

ANALYTIC SHEAF COHOMOLOGY GROUPS OF DIMENSION n OF n -DIMENSIONAL COMPLEX SPACES

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In this paper we prove the following:

MAIN THEOREM. *Suppose \mathcal{F} is a coherent analytic sheaf on a σ -compact complex space X (not necessarily reduced).*

(A) _{n} *If $\dim X = n$ and X has no compact n -dimensional branch, then $H^n(X, \mathcal{F}) = 0$.*

(B) _{n} *If $\dim X = n$ and X has only a finite number of compact n -dimensional branches, then $\dim H^n(X, \mathcal{F}) < \infty$.*

(A) _{n} with the additional assumption that X is a manifold and \mathcal{F} is locally free was proved by Malgrange [12, p. 236, Problème 1]. In [9] Komatsu proved the following related result: if \mathcal{S} is a coherent analytic sheaf on an n -dimensional complex manifold U such that $H^n(U, \mathcal{S}) = 0$, then $H^n(V, \mathcal{S}) = 0$ for any open subset V of U (p. 83, Theorem 7). The author in [16] proved (A) _{n} with the additional assumption that X is a manifold.

The paper is divided into five sections. In §I some Lemmas about Fréchet spaces and LF-spaces are proved. In §II a duality concerning distributions with restricted supports is established. In §III by partial normalizations and results of [13] the proof of the Main Theorem is reduced to the proof of (A) _{n} with the additional assumption that X is reduced and normal and \mathcal{F} is torsion-free. In §IV we prove by local resolutions of singularities that $H^n(G, \mathcal{F}) = 0$ for $G \subset \subset X$ and also obtain a result on the approximation of $(n-1)$ -cocycles with coefficients in \mathcal{F} . §V sews up the proof of the Main Theorem.

All complex spaces here are σ -compact and are in the sense of Grauert [3, p. 9, Definition 2]. The structure sheaf of a complex space X is denoted by \mathcal{O}_X unless specified otherwise. The set of all singular points of X is denoted by $\sigma(X)$. The inverse image [4, p. 410, Definition 8] and the q th direct image [4, p. 413, Definition 9] of an analytic sheaf \mathcal{F} under a holomorphic map f of complex spaces are denoted respectively by $f^{-1}(\mathcal{F})$ and $R^q f(\mathcal{F})$. If \mathcal{G} is a subsheaf of \mathcal{F} , then $R^0 f(\mathcal{G})$ is regarded as a subsheaf of $R^0 f(\mathcal{F})$. A covering \mathfrak{U} of a complex space X is called a Stein covering if \mathfrak{U} is countable and every member of \mathfrak{U} is a Stein open subset. If Y is an open subset of X , then $\mathfrak{U}|Y = \{U \in \mathfrak{U} | U \subset Y\}$ is called the restriction of \mathfrak{U}

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to Y . If \mathfrak{B} is a refinement of a subcollection of \mathfrak{U} under some map $\tau: \mathfrak{B} \rightarrow \mathfrak{U}$ and \mathcal{F} is a sheaf on X , then τ induces $C^p(\mathfrak{U}, \mathcal{F}) \rightarrow C^p(\mathfrak{B}, \mathcal{F})$ and we call the image in $C^p(\mathfrak{B}, \mathcal{F})$ of $f \in C^p(\mathfrak{U}, \mathcal{F})$ the restriction of f to \mathfrak{B} and denote it by $f|_{\mathfrak{B}}$. Whenever we have a refinement \mathfrak{B} , we assume that a fixed suitable map τ is chosen once for all. The supports of sheaves, functions, distributions, etc. are denoted by Supp . All locally convex (linear topological) spaces are over C . The dual of a locally convex space E is denoted by E^* . The boundary, interior, and closure of a subset A of a topological space are denoted respectively by ∂A , A° , and A^- . Unless specified otherwise, the boundary, the interior, and the closure are with respect to the largest ambient topological space. Suppose B and C are subsets of a metric space Y with metric d , $x \in Y$, and $\varepsilon > 0$. Then

$$\begin{aligned} d(x, B) &= \inf_{y \in B} d(x, y), & d(B, C) &= \inf_{y \in B} d(y, C), \\ U_\varepsilon(B) &= \{y \in Y \mid d(y, B) < \varepsilon\}, & \bar{U}_\varepsilon(B) &= \{y \in Y \mid d(y, B) \leq \varepsilon\}. \end{aligned}$$

N = the set of all natural numbers.

I. Suppose E and F are LF-spaces [8, p. 18, Definition 8(d)] and are respectively the strict inductive limits of their closed Fréchet spaces $\{E_n\}_{n \in N}$ and $\{F_n\}_{n \in N}$. Suppose $\phi: E \rightarrow F$ is a continuous linear map.

LEMMA 1. *If $0 < \dim \text{Coker } \phi < \infty$, then the transpose $\phi^*: F^* \rightarrow E^*$ is not injective.*

Proof. We first prove that $\phi(E)$ is closed in F . Since $\dim \text{Coker } \phi < \infty$, $F = \phi(E) \oplus G$ for some finite dimensional subspace G of F . Define $\tilde{\phi}: E \oplus G \rightarrow F$ by $\tilde{\phi}(a \oplus b) = \phi(a) + b$ for $a \in E$ and $b \in G$. $\tilde{\phi}$ is a continuous linear surjection. Since $E \oplus G$ is an LF-space, $\tilde{\phi}$ is open [8, p. 44, Theorem 10]. Since $E \oplus 0$ is closed in $E \oplus G$ and $\tilde{\phi}^{-1}\tilde{\phi}(E \oplus 0) = E \oplus 0$, $\phi(E) = \tilde{\phi}(E \oplus 0)$ is closed in F .

Since $\dim \text{Coker } \phi > 0$, there exists $c \in F - \phi(E)$. Since F is locally convex and $\phi(E)$ is closed, by the theorem of Hahn-Banach there exists $f \in F^*$ such that $f(c) \neq 0$ and $f \equiv 0$ on $\phi(E)$. f is a nonzero element of $\text{Ker } \phi^*$. Q.E.D.

LEMMA 2. *If ϕ is surjective, then for any $k \in N$, there exists $l \in N$ such that $\phi(E_l) \supset F_k$.*

Proof. Suppose the lemma is not true. Fix $k \in N$. Let $G_n = E_n \cap \phi^{-1}(F_k)$. G_n is a Fréchet space. Let $\psi_n: G_n \rightarrow F_k$ be induced by ϕ . Since $\psi_n(G_n) \neq F_k$, $\psi_n(G_n)$ is of the first category in F_k [8, p. 41, Corollary 6]. $\bigcup_{n \in N} \psi_n(G_n) = \bigcup_{n \in N} \phi(G_n) = F_k$ is of the first category, contradicting that every Fréchet space is of the second category. Q.E.D.

LEMMA 3. *Suppose for $n \in N$*

$$(1)_n \quad \begin{array}{ccccc} E_n^1 & \xrightarrow{\alpha_n^1} & E_n^2 & \xrightarrow{\alpha_n^2} & E_n^3 \\ \phi_n^1 \downarrow & & \phi_n^2 \downarrow & & \phi_n^3 \downarrow \\ F_n^1 & \xrightarrow{\beta_n^1} & F_n^2 & \xrightarrow{\beta_n^2} & F_n^3 \end{array}$$

is a commutative diagram of continuous linear maps and Fréchet spaces such that E_n^i and F_n^i are closed subspaces of E_{n+1}^i and F_{n+1}^i respectively, $1 \leq i \leq 3$, and $(1)_n$ is induced from $(1)_{n+1}$, $n \in N$. Suppose in the following diagram which is the direct limit of $(1)_n$, $n \in N$, ϕ^1 is surjective:

$$(2) \quad \begin{array}{ccccc} E^1 & \xrightarrow{\alpha^1} & E^2 & \xrightarrow{\alpha^2} & E^3 \\ \phi^1 \downarrow & & \phi^2 \downarrow & & \phi^3 \downarrow \\ F^1 & \xrightarrow{\beta^1} & F^2 & \xrightarrow{\beta^2} & F^3 \end{array}$$

Assume that for every $p \in N$ there exists $r \geq p$ such that,

$$(3)_{p,r} \quad \begin{array}{l} \text{if } f \in (E_r^3)^* \text{ and } g \in (F_r^1)^* \text{ such that } f\alpha_r^2\alpha_r^1 = g\phi_r^1, \\ \text{then for some } h \in (F_p^2)^* \text{ we have } f\alpha_p^2 = h\phi_p^2 \text{ on } E_p^2. \end{array}$$

Let $Z_n^i = \text{Ker } \phi_n^i$, $i = 1, 2$, $n \in N$. Then for every $n \in N$ there exists $m \in N$ such that $(\alpha^2\alpha^1(Z_m^1))^\perp \supset \alpha^2(Z_n^2)$.

Proof. Fix $n \in N$. There exists $r \geq n$ satisfying $(3)_{n,r}$. Since ϕ^1 is surjective, $\phi^1(E_m^1) \supset F_r^1$ for some $m \geq r$ (Lemma 2). We claim that m satisfies the requirement.

