, \ldots$). Then the number of zeros of $g(z)$ besides the trivial zeros at a_n and besides one zero between each pair a_n, a_{n+1} ($n=1, 2, \ldots$) is

1. *exactly* $b+p+k$ *if* $p \leqq \rho < p+1$ *and* $a \leqq b+p-1$ *or if* $\rho=\sigma=p+1$, $q \leqq p$ *and* $a \leqq b+p-1$,
2. *at most* $b+\rho+k$ *if* ρ *does not satisfy* 1 *and if* $a \leqq b+\rho-1$,
3. *at most* $a+k+1$ *if* a *does not satisfy* 1 *or* 2.

(*Always $p<\rho<p+1$ is true if ρ is not an integer.*)

Case II. *Let $f(z)$ have infinitely many positive, infinitely many negative and finitely many complex zeros ($f(z)$ may have a zero at $z=0$).*

Let a_1 denote a real zero of $f(z)$ with multiplicity α_1 such that all real zeros of $B(z)$ are less than a_1. All real zeros of $f(z)$ which are bigger than a_1 are denoted by a_n in increasing order and with multiplicities α_n ($n=2, 3, \ldots$). Similarly let $-b_1$ denote a real zero of $f(z)$ with multiplicity β_1 such that all real zeros of $B(z)$ are bigger than $-b_1$. All real zeros of $f(z)$ which are smaller than $-b_1$ are denoted by $-b_n$ in decreasing order with multiplicities β_n ($n=2, 3, \ldots$). If $B(z)$ does not have any real zeros, $-b_1$ and a_1 can be arbitrary real zeros of $f(z)$ with $-b_1<a_1$.

Then at each point a_n and $-b_n$ ($n=1, 2, \ldots$) $g(z)$ has a zero of exact multiplicity α_n-1 and β_n-1 respectively and an odd number of zeros between each pair a_n, a_{n+1} and $-b_{n+1}$, $-b_n$ ($n=1, 2, \ldots$).

Let k be the number of zeros of $f(z)$ which are different from a_n and $-b_n$ $(n=1, 2, \dots)$.

Then the number of zeros of $g(z)$ besides the trivial zeros at a_n and $-b_n$ and besides one zero between each pair a_n, a_{n+1} and $-b_{n+1}$, $-b_n$ $(n=1, 2, \dots)$ is

1. *exactly*	$p+b+k$	*if p is odd, $p \leqq \rho < p+1$ and $a \leqq b+p-1$, or if $\rho=\sigma=p+1$, $p \geqq q$ and $a \leqq b+p-1$,*
2. *exactly*	$p+b+k+1$	*if p is even, $p \leqq \rho \leqq p+1$, $a \leqq b+p$,*
3. *at most*	$b+k+\rho$	*if ρ is odd and does not satisfy* 1 *or* 2, $a \leqq b+\rho-1$,
4. *at most*	$b+k+\rho+1$	*if ρ is even and does not satisfy* 1 *or* 2, $a \leqq b+\rho$,
5. *at most*	$a+k+1$	*if $a-b$ is even and a does not satisfy* 1–4,
6. *at most*	$a+k+2$	*if $a-b$ is odd and a does not satisfy* 1–4.

(Always $p<\rho<p+1$ is true if ρ is not an integer.)

REMARKS. A similar result as in Case I holds for functions $f(z)$ which have infinitely many negative and finitely many positive zeros. One only has to replace $g(z)$ by $g(-z)$.

The results in Cases I and II remain correct if $B(z)$ does have real zeros of even order between zeros a_n, a_{n+1} and $-b_{n+1}$, $-b_n$ as long as they are different from all a_n and $-b_n$.

Proof. In the following we assume that $a_1=1$ and $-b_1=-1$. This can be assumed without loss of generality, since we can replace the variable z by $\alpha z+\beta$ where α, β are suitable real constants.

From the assumptions made about $f(z)$ it follows that according to Hadamard's factorization theorem in Case I $f(z)$ can be written as $f(z)=K(z)e^{Q(z)}\Pi(z)$, where

$$\Pi(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)^{\alpha_n} \exp\left[\alpha_n\left(\frac{z}{a_n}+\cdots+\frac{1}{p}\left(\frac{z}{a_n}\right)^p\right)\right]$$

and in Case II as $f(z)=K(z)e^{Q(z)}\Pi_1(z)\Pi_2(z)$, where

$$\Pi_1(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)^{\alpha_n} \exp\left[\alpha_n\left(\frac{z}{a_n}+\cdots+\frac{1}{p}\left(\frac{z}{a_n}\right)^p\right)\right]$$

and

$$\Pi_2(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{b_n}\right)^{\beta_n} \exp\left[\beta_n\left(-\frac{z}{b_n}+\cdots+\frac{(-1)^p}{p}\left(\frac{z}{b_n}\right)^p\right)\right].$$

In Case I $K(z)$ is a real polynomial of degree k, whose zeros coincide with all zeros of $f(z)$ which are different from a_n $(n=1, 2, \dots)$. $Q(z)$ is a real polynomial of degree q with $q \leqq [\rho]$. p is the genus of the sequence a_n.

In Case II $K(z)$ is a real polynomial of degree k whose zeros coincide with all zeros of $f(z)$ which are different from a_n and $-b_n$ $(n=1, 2, \dots)$. $Q(z)$ is as in Case I and p is defined by

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p}+\frac{\beta_n}{b_n^p} = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^{p+1}}+\frac{\beta_n}{b_n^{p+1}} < \infty.$$

Substituting the expressions for $f(z)$ in $g(z)=A(z)f(z)+B(z)f'(z)$ we obtain in Case I

$$g = e^{Q}\Pi(AK+BK'+BKQ'+BK(\Pi'/\Pi))$$

and in Case II

$$g = e^{Q}\Pi_1\Pi_2[AK+BK'+BKQ'+BK((\Pi_1'/\Pi_1)+(\Pi_2'/\Pi_2))].$$

Here

$$\frac{\Pi'}{\Pi} = \sum_{n=1}^{\infty}\left(\frac{\alpha_n}{z-a_n}+\alpha_n\left(\frac{1}{a_n}+\cdots+\frac{z^{p-1}}{a_n}\right)\right) \quad \text{or} \quad \frac{\Pi'}{\Pi} = z^p\sum_{n=1}^{\infty}\frac{\alpha_n}{a_n^p(z-a_n)}.$$

Therefore in Case I

$$g = e^{Q}\Pi\left(AK+BK'+BKQ'+BKz^p\left(\sum_{n=1}^{\infty}\frac{\alpha_n}{a_n^p(z-a_n)}\right)\right) \tag{2}$$

and similarly in Case II

$$g = e^{Q}\Pi_1\Pi_2\left(AK+BK'+BKQ'+BKz^p\left(\sum_{n=1}^{\infty}\frac{\alpha_n}{a_n^p(z-a_n)}+(-1)^p\sum_{n=1}^{\infty}\frac{\beta_n}{b_n^p(z+b_n)}\right)\right). \tag{3}$$

Since in Case I $B(z)$ and $K(z)$ do not have zeros for $z\geqq a_1$ it follows that $g(z)$ has an odd number of zeros between each pair a_n, a_{n+1} $(n=1,2,\ldots)$. Similarly in Case II $B(z)$ and $K(z)$ do not have zeros for $z\geqq a_1$ and $z\leqq -b_1$, so that $g(z)$ has an odd number of zeros between each pair a_n, a_{n+1} and $-b_{n+1}, -b_n$ $(n=1,2,\ldots)$.

From now on we assume without loss of generality that in Case I $B(z)K(z)>0$ for $z\geqq a_1$ (otherwise replace $g(z)$ by $-g(z)$) and in Case II that (see the definition of p in Case II) $\sum_{n=1}^{\infty}\alpha_n/a_n^p=\infty$ and $\sum_{n=1}^{\infty}\alpha_n/a_n^{p+1}<\infty$, and that at the same time $B(z)K(z)>0$ for $z\geqq a_1$ (otherwise replace $g(z)$ by $g(-z)$ and/or $g(z)$ by $-g(z)$).

Now in Case I

$$\sum_{n=1}^{\infty}\frac{\alpha_n}{a_n^p(z-a_n)} = -\sum_{n=1}^{\infty}\frac{1}{1-z(1/a_n)}\left(\frac{\alpha_n}{a_n^{p+1}}\right) = -\int_0^1\frac{1}{1-zt}\,d\gamma(t)$$

with $\gamma(t)=\sum_{1/a_n\leqq t}\alpha_n/a_n^{p+1}$ for $0\leqq t\leqq 1$, $\gamma(t)$ being nondecreasing for $0\leqq t\leqq 1$. This gives

$$g = e^{Q}\Pi\left(AK+BK'+BKQ'-BKz^p\int_0^1\frac{1}{1-zt}\,d\gamma(t)\right).$$

We now use Lemma 2 to obtain

$$B(z)K(z)\int_0^1\frac{1}{1-zt}\,d\gamma(t) = P_{b+k-1}(z)+z^{b+k}\int_0^1\frac{t^{b+k}B(1/t)K(1/t)}{1-zt}\,d\gamma(t)$$

where $P_{b+k-1}(z)$ is a real polynomial of degree $b+k-1$ at most. With $\eta(t)=\int_0^t\tau^{b+k}B(1/\tau)K(1/\tau)\,d\gamma(\tau)$, $\eta(t)$ being nondecreasing on $0\leqq t\leqq 1$, we finally obtain in Case I

$$g = e^{Q}\Pi\left(AK+BK'+BKQ'-z^pP_{b+k-1}(z)-z^{p+b+k}\int_0^1\frac{1}{1-zt}\,d\eta(t)\right). \tag{4}$$

In Case II we put

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p(z-a_n)} = -\int_0^1 \frac{1}{1-zt}\, d\gamma_1(t) \quad \text{with } \gamma_1(t) = \sum_{1/a_n \leqq t} \frac{\alpha_n}{a_n^{p+1}} \text{ for } 0 \leqq t \leqq 1,$$

$$\sum_{n=1}^{\infty} \frac{\beta_n}{b_n^p(z+b_n)} = \int_0^1 \frac{1}{1+zt}\, d\gamma_2(t) \quad \text{with } \gamma_2(t) = \sum_{1/b_n \leqq t} \frac{\beta_n}{b_n^{p+1}} \text{ for } 0 \leqq t \leqq 1,$$

$\gamma_1(t)$, $\gamma_2(t)$ being nondecreasing for $0 \leqq t \leqq 1$.

Now

$$\int_0^1 \frac{1}{1+zt}\, d\gamma_2(t) = \int_{-1}^0 \frac{1}{1-zt}\, d(-\gamma_2(-t))$$

and from (3) follows

$$g = e^Q \Pi_1 \Pi_2 \Big(AK + BK' + BKQ' - BKz^p \times \Big(\int_0^1 \frac{1}{1-zt}\, d\gamma_1(t) + \int_{-1}^0 \frac{1}{1-zt}\, d(-1)^p \gamma_2(-t) \Big) \Big). \tag{5}$$

Here $(-1)^p \gamma_2(-t)$ is increasing for $-1 \leqq t \leqq 0$ if p is odd and decreasing if p is even. Define

$$\left.\begin{aligned} \gamma(t) &= -\gamma_2(-t) && \text{for } -1 \leqq t \leqq 0 \\ &= \gamma_1(t) && \text{for } 0 \leqq t \leqq 1 \end{aligned}\right\} \quad \text{if } p \text{ is odd}$$

and

$$\left.\begin{aligned} \gamma(t) &= \gamma_2(-t) && \text{for } -1 \leqq t \leqq 0 \\ &= \gamma_1(t) && \text{for } 0 \leqq t \leqq 1 \end{aligned}\right\} \quad \text{if } p \text{ is even.}$$

Then (5) gives

$$g = e^Q \Pi_1 \Pi_2 \Big(AK + BK' + BKQ' - BKz^p \int_{-1}^1 \frac{1}{1-zt}\, d\gamma(t) \Big). \tag{6}$$

Using Lemma 2 again we have

$$B(z)K(z) \int_{-1}^1 \frac{1}{1-zt}\, d\gamma(t) = P_{b+k-1}(z) + z^{b+k} \int_{-1}^1 \frac{t^{b+k} B(1/t)K(1/t)}{1-zt}\, d\gamma(t)$$

where $P_{b+k-1}(z)$ is a real polynomial of degree $b+k-1$ at most.

If p is even we use Lemma 3 in addition to obtain

$$\int_{-1}^1 \frac{t^{b+k} B(1/t)K(1/t)}{1-zt}\, d\gamma(t) = c + z \int_{-1}^1 \frac{t^{b+k+1} B(1/t)K(1/t)}{1-zt}\, d\gamma(t)$$

where c is a real constant.

Finally we put

$$\eta(t) = \int_{-1}^t \tau^{b+k} B(1/\tau)K(1/\tau)\, d\gamma(\tau) \quad \text{for } -1 \leqq t \leqq 1 \text{ if } p \text{ is odd,}$$

$$\eta(t) = \int_{-1}^t \tau^{b+k+1} B(1/\tau)K(1/\tau)\, d\gamma(\tau) \quad \text{for } -1 \leqq t \leqq 1 \text{ if } p \text{ is even.}$$

Then we obtain from (6)

$$g = e^{Q}\Pi_1\Pi_2\left(AK+BK'+BKQ'-z^{p}P_{b+k-1}-z^{p+b+k}\int_{-1}^{1}\frac{1}{1-zt}\,d\eta(t)\right)$$

if p is odd,

$$g = e^{Q}\Pi_1\Pi_2\Bigl(AK+BK'+BKQ'-z^{p}P_{b+k-1}-cz^{p+b+k} \tag{7}$$
$$-z^{p+b+k+1}\int_{-1}^{1}\frac{1}{1-zt}\,d\eta(t)\Bigr) \quad \text{if } p \text{ is even.}$$

From the assumptions made about $B(z)K(z)$ without loss of generality it follows that $\eta(t)$ is an increasing function for $-1\leqq t\leqq 1$.