Suppose the contrary. By the theorem of Hahn-Banach there exists $f \in (E^3)^*$ such that f is identically zero on $(\alpha^2\alpha^1(Z_m^1))^\perp$ and is not identically zero on $\alpha^2(Z_n^2)$. Let $G = E_m^1 \cap (\phi^1)^{-1}(F_r^1)$ and $\psi: G \rightarrow F_r^1$ be induced by ϕ^1 . Since ψ is surjective, ψ is open [8, p. 41, Theorem 8]. $f\alpha^2\alpha^1$ is zero on $Z_m^1 \supset \text{Ker } \psi$. Hence there exists $g \in (F_r^1)^*$ such that $f\alpha^2\alpha^1 = g\phi^1$ on $G \supset E_r^1$. By $(3)_{n,r}$ there exists $h \in (F_n^2)^*$ such that $f\alpha^2 = h\phi^2$ on E_n^2 . $f\alpha^2(Z_n^2) = h\phi^2(Z_n^2) = 0$, contradicting that f is not identically zero on $\alpha^2(Z_n^2)$. Q.E.D.

LEMMA 4. Suppose G and H are Fréchet spaces and $\psi: G \rightarrow H$ is a continuous linear surjection. Suppose $\{b_q\}_{q \in N}$ is a sequence in H converging to b and $a \in G$ with $\psi(a) = b$. Then there exists a sequence $\{a_q\}_{q \in N}$ in G converging to a such that $\psi(a_q) = b_q$, $q \in N$.

Proof. $\{b_q - b\}_{q \in N}$ is a sequence in H converging to zero. Suppose we can find a sequence $\{c_q\}_{q \in N}$ in G converging to zero such that $\psi(c_q) = b_q - b$. Then $a_q = c_q + a$, $q \in N$, satisfies the requirement. So we can assume without loss of generality that $a = 0$ and $b = 0$.

Since ψ is open, we can choose open neighborhood bases $\{V_q\}_{q \in N}$ of 0 in G and $\{W_q\}_{q \in N}$ of 0 in H such that $V_{q+1} \subset V_q$, $W_{q+1} \subset W_q$, and $\psi(V_q) \supset W_q$, $q \in N$.

Since $b_q \rightarrow 0$, we can find a strictly increasing function $p: N \rightarrow N$ such that if $q \in N$ and $k \geq p(q)$, then $b_k \in W_q$. We are going to define $\{a_q\}_{q \in N}$. Fix $q \in N$. If $q < p(1)$, choose any a_q such that $\psi(a_q) = b_q$. If $q \geq p(1)$, then there is a unique $r \in N$ such that $p(r) \leq q < p(r+1)$. $b_q \in W_r$. Choose $a_q \in V_r$ such that $\psi(a_q) = b_q$. $a_q \rightarrow 0$. Q.E.D.

II. Suppose B is a holomorphic vector bundle on an n -dimensional complex manifold M . Let $\lambda_M^{r,s}$ or simply $\lambda^{r,s}$ denote the C^∞ vector bundle of (r, s) -forms on

M . B^* denotes the dual of B (transition functions of B^* are the inverse transpose of those of B).

$\mathcal{O}(B)$ = the sheaf of germs of holomorphic sections of B .

$\mathcal{E}^{r,s}(B)$ = the sheaf of germs of C^∞ sections of $B \otimes \lambda^{r,s}$.

$\mathcal{D}^{r,s}(B^*)$ = the sheaf of germs of distribution sections of $B^* \otimes \lambda^{r,s}$.

$D^{r,s}(B^*) = \Gamma(M, \mathcal{D}^{r,s}(B^*))$.

If A is a closed subset of M , then $E_A^{r,s}(B)$ = the Fréchet space of all global C^∞ sections of $B \otimes \lambda^{r,s}$ whose supports are contained in A .

Suppose $\Lambda = \{\Lambda_k\}_{k \in N}$ is a sequence of closed subsets of M such that $\Lambda_k \subset \Lambda_{k+1}^0$, $k \in N$, and $M = \bigcup_{k \in N} \Lambda_k$. Let $E^{r,s}(B, \Lambda) = \bigcup_{k \in N} E_{\Lambda_k}^{r,s}(B)$ denote the strict inductive limit of $\{E_{\Lambda_k}^{r,s}(B)\}_{k \in N}$. $E^{r,s}(B, \Lambda)$ is an LF-space. If in addition every Λ_k is compact, then denote $E^{r,s}(B, \Lambda)$ by $E_{\text{cpct}}^{r,s}(B)$. $E_{\text{cpct}}^{r,s}(B)$ is independent of the choice of Λ and is the LF-space of all global C^∞ sections of $B \otimes \lambda^{r,s}$ with compact supports. It is well known that $D^{n-r,n-s}(B^*)$ is canonically the dual of $E_{\text{cpct}}^{r,s}(B)$ [15, p. 18, Proposition 4].

Let Λ^* denote $\{A \mid A \text{ is a closed subset of } M, A \cap \Lambda_k \text{ is compact for } k \in N\}$. $D^{r,s}(B^*, \Lambda^*)$ denotes the vector space of all distribution sections of $B^* \otimes \lambda^{r,s}$ whose supports are members of Λ^* . Suppose $T \in D^{n-r,n-s}(B^*, \Lambda^*)$ and $\phi \in E^{r,s}(B, \Lambda)$. Then $\text{Supp } \phi \subset \Lambda_k$ for some $k \in N$. Since $\Lambda_k \cap \text{Supp } T$ is compact, we can find a C^∞ function ρ on M with compact support such that $\rho \equiv 1$ on some neighborhood of $\Lambda_k \cap \text{Supp } T$. $\rho\phi \in E_{\text{cpct}}^{r,s}(B)$. Let $T(\rho\phi)$ be the value of T at $\rho\phi$ when T is regarded as an element of $D^{n-r,n-s}(B^*) = E_{\text{cpct}}^{r,s}(B)^*$. $\phi \mapsto T(\rho\phi)$ defines a continuous linear functional on $E^{r,s}(B, \Lambda)$, independent of the choices of k and ρ .

PROPOSITION 1. $D^{n-r,n-s}(B^*, \Lambda^*)$ is the dual of $E^{r,s}(B, \Lambda)$.

Proof. We have seen that every element of $D^{n-r,n-s}(B^*, \Lambda^*)$ defines a continuous linear functional on $E^{r,s}(B, \Lambda)$. Since $E_{\text{cpct}}^{r,s}(B, \Lambda) \subset E^{r,s}(B, \Lambda)$, if an element of $D^{n-r,n-s}(B^*, \Lambda^*)$ defines the zero functional on $E^{r,s}(B, \Lambda)$, then it must be zero.

Conversely, suppose F is a continuous linear functional on $E^{r,s}(B, \Lambda)$. Since $D^{n-r,n-s}(B^*)$ is the dual of $E_{\text{cpct}}^{r,s}(B)$, there exists $T \in D^{n-r,n-s}(B^*)$ such that T defines the continuous linear functional $F|_{E_{\text{cpct}}^{r,s}(B)}$. Let $A = \text{Supp } T$. We claim that $A \in \Lambda^*$. If $A \cap \Lambda_k$ is noncompact for some $k \in N$, then there exists a closed discrete sequence of distinct points $\{x_q\}_{q \in N} \subset A \cap \Lambda_k$. For every $q \in N$ take an open neighborhood U_q of x_q in Λ_{k+1}^0 such that every compact subset of M intersects only a finite number of U_q , $q \in N$. Since $x_q \in A$, there exists $\phi_q \in E_{\text{cpct}}^{r,s}(B)$, $q \in N$, such that $\text{Supp } \phi_q \subset U_q$ and $F(\phi_q) = 1$. $\phi_q \rightarrow 0$ in $E^{r,s}(B, \Lambda)$ as $q \rightarrow \infty$. $F(\phi_q) \rightarrow 0$ as $q \rightarrow \infty$, contradicting that $F(\phi_q) = 1$ for all $q \in N$. Hence $T \in D^{r,s}(B^*, \Lambda^*)$. T defines a continuous linear functional on $E^{r,s}(B, \Lambda)$. Since this functional agrees with F on the dense subset $E_{\text{cpct}}^{r,s}(B)$ of $E^{r,s}(B, \Lambda)$, it agrees with F on $E^{r,s}(B, \Lambda)$. Q.E.D.

Since B is a holomorphic vector bundle, the $\bar{\partial}$ -operator mapping $C^\infty(r, s)$ -forms

to $C^\infty(r, s+1)$ -forms induces a map $\bar{\partial}: \mathcal{E}^{r,s}(B) \rightarrow \mathcal{E}^{r,s+1}(B)$. If $i: \mathcal{O}(B) \rightarrow \mathcal{E}^{0,0}(B)$ denotes the inclusion map, then

$$(4) \quad 0 \longrightarrow \mathcal{O}(B) \xrightarrow{i} \mathcal{E}^{0,0}(B) \xrightarrow{\bar{\partial}} \mathcal{E}^{0,1}(B) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{E}^{0,n}(B) \longrightarrow 0$$

is exact [15, p. 14, Proposition 2].