In Case I we obtain

$$g = e^{Q}\Pi\left(P(z)-z^{l}\int_{0}^{1}\frac{1}{1-zt}\,d\eta(t)\right)$$

and in Case II

$$g = e^{Q}\Pi_1\Pi_2\left(P(z)-z^{l}\int_{-1}^{1}\frac{1}{1-zt}\,d\eta(t)\right) \quad \text{if } p \text{ is odd}$$

and

$$g = e^{Q}\Pi_1\Pi_2\left(P(z)-z^{l+1}\int_{-1}^{1}\frac{1}{1-zt}\,d\eta(t)\right) \quad \text{if } p \text{ is even.}$$

Here $l=p+b+k$ and in all three cases $P(z)$ is a real polynomial.

To these three expressions we apply Lemma 3, if necessary, and then Lemma 1 in order to obtain that *in* C_1 and *in* C_2 respectively $g(z)$ has at most as many zeros as stated in Theorem 1.

If now the same proof is carried out with a_2 instead of a_1, so that k has to be replaced by $k+\alpha_1$, the upper bound of zeros of $g(z)$ in C_1^* and C_2^* respectively, which is given by Lemma 1, is increased by α_1. But in the statement of Theorem 1 these additional α_1 zeros are already counted with the trivial (α_1-1)-fold zero of $g(z)$ at a_1 and with one zero of $g(z)$ between a_1 and a_2. Here the * in C_1^* and C_2^* refers to the fact that now we assume without loss of generality that $a_2=1$.

This argument can be repeated with all a_n $(n=3, 4, \ldots)$ and all $-b_n$ $(n=2, 3, \ldots)$, which shows that besides the trivial zeros $g(z)$ has at most as many zeros as stated in Theorem 1.

It remains to show that in some special cases listed in Theorem 1 the upper bound for the number of zeros of $g(z)$ actually is the exact number of zeros.

In Case I with $l=p+b+k$ we obtained in (4) that

$$g(z) = e^{Q(z)}\Pi(z)\left(P_{l-1}(z)-z^{l}\int_{0}^{1}\frac{1}{1-zt}\,d\eta(t)\right)$$

if $p\leqq\rho<p+1$ and $a\leqq b+p-1$, or $\sigma=\rho=p+1$, $q\leqq p$, and $a\leqq b+p-1$. Here $P_{l-1}(z)$ is a real polynomial of degree $l-1$ at most.

We now consider

$$h(z,\tau) = \tau P_{l-1}(z) - z^l \int_0^1 \frac{1}{1-zt}\,d\eta(t) \quad \text{for } 0 \leqq \tau \leqq 1.$$

Then

$$h(z,\tau) = \tau P_{l-1}(z) + z^p P_{b+k-1}(z) + B(z)K(z)z^p \sum_{n=1}^{\infty} \frac{\alpha_n}{a_n^p(z-a_n)},$$

which shows that for each τ, $0 \leqq \tau \leqq 1$, $h(z,\tau)$ has a simple pole at a_n and an odd number of zeros between each pair a_n, a_{n+1} $(n=1,2,\ldots)$.

We now want to determine a circle S around the origin such that $h(z,\tau)\neq 0$ for $z\in S$ and $0\leqq\tau\leqq 1$. With $z=re^{i\theta}$ we have for $r\geqq 1$ and all $0\leqq\tau\leqq 1$

$$\begin{aligned}\left|\frac{h(z,\tau)}{z^{l-1}}\right| &\geqq \left|\operatorname{Im}\frac{h(z,\tau)}{z^{l-1}}\right| \geqq \left|\operatorname{Im}\int_0^1 \frac{z(1-\bar z t)}{|1-zt|^2}\,d\eta(t)\right| - \left|\operatorname{Im}\tau\frac{P_{l-1}(z)}{z^{l-1}}\right| \\ &\geqq r|\sin\theta|\left(\int_0^1 \frac{d\eta(t)}{|1-zt|^2} - \frac{c_1}{r^2}\right) \geqq r|\sin\theta|\left(\int_0^1 \frac{t^{b+k}B(1/t)K(1/t)}{(1+rt)^2}\,d\gamma(t) - \frac{c_1}{r^2}\right) \\ &\geqq r|\sin\theta|\left(c_2\int_0^1 \frac{d\gamma(t)}{(1+rt)^2} - \frac{c_1}{r^2}\right) \quad \text{with } c_1, c_2 > 0,\end{aligned}$$

using the fact that $t^{b+k}B(1/t)K(1/t)\geqq c_2>0$ for $0\leqq t\leqq 1$.

Here

$$\int_0^1 \frac{d\gamma(t)}{(1+rt)^2} \geqq \int_{1/r}^1 \frac{d\gamma(t)}{(1+rt)^2} \geqq \int_{1/r}^1 \frac{d\gamma(t)}{(2rt)^2} \geqq \frac{1}{4r^2}\int_{1/r}^1 \frac{d\gamma(t)}{t}.$$

Therefore

$$|h(z,\tau)| \geqq |z|^{l-2}|\sin\theta|\left(\frac{c_2}{4}\int_{1/r}^1 \frac{d\gamma(t)}{t} - c_1\right).$$

From the definition of $\gamma(t)$ follows

$$\int_{1/r}^1 \frac{d\gamma(t)}{t} = \sum_{r\geqq a_n} \frac{\alpha_n}{a_n^p} \tag{8}$$

and therefore $\lim_{r\to\infty}\int_{1/r}^1 d\gamma(t)/t=\infty$. This shows that for $0\leqq\tau\leqq 1$

$$|h(z,\tau)| > 0 \quad \text{if } r \text{ is sufficiently large and } \sin\theta \neq 0. \tag{9}$$

If $z=-r$ then for all $r\geqq 1$ and $0\leqq\tau\leqq 1$

$$|h(-r,\tau)| \geqq r^{l-1}\left(r\int_0^1 \frac{1}{1+rt}\,d\eta(t) - c_3\right) \geqq r^{l-1}\left(c_2\int_0^1 \frac{r}{1+rt}\,d\gamma(t) - c_3\right)$$

which is >0 if r is sufficiently large according to (8)(3). The constants c_2, c_3 are >0.

(3) Since $r\int_0^1 (d\gamma(t)/(1+rt))\geqq\frac{1}{2}\int_{1/r}^1 (d\gamma(t)/t)$.

If $z=r$, we choose r_1 large enough and close to, but $<a_m$ for some $m\geqq 1$, so that (9) holds for $0<\theta<2\pi$ and such that

$$|h(r,\tau)| \geqq \left|r^l\int_0^1 \frac{d\eta(t)}{1-rt}\right| - |P_{l-1}(r)| > 0 \quad \text{for } r_1 \leqq r < a_m \text{ and } 0 \leqq \tau \leqq 1.$$

This is possible since $h(z,\tau)$ has a simple pole at each point a_n $(n=1,2,\dots)$. Let now S be the circle around the origin with this radius r_1. Then $|h(z,\tau)|>0$ for all $z\in S$ and $0\leqq\tau\leqq 1$ so that $|h(z,\tau)|\geqq c_4>0$ for these z and τ, since $|h(z,\tau)|$ is continuous as a function of $(z,\tau)\in S\times[0,1]$.

Therefore $(1/2\pi i)\int_S (h'(z,\tau)/h(z,\tau))\,dz$ is continuous as a function of τ for $0\leqq\tau\leqq 1$ and therefore a constant. Since $h(z,0)$ has an l-fold 0 at $z=0$ and at least one zero between each pair of poles a_n, a_{n+1} and since $h(z,\tau)$ has at least one zero between each pair a_n, a_{n+1} $(n=1,\dots,m-1)$ for all $0\leqq\tau\leqq 1$, we have

$$\frac{1}{2\pi i}\int_S \frac{h'(z,\tau)}{h(z,\tau)}\,dz \geqq l.$$

Together with what has been proved earlier we obtain that $h(z,1)$ or $g(z)=e^{Q(z)}\Pi(z)h(z,1)$ has exactly l zeros besides one zero between each pair a_n, a_{n+1} $(n=1,2,\dots)$ and besides trivial zeros at a_n of multiplicity α_n-1.

In Case II with $l=p+b+k$ we have obtained in (7)

$$g(z) = e^Q\Pi_1\Pi_2\left(P_{l-1}(z)-z^l\int_{-1}^1 \frac{1}{1-zt}\,d\eta(t)\right) \quad \text{if } p \text{ is odd,}$$

$$g(z) = e^Q\Pi_1\Pi_2\left(P_l(z)-z^{l+1}\int_{-1}^1 \frac{1}{1-zt}\,d\eta(t)\right) \quad \text{if } p \text{ is even}$$

and if the additional conditions of 1 or 2 in Case II of Theorem 1 are satisfied. Here $P_{l-1}(z)$ and $P_l(z)$ are real polynomials of degrees $l-1$ and l at most.

We now consider, similarly as in Case I for $0\leqq\tau\leqq 1$

$$\begin{aligned} h(z,\tau) &= \tau P_{l-1}(z)-z^l\int_{-1}^1 \frac{1}{1-zt}\,d\eta(t) \\ &= \tau P_{l-1}(z)+z^pP_{b+k-1}+z^pB(z)K(z) \\ &\quad\times\left(\sum_{n=1}^\infty \frac{\alpha_n}{a_n^p(z-a_n)}+(-1)^p\sum_{n=1}^\infty \frac{\beta_n}{b_n^p(z+b_n)}\right) \end{aligned} \tag{10}$$

if p is odd and

$$\begin{aligned} h(z,\tau) &= \tau P_l(z)-z^{l+1}\int_{-1}^1 \frac{1}{1-zt}\,d\eta(t) \\ &= \tau P_l(z)+cz^l+z^pP_{b+k-1}+z^pB(z)K(z) \\ &\quad\times\left(\sum_{n=1}^\infty \frac{\alpha_n}{a_n^p(z-a_n)}+(-1)^p\sum_{n=1}^\infty \frac{\beta_n}{b_n^p(z+b_n)}\right) \end{aligned} \tag{11}$$

if p is even.

In both cases $h(z, \tau)$ has for each τ, $0 \leqq \tau \leqq 1$, a simple pole at a_n and $-b_n$ ($n=1, 2, \dots$) and an odd number of zeros between each pair a_n, a_{n+1} and $-b_{n+1}$, $-b_n$. Similarly as in Case I we now want to determine a curve S around the origin such that $h(z, \tau) \neq 0$ for $z \in S$ and $0 \leqq \tau \leqq 1$.

With $z = re^{i\theta}$ we have

$$(12)\quad \left|\frac{h(z, \tau)}{z^{l-1}}\right| \geqq \left|\operatorname{Im} \frac{h(z, \tau)}{z^{l-1}}\right| \geqq r|\sin \theta|\left(\int_{-1}^{1} \frac{1}{|1-zt|^2}\, d\eta(t) - \frac{c_1}{r^2}\right) \quad \text{if } p \text{ is odd,}$$

$$(13)\quad \left|\frac{h(z, \tau)}{z^{l}}\right| \geqq \left|\operatorname{Im} \frac{h(z, \tau)}{z^{l}}\right| \geqq r|\sin \theta|\left(\int_{-1}^{1} \frac{1}{|1-zt|^2}\, d\eta(t) - \frac{c_2}{r^2}\right) \quad \text{if } p \text{ is even.}$$

This holds for all $r \geqq 1$ and all $0 \leqq \tau \leqq 1$ with constants c_1 and c_2 which are >0.

Now we use the fact that (w.l.o.g.) we have assumed $\sum_{n=1}^{\infty} \alpha_n / a_n^p = \infty$ and $\sum_{n=1}^{\infty} \alpha_n / a_n^{p+1} < \infty$ and that $t^{b+k} B(1/t) K(1/t) \geqq c_3 > 0$ for $-1 \leqq t \leqq 1$. Then in both expressions (12), (13) we have if p is odd

$$\int_{-1}^{1} \frac{1}{|1-zt|^2}\, d\eta(t) = \int_{-1}^{1} \frac{t^{b+k} B(1/t) K(1/t)}{|1-zt|^2}\, d\gamma(t) \geqq c_3 \int_{0}^{1} \frac{d\gamma_1(t)}{|1-zt|^2} \geqq c_3 \int_{0}^{1} \frac{d\gamma_1(t)}{(1+rt)^2}$$
$$\geqq c_3 \int_{1/r}^{1} \frac{d\gamma_1(t)}{(2rt)^2} \geqq \frac{c_3}{4r^2} \int_{1/r}^{1} \frac{d\gamma_1(t)}{t}.$$

Similarly if p is even

$$\int_{-1}^{1} \frac{1}{|1-zt|^2}\, d\eta(t) = \int_{-1}^{1} \frac{t^{b+k+1} B(1/t) K(1/t)}{|1-zt|^2}\, d\gamma(t) \geqq c_3 \int_{0}^{1} \frac{t\,d\gamma_1(t)}{|1-zt|^2}$$

and now this is

$$\geqq \frac{c_3}{4r^2} \int_{1/r}^{1} \frac{t\,d\gamma_1(t)}{t^2} = \frac{c_3}{4r^2} \int_{1/r}^{1} \frac{d\gamma_1(t)}{t}.$$

Consequently we obtain for all $0 \leqq \tau \leqq 1$ and $r \geqq 1$ from (12), (13)

$$|h(z, \tau)| \geqq r^{l-2} |\sin \theta| \left(\frac{c_3}{4} \int_{1/r}^{1} \frac{d\gamma_1(t)}{t} - c_1\right) \quad \text{if } p \text{ is odd,}$$

$$|h(z, \tau)| \geqq r^{l-1} |\sin \theta| \left(\frac{c_3}{4} \int_{1/r}^{1} \frac{d\gamma_1(t)}{t} - c_2\right) \quad \text{if } p \text{ is even.}$$

Similarly as in Case I $\int_{1/r}^{1} d\gamma_1(t)/t = \sum_{a_n \leqq r} \alpha_n / a_n^p$ and therefore $\lim_{r \to \infty} \int_{1/r}^{1} d\gamma_1(t)/t = \infty$ so that

$$(14)\qquad |h(z, \tau)| > 0 \quad \text{for all } 0 \leqq \tau \leqq 1 \text{ if } r \text{ is sufficiently large and } \sin \theta \neq 0.$$

We then choose r_1 large enough and close to, but $< a_{m_1}$ for some $m_1 \geqq 1$, such that for all $0 \leqq \tau \leqq 1$, $|h(z, \tau)| > 0$ for $r_1 \leqq z < a_{m_1}$ and such that (14) holds for all $r \geqq r_1$.