For any open subset U of M let $\bar{\partial}_U^*: \Gamma(U, \mathcal{D}^{n-r,n-s-1}(B^*)) \rightarrow \Gamma(U, \mathcal{D}^{n-r,n-s}(B^*))$ be the transpose of $\bar{\partial}: E_{\text{cpct}}^{r,s}(B|U) \rightarrow E_{\text{cpct}}^{r,s+1}(B|U)$, where $B|U$ denotes the restriction of B to U . $\{\bar{\partial}_U^*\}$ induces a sheaf-homomorphism

$$\bar{\partial}^*: \mathcal{D}^{n-r,n-s-1}(B^*) \rightarrow \mathcal{D}^{n-r,n-s}(B^*).$$

If $i: \mathcal{O}(B^* \otimes \lambda^{n,0}) \rightarrow \mathcal{D}^{n,0}(B^*)$ denotes the inclusion map, then

$$(5) \quad 0 \longrightarrow \mathcal{O}(B^* \otimes \lambda^{n,0}) \xrightarrow{i} \mathcal{D}^{n,0}(B^*) \xrightarrow{\bar{\partial}^*} \mathcal{D}^{n,1}(B^*) \xrightarrow{\bar{\partial}^*} \dots \xrightarrow{\bar{\partial}^*} \mathcal{D}^{n,n}(B^*) \longrightarrow 0$$

is exact [15, p. 14, Proposition 2]. Since $E_{\text{cpct}}^{r,q}(B)$ is dense in $E^{r,q}(B, \Lambda)$ for $q=s, s+1$,

$$(6) \quad \begin{aligned} \bar{\partial}^*: D^{n-r,n-s-1}(B^*, \Lambda^*) &\rightarrow D^{n-r,n-s}(B^*, \Lambda^*) \text{ is the transpose of} \\ \bar{\partial}: E^{r,s}(B, \Lambda) &\rightarrow E^{r,s+1}(B, \Lambda), \end{aligned}$$

where the first and second maps are induced by $\bar{\partial}^*: \mathcal{D}^{n-r,n-s-1}(B^*) \rightarrow \mathcal{D}^{n-r,n-s}(B^*)$ and $\bar{\partial}: \mathcal{E}^{r,s}(B) \rightarrow \mathcal{E}^{r,s+1}(B)$ respectively.

III. PROPOSITION 2. Suppose A is an analytic subvariety of codimension ≥ 1 in a reduced complex space (X, \mathcal{O}) . Then there exist, uniquely up to isomorphism, a reduced complex space X' and a proper nowhere degenerate holomorphic map $\pi: X' \rightarrow X$ such that

- (i) π induces a biholomorphic map $X' - \pi^{-1}(A) \approx X - A$,
- (ii) for every open subset U of X' the following holds: if f is a weakly holomorphic function on U and for every $x \in U - \pi^{-1}(A)$ f is (strongly) holomorphic at x , then f is a (strongly) holomorphic function on U .

Proof. Let \mathcal{S} be the sheaf of germs of weakly holomorphic functions on X which are (strongly) holomorphic outside A . We need only prove that \mathcal{S} is coherent. Then the construction and uniqueness of X' and π follows the same line as §4, pp. 118–122 of [10].

To prove the coherence of \mathcal{S} , we can assume without loss of generality the following:

- (i) X is a complex subspace of an open subset G of \mathbb{C}^n such that $\mathcal{O} \approx (\mathcal{O}/\mathcal{I})|_X$ for some coherent ideal-sheaf \mathcal{I} on G and $X = \{x \in G \mid \mathcal{I}_x \neq \mathcal{O}_x\}$.
- (ii) There is a holomorphic function u' on G such that $u = u'|_X$ is a universal denominator on X .

Let \mathcal{O}' be the sheaf of germs of weakly holomorphic functions on X . Let \mathcal{T} be the gap-sheaf of $u'_G \mathcal{O} + \mathcal{I}$ with A as the exceptional subvariety [17, p. 381, Definition 10]. Let $\lambda: \mathcal{O}' \rightarrow \mathcal{O}$ be the sheaf-monomorphism defined by multiplication by u .

$\mathcal{S} = \lambda^{-1}(u\mathcal{O}' \cap (\mathcal{T}/\mathcal{I})|X)$. Since \mathcal{T} is coherent on G [17, p. 392, Satz 9], \mathcal{S} is coherent. Q.E.D.

DEFINITION 1. X' is called the *partial normalization* of X with respect to A .

PROPOSITION 3. Suppose (X, \mathcal{H}) is a complex space and (X, \mathcal{O}) is its reduction. Suppose $p \in N$. If $H^p(X, \mathcal{F}) = 0$ for every coherent analytic sheaf \mathcal{F} on (X, \mathcal{O}) , then $H^p(X, \mathcal{G}) = 0$ for every coherent analytic sheaf \mathcal{G} on (X, \mathcal{H}) .

Proof. The proof of Satz 3, p. 17 of [3], with trivial modifications yields this result. Q.E.D.

We introduce the following statements (where \mathcal{F} is a coherent analytic sheaf on a complex space X):

- (A)_n^{*} If $\dim X = n$ and X is reduced and has no compact n -dimensional branch, then $H^n(X, \mathcal{F}) = 0$.
- (B)_n^{*} If $\dim X = n$ and X is reduced and has only a finite number of compact n -dimensional branches, then $\dim H^n(X, \mathcal{F}) < \infty$.
- (A)_n[†] If $\dim X = n$ and X is reduced and normal and has no compact n -dimensional branch, then $H^n(X, \mathcal{F}) = 0$.
- (A)_n[#] If $\dim X = n$ and X is reduced, normal, connected, and noncompact and \mathcal{F} is torsion-free, then $H^n(X, \mathcal{F}) = 0$.

LEMMA 5. (A)_n^{*} and (B)_n^{*} \Rightarrow (A)_n and (B)_n.

Proof. (A)_n follows from Proposition 3. To prove (B)_n, suppose (X, \mathcal{H}) is an n -dimensional complex space having only a finite number of compact n -dimensional branches whose union is K and suppose \mathcal{F} is a coherent analytic sheaf on X . Let \mathcal{K} be the subsheaf of all nilpotent elements of \mathcal{H} . Since K is compact, there exists $m \in N$ such that $\mathcal{K}^m = 0$ on K . Let $Y = \text{Supp } \mathcal{K}^m \mathcal{F}$. Y is a subvariety disjoint from K . By Satz 2, p. 275 of [13], and (A)_n, $H^n(X, \mathcal{K}^m \mathcal{F}) = H^n(Y, \mathcal{K}^m \mathcal{F}) = 0$. The short exact sequences

$$0 \rightarrow \mathcal{K}^r \mathcal{F} / \mathcal{K}^{r+1} \mathcal{F} \rightarrow \mathcal{F} / \mathcal{K}^{r+1} \mathcal{F} \rightarrow \mathcal{F} / \mathcal{K}^r \mathcal{F} \rightarrow 0, \quad 1 \leq r \leq m-1,$$

yield exact sequences

$$H^n(X, \mathcal{K}^r \mathcal{F} / \mathcal{K}^{r+1} \mathcal{F}) \rightarrow H^n(X, \mathcal{F} / \mathcal{K}^{r+1} \mathcal{F}) \rightarrow H^n(X, \mathcal{F} / \mathcal{K}^r \mathcal{F}), \quad 1 \leq r \leq m-1.$$

Since $\mathcal{K}^r \mathcal{F} / \mathcal{K}^{r+1} \mathcal{F}$, $1 \leq r \leq m-1$, and $\mathcal{F} / \mathcal{K}^r \mathcal{F}$ can be regarded as coherent analytic sheaves on the reduction (X, \mathcal{O}) of (X, \mathcal{H}) , by (B)_n^{*} and by induction on r we obtain $\dim H^n(X, \mathcal{F} / \mathcal{K}^r \mathcal{F}) < \infty$ for $1 \leq r \leq m$. From the exact sequence $0 = H^n(X, \mathcal{K}^m \mathcal{F}) \rightarrow H^n(X, \mathcal{F}) \rightarrow H^n(X, \mathcal{F} / \mathcal{K}^m \mathcal{F})$, we conclude that

$$\dim H^n(X, \mathcal{F}) < \infty. \quad \text{Q.E.D.}$$

LEMMA 6. (A)_{n-1} and (A)_n[†] \Rightarrow (A)_n and (B)_n.

Proof. By Lemma 5, we need only prove $(A)_n^*$ and $(B)_n^*$. Let \mathcal{F} be a coherent analytic sheaf on an n -dimensional *reduced* complex space (X, \mathcal{O}) . Let $B = \sigma(X)$.

(i) Assume that B has no compact $(n-1)$ -dimensional branch. Let $\pi: Y \rightarrow X$ be the normalization of X . Since π is proper and nowhere degenerate, $R^q\pi(\pi^{-1}(\mathcal{F})) = 0$ for $q \geq 1$. Hence

$$H^n(X, R^0\pi(\pi^{-1}(\mathcal{F}))) \approx H^n(Y, \pi^{-1}(\mathcal{F}))$$

[4, p. 418, Satz 6]. There is a natural sheaf-homomorphism $\lambda: \mathcal{F} \rightarrow R^0\pi(\pi^{-1}(\mathcal{F}))$ [4, p. 418, Satz 7(b)] inducing a sheaf-isomorphism on the restrictions to $X-B$. Let $\mathcal{K} = \text{Ker } \lambda$, $\mathcal{L} = \text{Im } \lambda$, and $\mathcal{Q} = \text{Coker } \lambda$. $\text{Supp } \mathcal{K} \subset B$ and $\text{Supp } \mathcal{Q} \subset B$. By $(A)_{n-1}$ and Satz 2, p. 275 of [13], $H^p(X, \mathcal{Q}) = H^p(\text{Supp } \mathcal{Q}, \mathcal{Q}) = 0$ for $p = n-1, n$, and $H^p(X, \mathcal{K}) = H^p(\text{Supp } \mathcal{K}, \mathcal{K}) = 0$ for $p = n, n+1$. The short exact sequences $0 \rightarrow \mathcal{K} \rightarrow \mathcal{F} \rightarrow \mathcal{L} \rightarrow 0$ and $0 \rightarrow \mathcal{L} \rightarrow R^0\pi(\pi^{-1}(\mathcal{F})) \rightarrow \mathcal{Q} \rightarrow 0$ yield

$$H^n(X, \mathcal{F}) \approx H^n(X, \mathcal{L}) \approx H^n(X, R^0\pi(\pi^{-1}(\mathcal{F}))).$$

By $(A)_n^*$ and [5, p. 245, VIII.A.19], $H^n(X, \mathcal{F}) \approx H^n(Y, \pi^{-1}(\mathcal{F})) = 0$ if X has no compact n -dimensional branch, and $H^n(X, \mathcal{F}) \approx H^n(Y, \pi^{-1}(\mathcal{F}))$ is finite-dimensional if X has only a finite number of compact n -dimensional branches.

(ii) The general case. Let A be the union of all compact $(n-1)$ -dimensional branches of B . Let $\pi: X' \rightarrow X$ be the partial normalization of X with respect to A . $\mathcal{O}^* = R^0\pi(\pi^{-1}\mathcal{O})$ is precisely the sheaf of germs of all weakly holomorphic functions on X which are (strongly) holomorphic outside A . $\sigma(X')$ has no compact $(n-1)$ -dimensional branch, because any compact $(n-1)$ -dimensional branch K of $\sigma(X')$ is contained in $\pi^{-1}(A)$ and hence X' is normal at any point x of K that is a regular point of $\sigma(X')$, which implies that $\dim K \leq \dim_x \sigma(X') \leq n-2$ [10, p. 115, Lemma 3]. Let $\mathcal{J} \subset \mathcal{O}$ be the ideal-sheaf of A and $\mathcal{J}' = \pi^{-1}(\mathcal{J})$. Let \mathcal{C} be the ideal-sheaf on X defined by $\mathcal{C}_x = \{s \in \mathcal{O}_x \mid s\mathcal{O}_x^* \subset \mathcal{O}_x\}$ for $x \in X$. \mathcal{C} is coherent and

$$\{x \in X \mid \mathcal{C}_x \neq \mathcal{O}_x\} \subset A.$$

Let $\{A_i\}_{i \in I}$ be the set of all branches of A . Since A_i , $i \in I$, is compact, the Hilbert Nullstellensatz [5, p. 97, III.A.7] implies that there exists $p: I \rightarrow N$ such that $\mathcal{J}_x^{p(i)} \subset \mathcal{C}_x$ for $x \in A_i$ and $i \in I$. Let $\mathcal{F}' = \pi^{-1}(\mathcal{F})$ and $\mathcal{F}'' = R^0\pi(\mathcal{F}')$. We have a natural map $\lambda: \mathcal{F} \rightarrow \mathcal{F}''$. It is easily verified that

$$(7) \quad \lambda(\mathcal{F}_x) \supset R^0\pi((\mathcal{J}')^{p(i)}\mathcal{F}')_x$$

for $x \in A_i$ and $i \in I$.

Let $\mathcal{J}_i \subset \mathcal{O}$ be the ideal-sheaf of A_i , $i \in I$. $\mathcal{G} = \prod_{i \in I} \mathcal{J}_i^{p(i)}$ is a coherent ideal-sheaf on X . Let $\mathcal{G}' = \pi^{-1}(\mathcal{G})$. By (7) $\lambda(\mathcal{F}_x) \supset R^0\pi(\mathcal{G}'\mathcal{F}')_x$ for $x \in X - \sigma(A)$. Let $\mathcal{G}'' = \lambda(\mathcal{F}) \cap R^0\pi(\mathcal{G}'\mathcal{F}')$. Then

$$\text{Supp } (R^0\pi(\mathcal{G}'\mathcal{F}')/\mathcal{G}'') \subset \sigma(A).$$

$H^r(X, R^0\pi(\mathcal{G}'\mathcal{F}')/\mathcal{G}'') = 0$ for $r = n-1, n$. The cohomology sequence of

$$0 \rightarrow \mathcal{G}'' \rightarrow R^0\pi(\mathcal{G}'\mathcal{F}') \rightarrow R^0\pi(\mathcal{G}'\mathcal{F}')/\mathcal{G}'' \rightarrow 0$$

yields

$$(8) \quad H^n(X, \mathcal{G}'') \approx H^n(X, R^0\pi(\mathcal{G}'\mathcal{F}')).$$

Let $\mathcal{K} = \text{Ker } \lambda$. Let $\tilde{\lambda}: \mathcal{F}/\mathcal{K} \rightarrow \mathcal{F}''$ be the sheaf-monomorphism induced by λ and let $\lambda' = \tilde{\lambda}^{-1}|_{\mathcal{G}''}$. Let $\mathcal{Q} = \text{Coker } \lambda$. Since outside A λ is isomorphic and $\mathcal{G}'' = \mathcal{F}''$, $\text{Supp } \mathcal{Q} \subset A$. $H^n(X, \mathcal{Q}) = 0$. The cohomology sequence of

$$0 \longrightarrow \mathcal{G}'' \xrightarrow{\lambda'} \mathcal{F}/\mathcal{K} \longrightarrow \mathcal{Q} \longrightarrow 0$$

yields

$$(9) \quad \dim H^n(X, \mathcal{F}/\mathcal{K}) \leq \dim H^n(X, \mathcal{G}'').$$

Since $\text{Supp } \mathcal{K} \subset A$, $H^r(X, \mathcal{K}) = 0$ for $r = n, n+1$. The cohomology sequence of $0 \rightarrow \mathcal{K} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{K} \rightarrow 0$ yields

$$(10) \quad H^n(X, \mathcal{F}) \approx H^n(X, \mathcal{F}/\mathcal{K}).$$

Since π is proper and nowhere degenerate,

$$(11) \quad H^n(X, R^0\pi(\mathcal{G}'\mathcal{F}')) \approx H^n(X', \mathcal{G}'\mathcal{F}').$$

From (8), (9), (10), and (11), we conclude that $\dim H^n(X, \mathcal{F}) \leq \dim H^n(X', \mathcal{G}'\mathcal{F}')$. Since $\sigma(X')$ has no compact $(n-1)$ -dimensional branch, the result follows from (i). Q.E.D.

PROPOSITION 4. $(A)_k^\#, 1 \leq k \leq n \Rightarrow (A)_n$ and $(B)_n$.

Proof. By Lemma 6 and by induction on n , we conclude that it suffices to prove that $(A)_n^\# \Rightarrow (A)_n^+$.

Suppose \mathcal{F} is a coherent analytic sheaf on an n -dimensional reduced normal complex space X having no compact n -dimensional branch. We need only prove that $H^n(X, \mathcal{F}) = 0$. We can assume without loss of generality that X is connected. Let \mathcal{T} be the torsion-subsheaf of \mathcal{F} . Then \mathcal{T} is coherent, \mathcal{F}/\mathcal{T} is torsion-free, and $\dim \text{Supp } \mathcal{T} < n$ [1, pp. 14-15, Propositions 6, 7]. $H^n(X, \mathcal{T}) = 0$. By $(A)_n^\#$, $H^n(X, \mathcal{F}/\mathcal{T}) = 0$. The cohomology sequence of $0 \rightarrow \mathcal{T} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/\mathcal{T} \rightarrow 0$ yields $H^n(X, \mathcal{F}) = 0$. Q.E.D.

IV. LEMMA 7. Suppose $f: X \rightarrow Y$ is a monoidal transformation with center $D \subset \sigma(Y)$ [7, p. 315, Definition 1], where X and Y are n -dimensional reduced complex spaces. Suppose there is a holomorphic function u on Y vanishing identically on no branch of Y but vanishing identically on $\sigma(Y)$. If, for some relatively compact open subset Q of Y $H^n(f^{-1}(Q), {}_X\mathcal{O}) = 0$, then $H^n(Q, {}_Y\mathcal{O}) = 0$.

Proof. By replacing u by its sufficiently high power, we can assume without loss of generality that $u|_Q$ is a section of the ideal-sheaf of the complex subspace D and u is a universal denominator on Q . Let \mathcal{I} be the ideal-sheaf of the complex subspace $f^{-1}(D)$ on X . Let $v = u \circ f$. Since $\dim \text{Supp } (\mathcal{I}^k/v^k {}_X\mathcal{O}) < n$, $H^n(f^{-1}(Q), \mathcal{I}^k/v^k {}_X\mathcal{O}) = 0$

for $k \in N$. The cohomology sequence of $0 \rightarrow {}_X\mathcal{O} \xrightarrow{g} \mathcal{J}^k \rightarrow \mathcal{J}^k/v^k {}_X\mathcal{O} \rightarrow 0$, where g is defined by multiplication by v^k , yields $H^n(f^{-1}(Q), \mathcal{J}^k) = 0$, $k \in N$. For some $k \in N$, $R^q f(\mathcal{J}^k)|_Q = 0$ for $q \geq 1$ [7, p. 317, Lemma 2].

$$H^n(Q, R^0 f(\mathcal{J}^k)) \approx H^n(f^{-1}(Q), \mathcal{J}^k) = 0.$$

$uR^0 f(\mathcal{J}^k) \subset {}_Y\mathcal{O}$ on Q . Since $\dim \text{Supp } ({}_Y\mathcal{O}/uR^0 f(\mathcal{J}^k)) < n$,

$$H^n(Q, {}_Y\mathcal{O}/uR^0 f(\mathcal{J}^k)) = 0.$$

The cohomology sequence of

$$0 \longrightarrow R^0 f(\mathcal{J}^k) \xrightarrow{h} {}_Y\mathcal{O} \longrightarrow {}_Y\mathcal{O}/uR^0 f(\mathcal{J}^k) \longrightarrow 0,$$

where h is defined by multiplication by u , yields $H^n(Q, {}_Y\mathcal{O}) = 0$. Q.E.D.