Similarly we choose $r_2 \geqq r_1$ such that $-r_2$ is close enough to but $> -b_{m_2}$ for some $m_2 \geqq 1$ with $|h(z, \tau)| > 0$ for $-b_{m_2} < z \leqq -r_2$ and all $0 \leqq \tau \leqq 1$.

Now we can define the curve S as follows:

$$S: z = r_1 e^{i\theta} \quad \text{for } -\pi/2 \leqq \theta \leqq \pi/2, \qquad z = r_2 e^{i\theta} \quad \text{for } \pi/2 \leqq \theta \leqq 3\pi/2,$$
$$z = re^{i(\pi/2)} \quad \text{for } r_1 \leqq r \leqq r_2, \qquad z = re^{-i(\pi/2)} \quad \text{for } r_1 \leqq r \leqq r_2.$$

Then $|h(z, \tau)| > 0$ for $z \in S$ and $0 \leqq \tau \leqq 1$ and since $|h(z, \tau)|$ is continuous as a function of $(z, \tau) \in S \times [0, 1]$ we actually have $|h(z, \tau)| \geqq c_4 > 0$ so that

$$\frac{1}{2\pi i} \int_S \left(\frac{h'(z, \tau)}{h(z, \tau)} \right) dz$$

is constant with respect to τ for $0 \leqq \tau \leqq 1$. Similarly as in Case I this integral is $\geqq l$ if p is odd and $\geqq l+1$ if p is even. Together with what has been proved earlier it follows that $g(z)$ has exactly l zeros (exactly $l+1$ zeros) besides at least one zero between each pair a_n, a_{n+1} and $-b_{n+1}, -b_n$ and besides the trivial zeros at a_n and $-b_n$ of multiplicities $\alpha_n - 1$ and $\beta_n - 1$ respectively if p is odd (if p is even).

Finally we want to end this section by giving some examples which show that the results of Theorem 1, Case II cannot be sharpened.

Let $\varphi(z) = e^{(\beta/2)z^2} \cos z$ with real $\beta \neq 0$. Here $\rho = 2$ and $p = 1$. Furthermore let

$$\gamma(z) = \alpha\varphi(z) + z\varphi'(z) = e^{(\beta/2)z^2}((\alpha + \beta z^2) \cos z - z \sin z) \quad \text{with real } \alpha \neq 0.$$

We now apply Theorem 1, Case II, 1 to $g(z) = z \sin z - (\alpha + \beta z^2) \cos z$, where in the notation of Theorem 1 $f(z) = \sin z$, $a = 1$, $b = 2$, $a_1 = \pi$, $-b_1 = -\pi$. From this theorem and from $g(-z) = g(z)$ follows that $g(z)$ has exactly one zero between each pair of consecutive positive and negative zeros of $\sin z$ and that $g(z)$ has exactly two complex zeros and exactly two real zeros in $(-\pi, \pi)$ if α, β have the same sign. In particular in $(-\pi/2, \pi/2)$ $g(z)$ has no zero if $\alpha, \beta < 0$ and exactly two zeros if $\alpha, \beta > 0$.

We now apply Theorem 1, Case II,6 to $g(z) = -(\alpha + \beta z^2) \cos z + z \sin z$ where we consider $\cos z$ as $f(z)$, $a = 2$, $b = 1$, and $a_1 = \pi/2$, $-b_1 = -\pi/2$. Then $g(z)$ also has exactly one zero between each pair of consecutive positive and negative zeros of $\cos z$. Besides these zeros $g(z)$ has exactly four zeros (2 complex and 2 in $(-\pi/2, \pi/2)$) if $\alpha, \beta > 0$ and exactly two zeros (2 complex and no zeros in $(-\pi/2, \pi/2)$) if $\alpha, \beta < 0$.

Therefore if $\alpha, \beta > 0$ the upper bound, as given in Case II,6 is reached by $g(z)$ and if $\alpha, \beta < 0$ it is not.

At the same time $\varphi(z)$ and $\gamma(z)$ are examples which show that Case II,1 and Case II,4 are best possible.

In Case II,1 the condition $p \leqq \rho < p+1$ cannot be replaced by $p \leqq \rho \leqq p+1$, since in the above example $\rho = 2$, $p = 1$, but if $\alpha, \beta > 0$, $\gamma(z)$ has *exactly* four zeros besides one between each pair of consecutive positive and negative zeros of $\varphi(z)$.

In addition this example shows that the upper bound, as given by Case II,4 is reached. On the other hand if $\alpha, \beta < 0$ this upper bound is not reached.

Finally we want to show that Case II,3 and Case II,5 are best possible also. Let $\varphi(z)=e^{(\beta/3)z^3}\cos z$ with real $\beta\neq 0$. Here $\rho=3$, $p=1$. Furthermore let $\gamma(z)=A(z)\varphi(z)+B(z)\varphi'(z)$, where $A(z)$ and $B(z)$ are real polynomials of exact degrees 3 and 1 respectively.

Then $\gamma(z)=e^{(\beta/3)z^3}g(z)$ with $g(z)=(A(z)+\beta z^2B(z))\cos z-B(z)\sin z$ and we also assume that $P(z)=A(z)+\beta z^2B(z)$ shall have degree 3 exactly. From Theorem 1, Case II,1, applied to $g(z)$ with $f(z)=\sin z$, $a=1$, $b=3$, $a_1=\pi$, $-b_1=-\pi$, follows that $g(z)$ has exactly five zeros besides one between each pair of consecutive positive and negative zeros of $\sin z$ if $P(z)$ has its real zeros in $(-\pi,\pi)$. If we choose $B(z)$ and $P(z)$ such that $B(-\pi/2)>0$, $B(\pi/2)<0$ and $P(0)>0$, $P(\pi/4)-B(\pi/4)<0$, $P(-\pi/4)+B(-\pi/4)<0$, $P(-\pi)<0$, $P(\pi)>0$, then $g(z)$ has exactly four zeros in $(-\pi/2,\pi/2)$ and exactly five zeros in $(-\pi,\pi)$.

If we choose $B(z)$ and $P(z)$ such that $B(-\pi/2)<0$, $B(\pi/2)>0$ and $P(z)\equiv 0$ then $g(z)$ has exactly two zeros in $(-\pi/2,\pi/2)$.

Now Theorem 1, Case II,5 applied to $g(z)$ with $f(z)=\cos z$, $a=3$, $b=1$, $a_1=\pi/2$, $-b_1=-\pi/2$ shows that $g(z)$, for both choices of the pair $B(z)$, $P(z)$ also has exactly one zero between each pair of consecutive positive and negative zeros of $\cos z$. In addition the upper bound, as given by Case II,5 is reached by $g(z)$ if $g(z)$ has four zeros in $(-\pi/2,\pi/2)$.