Suppose (X, \mathcal{O}) is a connected, reduced, normal, noncompact complex space of dimension $n \geq 1$. Suppose \mathcal{F} is a torsion-free coherent analytic sheaf on X . Let

$$Z = \sigma(X) \cup \{x \in X \mid \mathcal{F}_x \text{ is not free over } \mathcal{O}_x\}.$$

Z is a subvariety of X [1, p. 15, Proposition 8]. Let B be the holomorphic vector bundle on $X - Z$ such that $\mathcal{O}(B) \approx \mathcal{F}$ on $X - Z$. Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{S}_0 \xrightarrow{\phi_0} \mathcal{S}_1 \xrightarrow{\phi_1} \dots$ be a flabby sheaf resolution of \mathcal{F} on X . Let $\mathcal{Z}_q = \text{Ker } \phi_q$, $q \in N$. Let d be a metric on X defining the topology of X .

We introduce the following notation: if G is an open subset of X , then $\Phi(G) = \{A \mid A \text{ is a closed subset of } G \text{ and } A \cap Z = \emptyset\}$ and $\Psi(G) = \{A \mid A \text{ is a closed subset of } G \text{ and } d(A, Z) > 0\}$. $\Phi(G)$ and $\Psi(G)$ are (paracompactifying) families of supports for $G - Z$ [13, p. 273, Definition 1].

LEMMA 8. *For every $x \in X$ there exists an open neighborhood U of x in X such that for every open subset W of U $H^n(W, \mathcal{F}) = 0$.*

Proof. Fix $x \in X$. By Main Theorem I', p. 151 of [6], there exist an open neighborhood V of x in X and a finite succession of monoidal transformations $f_i: V_{i+1} \rightarrow V_i$ with centers D_i for $0 \leq i < r$ and $V_0 = V$ such that $D_i \subset \sigma(V_i)$ and $\sigma(V_r) = \emptyset$. Let ${}_i\mathcal{O}$ be the structure sheaf of V_i , $0 \leq i \leq r$. ${}_0\mathcal{O} = \mathcal{O}$. Choose two Stein open neighborhoods U, U' of x such that $U \subset \subset U' \subset V$ and on U we have a sheaf-epimorphism $g: \mathcal{O}^p \rightarrow \mathcal{F}$. Since U' is Stein, we can find a holomorphic function u on U' which vanishes identically on $\sigma(U')$ but does not vanish identically on any branch of U' . We claim that U satisfies the requirement. Take an open subset W of U . Let $W_i = (f_0 \circ \dots \circ f_{i-1})^{-1}(W)$ for $1 \leq i \leq r$ and let $W_0 = W$. Then $H^n(W_r, {}_r\mathcal{O}) = 0$ ([12, p. 236, Problème 1]; or [16, Theorem]). By Lemma 7 and backward induction on i , we have $H^n(W_i, {}_i\mathcal{O}) = 0$, $0 \leq i \leq r$. Hence $H^n(W, \mathcal{O}^p) = 0$. Let $\mathcal{K} = \text{Ker } g$. $H^{n+1}(W, \mathcal{K}) = 0$. The cohomology sequence of $0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}^p \rightarrow \mathcal{F} \rightarrow 0$ yields $H^n(W, \mathcal{F}) = 0$. Q.E.D.

PROPOSITION 5. *If G is a relatively compact open subset of X , then there exists a relatively compact open neighborhood \tilde{G} of G^- in X such that for any open*

neighborhood D of G in \tilde{G} the restriction map $H^n(\tilde{G}, \mathcal{F}) \rightarrow H^n(D, \mathcal{F})$ is surjective. Consequently $\dim H^n(G, \mathcal{F}) < \infty$.

Proof. Let $\mathcal{U} = \{U_i\}_{i=1}^m$ be a finite Stein covering of some neighborhood of G^- such that (i) U_i^- , $1 \leq i \leq m$, is compact, and (ii) for any open subset W of U_i , $1 \leq i \leq m$, $H^n(W, \mathcal{F}) = 0$. We claim that $\tilde{G} = \bigcup_{i=1}^m U_i$ satisfies the requirement. Let D be an open neighborhood of G in \tilde{G} . Define inductively $D_0 = D$ and $D_i = U_i \cup D_{i-1}$, $1 \leq i \leq m$. The following portions of Mayer-Vietoris sequences are exact:

$$H^n(D_i, \mathcal{F}) \rightarrow H^n(U_i, \mathcal{F}) \oplus H^n(D_{i-1}, \mathcal{F}) \rightarrow H^n(U_i \cap D_{i-1}, \mathcal{F}),$$

$1 \leq i \leq m$ [2, p. 236, §17(a)]. $H^n(U_i \cap D_{i-1}, \mathcal{F}) = 0$ implies that $H^n(D_i, \mathcal{F}) \rightarrow H^n(D_{i-1}, \mathcal{F})$ is surjective, $1 \leq i \leq m$. The surjectivity of $H^n(\tilde{G}, \mathcal{F}) \rightarrow H^n(D, \mathcal{F})$ follows from $\tilde{G} = D_m$. In particular, $H^n(\tilde{G}, \mathcal{F}) \rightarrow H^n(G, \mathcal{F})$ is surjective.

$$\dim H^n(G, \mathcal{F}) < \infty$$

(cf. proof of Theorem 11, p. 239 of [2]). Q.E.D.

LEMMA 9. $\dim Z \leq n - 2$.

Proof. Suppose $\dim Z \geq n - 1$. Since $\sigma(X) \leq n - 2$, we can take a connected Stein open subset W of $X - \sigma(X)$ such that $\dim(W \cap Z) \geq n - 1$. Take a holomorphic function $f \neq 0$ on W vanishing on $W \cap Z$. Since \mathcal{F} is torsion-free, f_x is not a zero-divisor for \mathcal{F}_x for $x \in W$. $V = \{x \in W \mid \text{homological codimension of } (\mathcal{F}/f\mathcal{F})_x \text{ is } \leq n - 2\}$ is a subvariety of dimension $\leq n - 2$ in W [14, p. 81, Satz 5]. There exists $x \in Z \cap W - V$. \mathcal{F}_x is free over \mathcal{O}_x , contradicting $x \in Z$. Q.E.D.

PROPOSITION 6. If G is a relatively compact open subset of X , then

$$\dim H_{\Psi(G)}^n(G - Z, \mathcal{F}) < \infty.$$

Proof. By Proposition 5 there exists a relatively compact open neighborhood \tilde{G} of G^- in X such that $H^n(\tilde{G}, \mathcal{F}) \rightarrow H^n(D, \mathcal{F})$ is surjective for any open neighborhood D of G in \tilde{G} . Since $\dim Z \leq n - 2$ (Lemma 9),

$$H^n(\tilde{G}, \mathcal{F}) \approx H_{\Phi(\tilde{G})}^n(\tilde{G} - Z, \mathcal{F}) \quad \text{and} \quad H^n(D, \mathcal{F}) \approx H_{\Phi(D)}^n(D - Z, \mathcal{F})$$

[13, p. 273, Satz 3]. Hence

$$(12) \quad \begin{array}{l} \text{the restriction map } H_{\Phi(\tilde{G})}^n(\tilde{G} - Z, \mathcal{F}) \rightarrow H_{\Phi(D)}^n(D - Z, \mathcal{F}) \\ \text{is surjective for any open neighborhood } D \text{ of } G \text{ in } \tilde{G}. \end{array}$$

$\dim H_{\Phi(\tilde{G})}^n(\tilde{G} - Z, \mathcal{F}) = \dim H^n(\tilde{G}, \mathcal{F}) < \infty$ (Proposition 5). $H_{\Phi(\tilde{G})}^n(\tilde{G} - Z, \mathcal{F})$ is generated by the cohomology classes defined by a finite number of elements $t_1, \dots, t_k \in \Gamma_{\Phi(\tilde{G})}(\tilde{G} - Z, \mathcal{F}_n)$. Let $s_i = t_i|_{G - Z}$, $1 \leq i \leq k$. Then $\text{Supp } s_i \subset G^- \cap \text{Supp } t_i$, $1 \leq i \leq k$. Since $G^- \cap \text{Supp } t_i$ is compact and disjoint from Z , $d(\text{Supp } s_i, Z) > 0$, $1 \leq i \leq k$. $s_1, \dots, s_k \in \Gamma_{\Psi(G)}(G - Z, \mathcal{F}_n)$. We claim that the cohomology classes defined by s_1, \dots, s_k generate $H_{\Psi(G)}^n(G - Z, \mathcal{F})$.