At the same time $\varphi(z)$ and $\gamma(z)$ are examples which show that the upper bound, as given by Case II,3 is reached in one case, while it is not reached in the other case.

In a similar way one can find examples which show that Cases I,2 and 3 are best possible by using $\cos\sqrt{z}$ or $\sqrt{z}\sin\sqrt{z}$.

3. In the following let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ denote a *real*, nonconstant, entire function of finite order ρ and with $f(0)\neq 0$. Especially we want to restrict $f(z)$ to the following two types (for the definition of p and q see the beginning of §2):

$$f(z)=c\prod_{n=1}^{\infty}\left(1-\frac{z}{c_n}\right)\quad\text{with}\quad\sum_{n=1}^{\infty}|c_n|^{-1}<\infty \tag{1}$$

(i.e. either $\rho<1$ or $\rho=1$ and $p=q=0$),

$$f(z)=ce^{bz}\prod_{n=1}^{\infty}\left(1-\frac{z}{c_n}\right)e^{z/c_n}\quad\text{with}\quad\sum_{n=1}^{\infty}|c_n|^{-2}<\infty \tag{2}$$

(i.e. either $\rho<2$ or $\rho=2$ and $q\leqq p=1$).

Here b and c shall be real and $c\neq 0$. Furthermore let $F(z)=\sum_{n=0}^{\infty}a_nG(n)z^n$, where $G(z)$ is a real entire function of the following two types:

(A) $G(z)=e^{\beta z}\prod_{n=1}^{\infty}(1+z/\alpha_n)e^{-z/\alpha_n}$, where each $\alpha_n>0$, β real and $\sum_{n=1}^{\infty}\alpha_n^{-2}<\infty$. It is also allowed that $G(z)$ has finitely many (negative) zeros only.

(B) $G(z)=(z-\alpha_1)\cdot\cdots\cdot(z-\alpha_m)$, where $\alpha_1,\ldots,\alpha_m>0$.

Then always $F(z)$ is a real entire function of order $\leqq \rho$. Having some information on the zeros of $f(z)$ we want to gain information on the zeros of $F(z)$. The following theorems, with the exception of Theorem 4, can be deduced from Theorem 1. The proof of Theorem 4 is independent of Theorem 1.

THEOREM 2. *Let $f(z)$ have infinitely many zeros but only k zeros which are not* >0.

(a) *If $f(z)$ is of type* (1) *and $G(z)$ of type* (A) *then $F(z)$ has at most k zeros which are not* >0.

(b) *If $f(z)$ is of type* (1) *and $G(z)$ of type* (B) *then $F(z)$ has at most $k+m$ zeros which are not* >0.

If $f(z)$ is of type (2) *and $G(z)$ of type* (B) *then $F(z)$ has at most $k+2m$ zeros which are not* >0. *Besides $F(z)$ is of the same type as $f(z)$ and has infinitely many positive zeros. A corresponding result can be obtained if $f(z)$ has k zeros which are not* <0.

THEOREM 3. *Let $f(z)$ be of type* (2) *with infinitely many real zeros of both signs and $2k$ complex zeros.*

(a) *If $G(z)$ is of type* (A) *then $F(z)$ has at most $2k$ complex zeros.*

(b) *If $G(z)$ is of type* (B) *then $F(z)$ has at most $2k+2m$ complex zeros and is of the same type as $f(z)$. Besides $F(z)$ has infinitely many real zeros of both signs.*

THEOREM 4. *Let $f(z)=\exp(az^2)h(z)$ where $h(z)$ is of type* (2) *and $a\leqq 0$. Furthermore let $f(z)$ have real zeros only* (*or no zeros at all*). *If $G(z)$ is of type* (A) *then $F(z)$ has real zeros only.*

THEOREM 5. *Let $f(z)$ have infinitely many zeros but only k zeros which are not* >0. *Furthermore assume that $G(z)$ is of type* (B) *having the positive zeros $\alpha_1, \ldots, \alpha_m$. With $f_0(z)=f(z)$ and $f_j(z)=-\alpha_j f_{j-1}(z)+zf'_{j-1}(z)$ for $j=1,\ldots,m$ we assume that for each j, $f_j(z)$ has exactly one zero between each pair of consecutive positive zeros of $f_{j-1}(z)$ and no zero between $z=0$ and the smallest positive zero of $f_{j-1}(z)$.*

If $f(z)$ is of type (1) *then $F(z)$ has exactly $k+m$ zeros which are not* >0. *If $f(z)$ is of type* (2) *and $p=1$ then $F(z)$ has exactly $k+2m$ zeros which are not* >0.

In both cases $F(z)$ is of the same type as $f(z)$. A corresponding result holds if $f(z)$ has k zeros which are not <0.

THEOREM 6. *Let $f(z)$ be of type* (2) *with infinitely many real zeros of both signs and with exactly $2k$ complex zeros. Furthermore assume that $G(z)$ is of type* (B) *having the positive zeros $\alpha_1, \ldots, \alpha_m$. With $f_0(z)=f(z)$ and $f_j(z)=-\alpha_j f_{j-1}(z)+zf'_{j-1}(z)$ for $j=1,\ldots,m$ we assume that for each j, $f_j(z)$ has exactly one zero between each pair of consecutive positive and negative zeros of $f_{j-1}(z)$ and that $f_j(z)$ has no zero between the smallest positive and the smallest negative zero of $f_{j-1}(z)$.*

Under these conditions $F(z)$ has exactly $2k+2m$ complex zeros. Besides $F(z)$ is of the same type as $f(z)$.

In order to prove Theorems 2–6 we need the following

LEMMA 4. *Let $g(z)=\alpha f(z)+zf'(z)$ with real $\alpha\neq 0$, a, b and $f(z)$ real.*

(1) *If* $f(z)=e^{bz}h(z)$ *where* $h(z)$ *is of type* (1), *then* $g(z)=e^{bz}h^*(z)$ *where* $h^*(z)$ *is of type* (1).

(2) *If* $f(z)=e^{az^2}h(z)$ *where* $h(z)$ *is of type* (2), *then* $g(z)=e^{az^2}h^*(z)$ *where* $h^*(z)$ *is of type* (2).