Take $u \in \Gamma_{\Psi(G)}(G-Z, \mathcal{F}_n)$. Let $\delta = d(\text{Supp } u, Z) > 0$. Let $D = G \cup (U_\delta(Z) \cap \tilde{G})$. Define $\tilde{u} \in \Gamma(D-Z, \mathcal{F}_n)$ by setting $\tilde{u} = u$ on $G-Z$ and $\tilde{u} = 0$ on $U_\delta(Z) \cap \tilde{G}-Z$. Since $\text{Supp } \tilde{u} \in \Phi(D)$, by (12) there exist $\tilde{v} \in \Gamma_{\Phi(D)}(D-Z, \mathcal{S}_{n-1})$ and $c_1, \dots, c_k \in C$ such that $\tilde{u} - \phi_{n-1}(\tilde{v}) = \sum_{i=1}^k c_i(t_i|D-Z)$. Let $v = \tilde{v}|G-Z$. Then $u - \phi_{n-1}(v) = \sum_{i=1}^k c_i s_i$. Let $A = \text{Supp } v$. To complete the proof, we need only show that $d(A, Z) > 0$. Let $\tilde{A} = \text{Supp } \tilde{v}$. $\tilde{A} = \hat{A} \cap D$ for some closed subset \hat{A} of X . Take $0 < \eta < \delta$. Suppose $d(A, Z) = 0$. Then $d(A \cap \bar{U}_\eta(Z), Z) = 0$.

$$\tilde{A} \cap \bar{U}_\eta(Z) \cap G^- = \hat{A} \cap D \cap \bar{U}_\eta(Z) \cap G^- = \hat{A} \cap \bar{U}_\eta(Z) \cap G^-,$$

because $\bar{U}_\eta(Z) \cap G^- \subset U_\delta(Z) \cap \tilde{G} \subset D$. $\tilde{A} \cap \bar{U}_\eta(Z) \cap G^-$ is therefore compact. $\tilde{A} \cap Z = \emptyset$ implies $d(\tilde{A} \cap \bar{U}_\eta(Z) \cap G^-, Z) > 0$. Since $A \cap \bar{U}_\eta(Z) \subset \tilde{A} \cap \bar{U}_\eta(Z) \cap G^-$, $d(A \cap \bar{U}_\eta(Z), Z) > 0$. Contradiction. Q.E.D.

LEMMA 10. *If G is a nonempty relatively compact open subset of X , then $G - U_{1/k}(Z)$ is noncompact for some $k \in \mathbb{N}$.*

Proof. Suppose $G - U_{1/k}(Z)$ is compact for all $k \in \mathbb{N}$. We claim that $\partial G \subset Z$. Suppose $x \in \partial G - Z$. Then $d(x, Z) > 1/k$ for some $k \in \mathbb{N}$. $x \in \partial(G - U_{1/k}(Z))$. Being compact, $G - U_{1/k}(Z)$ is closed. $x \in G - U_{1/k}(Z) \subset G = G^\circ$, contradicting $x \in \partial G$. Hence $\partial G \subset Z$. $G^- \subset G \cup Z$. $G^- - Z = G - Z$. Therefore $G - Z$ is both open and closed in $X - Z$. Since $G - Z \neq \emptyset$ and $X - Z$ is connected, $G - Z = X - Z$. $(G - Z)^- = (X - Z)^- = X$ is compact, contradicting that X is noncompact. Q.E.D.

PROPOSITION 7. *If G is a relatively compact open subset of X , then*

$$H_{\Psi(G)}^n(G-Z, \mathcal{F}) = 0.$$

Proof. We can assume without loss of generality that G is connected and nonempty. Let $M = G - Z$ and $\tilde{B} = B|M$. Let $\Lambda_k = \{G - U_{1/k}(Z) \mid k \in \mathbb{N}\}$ and $\Lambda = \{\Lambda_k\}_{k \in \mathbb{N}}$. Then

$$E^{r,s}(\tilde{B}, \Lambda) = \Gamma_{\Psi(G)}(M, \mathcal{E}^{r,s}(\tilde{B})).$$

By (4) $H_{\Psi(G)}^n(M, \mathcal{F})$ is isomorphic to the cokernel of $\bar{\partial}: E^{0,n-1}(\tilde{B}, \Lambda) \rightarrow E^{0,n}(\tilde{B}, \Lambda)$. Suppose $H_{\Psi(G)}^n(M, \mathcal{F}) \neq 0$. Then by Proposition 6 and Lemma 1 $\bar{\partial}^*: D^{n,0}(\tilde{B}^*, \Lambda^*) \rightarrow D^{n,1}(\tilde{B}^*, \Lambda^*)$ is not injective. There exists $f \in D^{n,0}(\tilde{B}^*, \Lambda^*)$ such that $\bar{\partial}^* f = 0$ and $\text{Supp } f \neq \emptyset$. By (5) $f \in \Gamma(M, \mathcal{O}(\tilde{B} \otimes \lambda_M^{n,0}))$. Since f is a holomorphic section of the holomorphic vector bundle $\tilde{B} \otimes \lambda_M^{n,0}$ and M is connected, $\text{Supp } f = M$. Hence $M \in \Lambda^*$, contradicting Lemma 10. Q.E.D.

PROPOSITION 8. *If G is a relatively compact open subset of X , then*

$$H^n(G, \mathcal{F}) = 0.$$

Proof. By Proposition 5 there exists a relatively compact open neighborhood \tilde{G} of G^- in X such that $H^n(\tilde{G}, \mathcal{F}) \rightarrow H^n(G, \mathcal{F})$ is surjective. Take $s \in \Gamma(G, \mathcal{F}_n)$. There exist $a \in \Gamma(G, \mathcal{S}_{n-1})$ and $t \in \Gamma(\tilde{G}, \mathcal{F}_n)$ such that $t|G = s - \phi_{n-1}(a)$. Since $H_{\Phi(\tilde{G})}^n(\tilde{G}-Z, \mathcal{F}) \approx H^n(\tilde{G}, \mathcal{F})$, there exist $b \in \Gamma(\tilde{G}, \mathcal{S}_{n-1})$ and $u \in \Gamma_{\Phi(\tilde{G})}(\tilde{G}-Z, \mathcal{F}_n)$

$= \Gamma_{\Phi(\tilde{G})}(\tilde{G}, \mathcal{Z}_n)$ such that $t - \phi_{n-1}(b) = u$ on \tilde{G} . Let $v = u|_{G-Z}$. Then $d(\text{Supp } v, Z) \geq d(G^- \cap \text{Supp } u, Z) > 0$. $v \in \Gamma_{\Psi(G)}(G-Z, \mathcal{Z}_n)$. By Proposition 7 there exists $c \in \Gamma_{\Psi(G)}(G-Z, \mathcal{S}_{n-1}) = \Gamma_{\Psi(G)}(G, \mathcal{S}_{n-1})$ such that $\phi_{n-1}(c) = v$ on $G-Z$.

$$s = \phi_{n-1}(a + (b|_G) + c). \quad \text{Q.E.D.}$$

DEFINITION 2. If A is a subset of X and $X-A = (\bigcup_{i \in I} B_i) \cup (\bigcup_{j \in J} C_j)$ is the decomposition into topological components, where B_i^- , $i \in I$, is compact, and C_j^- , $j \in J$, is noncompact, then $A \cup (\bigcup_{i \in I} B_i)$, denoted by $\text{Env}_X(A)$, is called the *envelope* of A .

LEMMA 11. Suppose K and L are compact subsets of X and Y is an open subset of X such that $K \subset Y \subset L$, and $\text{Env}_X(K) = K$. If $\varepsilon > 0$, then there exist $0 < \delta < \varepsilon$ and a compact subset \tilde{K} in Y containing K such that, if s is a global holomorphic section of a holomorphic vector bundle on $X - K - \bar{U}_\delta(Z)$ and s is identically zero on $X - L - \bar{U}_\delta(Z)$, then s is identically zero on $Y - \tilde{K} - \bar{U}_\varepsilon(Z)$.

Proof. Let \tilde{K} be a compact neighborhood of K in Y . Let $X - K = \bigcup_{i \in I} B_i$ be the decomposition into topological components. $B_i \not\subset L$, $i \in I$. $J = \{i \in I \mid B_i \cap \partial L \neq \emptyset\}$ is finite, because ∂L is compact, $\partial L \subset \bigcup_{i \in I} B_i$, and $B_i \cap B_j = \emptyset$ for $i \neq j$. For $i \in I - J$, $B_i \cap L = \emptyset$, because $B_i \cap \partial L = \emptyset$, $B_i \not\subset L$ and B_i is connected. Hence $L - K \subset \bigcup_{i \in J} B_i$. Take $x_i \in B_i - L - Z$, $i \in J$. For $i \in J$ and $\eta > 0$, $A_{i,\eta} = \{x \in B_i - Z \mid x \text{ can be joined to } x_i \text{ by a path } \gamma \text{ in } B_i - Z \text{ such that } d(\gamma, Z) > \eta\}$ is an open connected subset of $B_i - Z$. $B_i - Z = \bigcup_{\eta > 0} A_{i,\eta}$ for $i \in J$. $A_{i,\xi} \subset A_{i,\eta}$ for $i \in J$ and $0 < \eta < \xi$. Since $B_i \cap (L - \tilde{K}^0 - U_\varepsilon(Z))$ is a compact subset of $B_i - Z$ for $i \in J$ and J is finite, there exists $0 < \delta < \varepsilon$ such that $d(x_i, Z) > \delta$ and $B_i \cap (L - \tilde{K}^0 - U_\varepsilon(Z)) \subset A_{i,\delta}$ for $i \in J$. We claim that \tilde{K} and δ satisfy the requirement. Suppose s is a global holomorphic section of a holomorphic vector bundle on $X - K - \bar{U}_\delta(Z)$ and s is identically zero on $X - L - \bar{U}_\delta(Z)$. Since for $i \in J$ $A_{i,\delta}$ is a connected open subset of $X - K - \bar{U}_\delta(Z)$ and $x_i \in A_{i,\delta} \cap (X - L - \bar{U}_\delta(Z)) \neq \emptyset$, s is identically zero on $A_{i,\delta}$. The result follows from

$$Y - \tilde{K} - \bar{U}_\varepsilon(Z) \subset L - K - \bar{U}_\varepsilon(Z) \subset \bigcup_{i \in J} B_i \cap (L - \tilde{K}^0 - U_\varepsilon(Z)) \subset \bigcup_{i \in J} A_{i,\delta}. \quad \text{Q.E.D.}$$

LEMMA 12. Suppose A is a subset of $X - Z$ such that $A - U_{1/k}(Z)$ is compact for $k \in \mathbb{N}$. Then A is a closed subset of $X - Z$.