In both cases we assume that all but finitely many zeros of $h(z)$ *are real. Then the same is true for* $h^*(z)$. *It is allowed that* $h(z)$ *has finitely many zeros only.*

Proof. (1) We have $f'(z)/f(z)=b+\sum_{n=1}^{\infty} 1/(z-c_n)$ where all but finitely many c_n are real. As z tends to infinity along the imaginary axis $f'(z)/f(z)$ tends to b. According to Hadamard's factorization theorem $g(z)$ is of the same form as $f(z)$. It remains to show that $g'(z)/g(z)$ also tends to b as z tends to infinity along the imaginary axis. We have

$$\frac{g'}{g}=\frac{f'}{f}+\frac{(f'/f)+z(f'/f)'}{\alpha+z(f'/f)}. \tag{3}$$

Here $z(f'/f)'$ tends to 0 as z tends to infinity along the imaginary axis. In addition we have for $z=iy$, y real:

$$\left|\alpha+z\frac{f'}{f}\right| \geqq |z|\left|b+\sum_{n=N+1}^{\infty}\frac{1}{z-c_n}\right|-\left|\sum_{n=1}^{N}\frac{z}{z-c_n}\right|-|\alpha|$$

where we assume that $c_1,\ldots,c_N$ consist of all complex zeros of $f(z)$. Here the first term on the right side is

$$\geqq |z|\left|\operatorname{Im}\sum_{n=N+1}^{\infty}\frac{1}{z-c_n}\right| = y^2\sum_{n=N+1}^{\infty}\frac{1}{y^2+c_n^2},$$

and this tends to infinity with y. Therefore g'/g also tends to b as y tends to infinity.

(2) Here we have

$$\frac{1}{z}\frac{f'}{f} = 2a+\frac{b}{z}+\frac{1}{z}\sum_{n=1}^{\infty}\left(\frac{1}{z-c_n}+\frac{1}{c_n}\right)$$

where again all but finitely many c_n are real. Therefore f'/zf tends to $2a$ as z tends to infinity along the imaginary axis. We divide equation (3) by z and observe that now $(f'/f)'$ tends to $2a$ as z tends to infinity along the imaginary axis and that for the denominator $\alpha+z(f'/f)$ we have for $a=0$ (the case $a\neq 0$ being trivial because of (3)):

$$\left|\alpha+z\frac{f'}{f}\right| \geqq |z|\left|b+\sum_{n=1}^{N}\frac{1}{c_n}+\sum_{n=N+1}^{\infty}\left(\frac{1}{z-c_n}+\frac{1}{c_n}\right)\right|-\left|\sum_{n=1}^{N}\frac{z}{z-c_n}\right|-|\alpha|. \tag{4}$$

Here again we assume that $c_1,\ldots,c_N$ consists of all complex zeros of $f(z)$. Now if $z=iy$ is purely imaginary, the first term on the right side of (4) is

$$\geqq |z|\left|\operatorname{Im}\sum_{n=N+1}^{\infty}\frac{1}{z-c_n}\right|$$

which similarly as in the proof of (1) tends to infinity with y. This shows that g'/zg also tends to $2a$ as y tends to infinity.

Proof of Theorems 2 and 3. Let α be real and $\neq 0$. Observing that with $f(z)=\sum_{n=0}^{\infty} a_n z^n$ one has $g(z)=\alpha f(z)+zf'(z)=\sum_{n=0}^{\infty} a_n(\alpha+n)z^n$ we first prove the statement of the Theorem for the function $g(z)$.

From $g(z)=\alpha f(z)+zf'(z)$ we see that always $g(z)$ has an odd (even) number of zeros between $z=0$ and the smallest positive and the smallest negative zero of $f(z)$ if $\alpha>0$ ($\alpha<0$).

To prove Theorem 2(a) we apply Theorem 1, Case I,1 to $g(z)$ with $\alpha>0$. Since $g(z)$ has at least one zero between $z=0$ and the smallest positive zero of $f(z)$ it follows that $g(z)$ has at most k zeros which are not >0.

In order to prove part (b) of Theorem 2 we apply Theorem 1, Case I,1 to $g(z)$ again but now with $\alpha<0$, so that $g(z)$ has at most $k+1$ ($k+2$) zeros which are not >0 if $f(z)$ is of type (1) (type (2)).

In all cases $g(z)$ has infinitely many real zeros >0 and according to Lemma 4 $g(z)$ is of the same type as $f(z)$. Therefore $g(z)$ satisfies the same conditions of Theorem 2 as $f(z)$ does, the only difference being that in part (b) k has to be replaced by $k+1$ ($k+2$) if $f(z)$ is of type (1) (type (2)).

In order to finish the proof of part (b) we apply the above arguments m times where successively α is replaced by $-\alpha_j$ ($j=1,\dots,m$). Then we obtain that

$$\sum_{n=0}^{\infty} a_n(n-\alpha_1)\cdot\dots\cdot(n-\alpha_m)z^n \quad \text{or} \quad F(z) = \sum_{n=0}^{\infty} a_n G(n) z^n$$

has at most $k+m$ ($k+2m$) zeros which are not >0 if $f(z)$ is of type (1) (type (2)).

To finish the proof of part (a) we repeat the above arguments with $\alpha_1,\dots,\alpha_l>0$ and obtain that

$$\sum_{n=0}^{\infty} a_n(\alpha_1+n)\cdot\dots\cdot(\alpha_l+n)z^n$$

for each l has at most k zeros which are not >0. This remains true if we divide this series by $\alpha_1\cdot\dots\cdot\alpha_l$ and replace z by $z\exp(\beta-\sum_{j=1}^{l}\alpha_j^{-1})$ which gives the function $g_l(z)=\sum_{n=0}^{\infty} a_n G_l(n)z^n$ where $G_l(z)$ converges to $G(z)$ for each z as l tends to infinity. Since $|a_n G_l(n)|$ and $|a_n G(n)|$ are $\leqq |a_n|e^{\beta n}$ it follows that $g_l(z)$ converges, uniformly on compact domains, to the entire function $F(z)=\sum_{n=0}^{\infty} a_n G(n)z^n$, which according to a theorem of Hurwitz [6] still has at most k zeros which are not >0.

Theorem 3 is proved in a similar way. The only difference is that now Theorem 1, Case II has to be applied instead of Case I.

Proof of Theorem 4. From the assumptions made about $f(z)$ it follows that

$$\frac{f'}{f} = 2az+b+\sum_{n=1}^{\infty}\left(\frac{1}{z-c_n}+\frac{1}{c_n}\right)$$

where all c_n are real. Then we have $g(z)=\alpha f(z)+zf'(z)=zf(z)\varphi(z)$ where

$$\varphi(z) = \frac{\alpha}{z}+\frac{f'}{f} = \frac{\alpha}{z}+2az+b+\sum_{n=1}^{\infty}\left(\frac{1}{z-c_n}+\frac{1}{c_n}\right).$$

Suppose that z is not real, then

$$\frac{\varphi(z)-\varphi(\bar{z})}{z-\bar{z}} = -\frac{\alpha}{|z|^2}+2a-\sum_{n=1}^{\infty}\frac{1}{|z-c_n|^2}$$

which is <0 if $a\leqq 0$ and $\alpha>0$. Consequently $g(z)$ does not have complex zeros if $\alpha>0$. According to Lemma 4 $g(z)$ is of the same type as $f(z)$ (with $a\leqq 0$!) and in order to finish the proof one only has to repeat the arguments of the proof of Theorem 2.

Theorems 5 and 6 finally are proved in the same way as Theorem 2(b). In addition one has to use the fact that Theorem 1 gives the *exact* number of zeros which, under the assumptions made in Theorems 5 and 6, cannot be >0 in Theorem 5 and which cannot be real in Theorem 6.

Next we want to apply the previous theorems in order to obtain Hurwitz's theorem on the zeros of Besselfunctions.

Let

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(\nu+n+1)}$$

denote the Besselfunction of order ν, which is defined for all $z\neq 0$ and with $-\pi<\arg z<\pi$. We assume that ν is real and $\neq -1, -2, \ldots$. The zeros of $J_\nu(z)$ are determined by the zeros of the entire function

$$\phi_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(\nu+n+1)}.$$

This entire function is of order 1 as can be seen from the coefficients of the series. Therefore according to Hadamard's factorization theorem $p\leqq 1$ (in fact $p=1$, but this is not needed here). Furthermore $\phi_\nu(z)$ satisfies: $\phi_\nu(0)\neq 0$ and

(i) $\phi_\nu'(z)=-(z/2)\phi_{\nu+1}(z)$,

(ii) $2\phi_\nu(z)=2(\nu+1)\phi_{\nu+1}(z)+z\phi_{\nu+1}'(z)$,

(iii) $z\phi_\nu''(z)+(2\nu+1)\phi_\nu'(z)+z\phi_\nu(z)=0$.

Since $\varphi_\nu(z)=\phi_\nu(2\sqrt{z})$ is an entire function of order 1/2, again according to Hadamard's factorization theorem $\phi_\nu(z)$ has infinitely many zeros, which are simple because of (iii).

THEOREM 7 (HURWITZ'S THEOREM). (a) *$\phi_\nu(z)$ has real zeros only if $\nu>-1$.*

(b) *If m is a natural number and $-(m+1)<\nu<-m$ then $\phi_\nu(z)$ has exactly $2m$ complex zeros.*

(c) *If m is odd $\phi_\nu(z)$ has exactly 2 purely imaginary zeros and if m is even $\phi_\nu(z)$ has no purely imaginary zeros.*

Proof. (a) follows from Theorem 4 with $f(z)=e^{-(z/2)^2}$, $(\rho=2,\ p=0)$ and $G(z)=1/\Gamma(z/2+\nu+1)$ which for $\nu>-1$ is of type (A).

(b) follows from Theorem 6 with

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(n+\nu+m+1)}, \qquad (\rho = 1, p \leqq 1)$$

and $G(z)=(\nu+1+z/2)\cdot\cdots\cdot(\nu+m+z/2)$ which is of type (B). The remaining conditions made in Theorem 6 are satisfied here because of the properties (i) and (ii) with $\alpha_j=-2(\nu+j)$, $j=1,\ldots,m$.

(c) follows from sign $\phi_\nu(0)=\text{sign}\,(1/\Gamma(\nu+1))$ and by using (i) and (ii) for purely imaginary z.

More generally we consider for arbitrary real α, $\alpha J_\nu(z)+zJ_\nu'(z)=(z/2)^\nu\psi_{\nu\alpha}(z)$ with $\psi_{\nu\alpha}(z)=(\alpha+\nu)\phi_\nu(z)+z\phi_\nu'(z)$. If, for the moment, we put $g(z)=z^\alpha J_\nu(z)$, then

$$g' = z^{\alpha-1}(\alpha J_\nu+zJ_\nu'),$$

and from the differential equation for $J_\nu(z)$ or $\phi_\nu(z)$ follows

$$(z^{1-2\alpha}g')' = -z^{1-2\alpha}(1+(\alpha^2-\nu^2)/z^2)g. \tag{5}$$

If $|\nu|\leqq|\alpha|$ (5) shows that $g'(z)$ or $\psi_{\nu\alpha}(z)$ has exactly one zero between each pair of consecutive positive zeros of $g(z)$ or $\phi_\nu(z)$ and that $g'(z)$ has exactly one (has no) zero between $z=0$ and the smallest positive zero of $\phi_\nu(z)$ if $(\alpha+\nu)>0$ (if $(\alpha+\nu)<0$).

With these results Theorem 1, Case II,1 gives the following

THEOREM 8. *Let $\phi_\nu(z)$ have exactly $2m$ complex zeros $(m\geqq 0)$. Then for any real α $\psi_{\nu\alpha}(z)$ has exactly $2m+2$ zeros besides one zero between each pair of consecutive positive and negative zeros of $\phi_\nu(z)$.*

Especially $\psi_{\nu\alpha}(z)$ has at most $2m$ $(2m+2)$ complex zeros if $(\alpha+\nu)>0$ (if $(\alpha+\nu)<0$).

If furthermore $|\nu|\leqq|\alpha|$, then $\psi_{\nu\alpha}(z)$ has exactly $2m$ (exactly $2m+2$) complex zeros if $(\alpha+\nu)>0$ (if $(\alpha+\nu)<0$).

The following theorem again is an application of Theorem 1, Case II.

THEOREM 9. *Let $\Pi(z)$ be a canonical product of genus p with infinitely many real zeros of both signs, no complex zeros and no zero at $z=0$. Furthermore consider $g(z)=\alpha f(z)+z^l f'(z)$, where $f(z)=z^m\Pi(z)$ with $m\geqq 0$, α real and $\geqq 0$ and l odd.*

(a) *If $\alpha>0$ and $m\geqq 0$ then $g(z)$ has exactly $p+l-2$ $(p+l-1)$ complex zeros when p is odd (even).*

(b) *If $\alpha=0$ and $m>0$ then $g(z)$ has exactly $p-1$ (p) complex zeros when p is odd (even).*

(c) *If $\alpha=m=0$ then $g(z)$ has no complex zeros.*

REMARK. A similar result can be obtained if $\Pi(z)$ still has infinitely many real zeros but not infinitely many of both signs. One has to apply Theorem 1, Case I instead of Case II.

Proof. $g(z)=z^{l+p}f(z)\varphi_1(z)$ if p is even and $g(z)=z^{l+p-1}f(z)\varphi_2(z)$ if p is odd. Here

$$\varphi_1(z) = \frac{\alpha}{z^{l+p}}+\frac{m}{z^{p+1}}+\sum_{n=1}^{\infty}\frac{1}{a_n^p(z-a_n)} \quad \text{where } p \text{ is even,}$$

and

$$\varphi_2(z) = \frac{\alpha}{z^{l+p-1}}+\frac{m}{z^p}+\sum_{n=1}^{\infty}\left(\frac{1}{a_n^{p-1}(z-a_n)}+\frac{1}{a_n^p}\right) \quad \text{where } p \text{ is odd.}$$

Next we form the derivatives and see that $\varphi_1'(z)$ and $\varphi_2'(z)$ are <0 for real $z\neq 0$ and $\neq a_n$. Consequently $g(z)$ has exactly one zero between each pair of consecutive positive and negative zeros of $f(z)$. Furthermore if $\alpha>0$ and $m\geqq 0$ $g(z)$ has exactly one zero between $z=0$ and the smallest positive and negative zero of $f(z)$. In addition $g(z)$ has an m-fold zero at $z=0$ and thus according to Theorem 1, Case II we obtain (a).

Cases (b) and (c) are proved similarly.

COROLLARY. *Let $\alpha>0$ and l odd, then the functions $\alpha \sin z+z^l \cos z$ and $\alpha \cos z - z^l \sin z$ have exactly $l-1$ complex zeros.*

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