Proof. Suppose $\{x_q\}_{q \in \mathbb{N}} \subset A$ is a sequence approaching $x \in X - Z$ as a limit. Since $K = \{x\} \cup (\bigcup_{q \in \mathbb{N}} \{x_q\})$ is a compact subset of $X - Z$, $d(K, Z) > 1/k$ for some $k \in \mathbb{N}$. $\{x_q\}_{q \in \mathbb{N}} \subset A - U_{1/k}(Z)$. $x \in A - U_{1/k}(Z) \subset A$, because $A - U_{1/k}(Z)$ is compact. Q.E.D.

PROPOSITION 9. Suppose X_i , $1 \leq i \leq 3$, are open subsets of X such that $X_3 \subset \subset X_2 \subset \subset X_1 \subset \subset X$ and $\text{Env}_X(X_3^-) \subset X_2$. Suppose $\varepsilon > 0$. Then there exists $\delta > 0$ satisfying

the following: if $a \in \Gamma(X_2 - Z, \mathcal{E}^{0, n-1}(B))$ such that $\bar{\partial}a = 0$ and $d(\text{Supp } a, Z) \geq \varepsilon$, then there exists a sequence $\{a_k\}_{k \in \mathbb{N}} \subset \Gamma(X_1 - Z, \mathcal{E}^{0, n-1}(B))$ such that $\bar{\partial}a_k = 0$ and

$$d(\text{Supp } a_k, Z) \geq \delta$$

for $k \in \mathbb{N}$ and $a_k|_{X_3 - Z} \rightarrow a|_{X_3 - Z}$ in $\Gamma(X_3 - Z, \mathcal{E}^{0, n-1}(B))$.

Proof. For $p \in \mathbb{N}$ and $1 \leq i \leq 3$, let E_p^i = the Fréchet space

$$\{b \in \Gamma(X_i - Z, \mathcal{E}^{0, n-1}(B)) \mid d(\text{Supp } b, Z) \geq 1/p\}$$

and F_p^i = the Fréchet space $\{b \in \Gamma(X_i - Z, \mathcal{E}^{0, n}(B)) \mid d(\text{Supp } b, Z) \geq 1/p\}$. Let $\phi_p^i: E_p^i \rightarrow F_p^i$ be induced by $\bar{\partial}$. Then we have $(1)_p$ for $p \in \mathbb{N}$, where $\alpha_p^i, \beta_p^i, i = 1, 2$, are the restriction maps. We obtain (2) as the direct limit of $(1)_p, p \in \mathbb{N}$. By Proposition 7 and Lemma 3, we need only prove that, for every $p \in \mathbb{N}$, there exists $r \geq p$ such that $(3)_{p,r}$ holds.

Fix $p \in \mathbb{N}$. Set $K = \text{Env}_X(X_3^-)$, $L = X_1^-$, $Y = X_2$, and $\varepsilon = 1/4p$. K is compact (cf. Lemma 1, p. 333 of [11]). We can find a compact subset \tilde{K} of Y containing K and $0 < \delta < \varepsilon$ satisfying the requirement of Lemma 11. Choose $1/\delta < r \in \mathbb{N}$. We claim that $(3)_{p,r}$ holds.

Suppose $f \in (E_r^3)^*$ and $g \in (F_r^1)^*$ such that

$$(13) \quad f\alpha_r^2\alpha_r^1 = g\phi_r^1.$$

f and g can be extended respectively to $\tilde{f} \in (E^3)^*$ and $\tilde{g} \in (F^1)^*$. By Proposition 1 \tilde{f} and \tilde{g} are represented respectively by

$$\tilde{f} \in \Gamma(X_3 - Z, \mathcal{D}^{n,1}(B^*)) \quad \text{and} \quad \tilde{g} \in \Gamma(X_1 - Z, \mathcal{D}^{n,0}(B^*))$$

such that $\text{Supp } \tilde{f} - U_\eta(Z)$ and $\text{Supp } \tilde{g} - U_\eta(Z)$ are compact for $\eta > 0$. By Lemma 12, $\text{Supp } \tilde{f}$ and $\text{Supp } \tilde{g}$ are closed subsets of $X - Z$. \tilde{f} and \tilde{g} can respectively be extended trivially to $f' \in \Gamma(X - Z, \mathcal{D}^{n,1}(B^*))$ and $g' \in \Gamma(X - Z, \mathcal{D}^{n,0}(B^*))$ such that $\text{Supp } f' = \text{Supp } \tilde{f}$ and $\text{Supp } g' = \text{Supp } \tilde{g}$. (13) and (6) imply that $\bar{\partial}^*g' = f'$ on $X_1 - Z - \bar{U}_{1/r}(Z)$. Hence $\bar{\partial}^*g' = f'$ on $X - Z - \bar{U}_{1/r}(Z)$. Since $\text{Supp } f' \subset K$ and $\text{Supp } g' \subset L$, by (5) g' is a holomorphic section of the holomorphic vector bundle $B^* \otimes \lambda_{X-Z}^{n,0}$ on $X - K - \bar{U}_\delta(Z)$ and is identically zero on $X - L - \bar{U}_\delta(Z)$. Hence g' is identically zero on $Y - K - \bar{U}_\varepsilon(Z) = X_2 - \text{Env}_X(X_3^-) - \bar{U}_{1/4p}(Z)$. Let ρ be a C^∞ function on $X_2 - Z$ such that $\rho \equiv 0$ on $\bar{U}_{1/3p}(Z)$ and $\rho \equiv 1$ on $X - U_{1/2p}(Z)$. Let

$$g'' = \rho(g'|_{X_2 - Z}) \in \Gamma(X_2 - Z, \mathcal{D}^{n,0}(B^*)).$$

$\text{Supp } g'' \subset \text{Env}_X(X_3^-) - U_{1/3p}(Z)$. Hence $\text{Supp } g''$ is compact. By Proposition 1 g'' defines $\tilde{h} \in (F^2)^*$. Let $h = \tilde{h}|_{F_p^2}$. Since $\bar{\partial}^*g'' = \bar{\partial}^*g' = f'$ on $X_2 - \bar{U}_{1/2p}(Z)$, $f\alpha_p^2 = h\phi_p^2$ on E_p^2 . Q.E.D.

PROPOSITION 10. Assume $n \geq 2$. Suppose $X_i, i = 1, 2$, are open subsets of X such that $X_1 \subset \subset X_2 \subset \subset X$. Let \mathcal{U} be a Stein covering of X_2 . Then for any $f \in Z^{n-1}(\mathcal{U}, \mathcal{F})$, there exists $g \in C^{n-2}(\mathcal{U}, \mathcal{F})$ such that, for some $\eta > 0$ the restriction of $f + \delta g$ to $\mathcal{U}|_{X_1 \cap U_\eta(Z)}$ is zero.

Proof. Since $H_{\Phi(X_2)}^{n-1}(X_2 - Z, \mathcal{F}) \rightarrow H^{n-1}(X_2, \mathcal{F})$ is surjective, [13, p. 278, Satz 3], there exists $s \in \Gamma_{\Phi(X_2)}(X_2, \mathcal{F}_{n-1})$ defining the same cohomology class as f . Since $X_1 \subset \subset X_2$, $d(X_1 \cap \text{Supp } s, Z) = \eta > 0$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \Gamma(X_2, \mathcal{S}_0) & \xrightarrow{\phi_0} & \Gamma(X_2, \mathcal{S}_1) & \xrightarrow{\phi_1} & \dots \\
 & & \downarrow \mu & & \downarrow \mu & & \\
 0 & \longrightarrow & C^0(\mathfrak{U}, \mathcal{F}) & \xrightarrow{\lambda} & C^0(\mathfrak{U}, \mathcal{S}_0) & \xrightarrow{\phi_0} & C^0(\mathfrak{U}, \mathcal{S}_1) \xrightarrow{\phi_1} \dots \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
 0 & \longrightarrow & C^1(\mathfrak{U}, \mathcal{F}) & \xrightarrow{\lambda} & C^1(\mathfrak{U}, \mathcal{S}_0) & \xrightarrow{\phi_0} & C^1(\mathfrak{U}, \mathcal{S}_1) \xrightarrow{\phi_1} \dots \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

All rows except the first and all columns except the first are exact. Since $\text{Supp } s \cap X_1 \cap U_\eta(Z) = \emptyset$, we can choose by diagram-chasing $s_j^i = C^i(\mathfrak{U}, \mathcal{S}_j)$ for $0 \leq i, j \leq n-1$, $i+j=n-1$, $n-2$, and $h \in Z^{n-1}(\mathfrak{U}, \mathcal{F})$ such that (i) $s_{n-1}^0 = \mu s$, $s_{j+1}^i = \phi_j s_j^i$ and $s_j^{i+1} = \delta s_j^i$ for $0 \leq i \leq n-2$ and $i+j=n-2$, and $s_0^{n-1} = \lambda h$, (ii) the restrictions of s_j^i ($0 \leq i, j \leq n-1$, $i+j=n-1, n-2$) and h to $\mathfrak{U}|_{X_1 \cap U_\eta(Z)}$ are zero. By diagram-chasing, we can find $g \in C^{n-1}(\mathfrak{U}, \mathcal{F})$ such that $h - f = \delta g$. Q.E.D.

PROPOSITION 11. Suppose $X_1 \subset X_2$ are open subsets of X . Suppose \mathfrak{U}_2 is a Stein covering of X_2 such that $\mathfrak{U}_1 = \mathfrak{U}_2|_{X_1}$ covers X_1 . Suppose $p \in N$ and $a \in Z^p(\mathfrak{U}_1, \mathcal{F})$. Let \mathfrak{B}_2 be a Stein refinement of \mathfrak{U}_2 such that $\mathfrak{B}_1 = \mathfrak{B}_2|_{X_1}$ covers X_1 and refines \mathfrak{U}_1 . In the following commutative diagram $C^*(\cdot, \mathcal{F}) = \bigoplus_{q \in N} C^{q-1}(\cdot, \mathcal{F})$ and ρ_1, ρ_2, α , and β denote the restriction maps:

$$\begin{array}{ccc}
 C^*(\mathfrak{U}_2, \mathcal{F}) & \xrightarrow{\rho_2} & C^*(\mathfrak{B}_2, \mathcal{F}) \\
 \downarrow \alpha & & \downarrow \beta \\
 C^*(\mathfrak{U}_1, \mathcal{F}) & \xrightarrow{\rho_1} & C^*(\mathfrak{B}_1, \mathcal{F}).
 \end{array}$$

If $\beta(b_q) \rightarrow \rho_1(a)$ in $Z^p(\mathfrak{B}_1, \mathcal{F})$ for some $\{b_q\}_{q \in N} \subset Z^p(\mathfrak{B}_2, \mathcal{F})$, then $\alpha(a_q) \rightarrow a$ in $Z^p(\mathfrak{U}_1, \mathcal{F})$ for some $\{a_q\}_{q \in N} \subset Z^p(\mathfrak{U}_2, \mathcal{F})$.

Proof. Since $\phi: C^{p-1}(\mathfrak{B}_1, \mathcal{F}) \oplus Z^p(\mathfrak{U}_1, \mathcal{F}) \rightarrow Z^p(\mathfrak{B}_1, \mathcal{F})$ defined by $\phi(u \oplus v) = \delta u + \rho_1(v)$ is surjective, by Lemma 4 there exist $c_q \in C^{p-1}(\mathfrak{B}_1, \mathcal{F})$ and

$$d_q \in Z^p(\mathfrak{U}_1, \mathcal{F}), \quad q \in N,$$

such that

$$(14) \quad \delta c_q + \rho_1 d_q = \beta b_q,$$

and $c_q \rightarrow 0$ and $d_q \rightarrow a$. Since $H^p(\mathfrak{U}_2, \mathcal{F}) \approx H^p(\mathfrak{B}_2, \mathcal{F})$, there exist $e_q \in C^{p-1}(\mathfrak{B}_2, \mathcal{F})$ and $f_q \in Z^p(\mathfrak{U}_2, \mathcal{F})$, $q \in N$, such that

$$(15) \quad \delta e_q + \rho_2 f_q = b_q.$$

From (14) and (15) we have $\rho_1(d_q - \alpha f_q) = \delta(\beta e_q - c_q)$ $q \in N$. Since $H^p(\mathfrak{U}_1, \mathcal{F}) \approx H^p(\mathfrak{B}_1, \mathcal{F})$, for some $g_q \in C^{p-1}(\mathfrak{U}_1, \mathcal{F})$, $d_q - \alpha f_q = \delta g_q$, $q \in N$. g_q can be extended trivially to $\tilde{g}_q \in C^{p-1}(\mathfrak{U}_2, \mathcal{F})$ such that $\alpha \tilde{g}_q = g_q$, $q \in N$. $d_q = \alpha(f_q + \delta \tilde{g}_q)$, $q \in N$. Set $a_q = f_q + \delta \tilde{g}_q$, $q \in N$. Then $a_q \in Z^p(\mathfrak{U}_2, \mathcal{F})$, $q \in N$ and $\alpha(a_q) \rightarrow a$. Q.E.D.

PROPOSITION 12. Assume $n \geq 2$. Suppose X_1 , X'_2 , and X_3 are open subsets of X such that $X_1 \subset \subset X'_2 \subset X_3 \subset \subset X$ and $\text{Env}_X(X_1^-) \subset X'_2$. Suppose \mathfrak{U} is a Stein covering of X_3 whose restrictions to X'_2 and X_1 cover X'_2 and X_1 respectively. Let

$$\alpha: Z^{n-1}(\mathfrak{U}, \mathcal{F}) \rightarrow Z^{n-1}(\mathfrak{U}|X_1, \mathcal{F})$$

and

$$\beta: Z^{n-1}(\mathfrak{U}|X'_2, \mathcal{F}) \rightarrow Z^{n-1}(\mathfrak{U}|X_1, \mathcal{F})$$

be the restriction maps. Then $\text{Im } \beta \subset (\text{Im } \alpha)^-$.

Proof. Let X_2 be a relatively compact open neighborhood of $\text{Env}_X(X_1^-)$ in X'_2 . Let $R_k = U_{2-k+1}(Z) - \bar{U}_{2-k-1}(Z)$, $k \in N$ and $R_0 = X - \bar{U}_{1/2}(Z)$. By Proposition 11 we can suppose without loss of generality that $\mathfrak{U}|X_2$ covers X_2 , $\mathfrak{U}|X_i - Z$ covers $X_i - Z$, $1 \leq i \leq 3$,

$$(16) \quad \mathfrak{U}|X_2 - Z = \bigcup_{k \geq 0} \mathfrak{U}|X_2 \cap R_k,$$

and

$$(17) \quad \mathfrak{U}|X_i \cap U_{1/k}(Z) \text{ covers } X_i \cap Z \text{ for } k \in N \text{ and } i = 1, 3.$$

Fix $f \in Z^{n-1}(\mathfrak{U}|X'_2, \mathcal{F})$. We have to prove that there exists $\{f_q\}_{q \in N} \subset Z^{n-1}(\mathfrak{U}, \mathcal{F})$ such that $\alpha(f_q) \rightarrow \beta(f)$. By Proposition 10 there exists $g \in C^{n-2}(\mathfrak{U}|X'_2, \mathcal{F})$ such that for some $k \in N$ $f + \delta g$ is zero when restricted to $\mathfrak{U}|U_{2-k}(Z) \cap X_2$. Suppose we have found $\{f_q\}_{q \in N} \subset Z^{n-1}(\mathfrak{U}, \mathcal{F})$ such that $\alpha(f_q) \rightarrow \beta(f + \delta g)$. Let $\tilde{g} \in C^{n-2}(\mathfrak{U}, \mathcal{F})$ be the trivial extension of g . Then $\alpha(f_q - \delta \tilde{g}) \rightarrow \beta(f)$. Hence we can assume without loss of generality that the restriction of f to $\mathfrak{U}|U_{2-k}(Z) \cap X_2$ is zero. Let $\mathfrak{U}_i = \mathfrak{U}|X_i - Z$, $1 \leq i \leq 3$. Let $h = f|_{\mathfrak{U}_2}$.

Suppose $\mathfrak{U}_2 = \{U_\nu\}_{\nu \in I}$. Let $\{\rho_\nu\}_{\nu \in I}$ be a partition of unity subordinate to \mathfrak{U}_2 . For $p \in N$ and $0 \leq q \leq n$ define

$$\phi_{p,q}: C^p(\mathfrak{U}_2, \mathcal{E}^{0,q}(B)) \rightarrow C^{p-1}(\mathfrak{U}_2, \mathcal{E}^{0,q}(B))$$

as follows: if $s = \{s_{\nu_0 \dots \nu_p}\} \in C^p(\mathfrak{U}_2, \mathcal{E}^{0,q}(B))$, then set $(\phi_{p,q}(s))_{\nu_0 \dots \nu_{p-1}} = \sum_{\nu \in I} t_\nu$, where t_ν is the trivial extension of $\rho_\nu s_{\nu \nu_0 \dots \nu_{p-1}}$ to $U_{\nu_0} \cap \dots \cap U_{\nu_{p-1}}$.

$$(18) \quad \delta \phi_{p,q} \text{ is the identity map on } Z^p(\mathfrak{U}_2, \mathcal{E}^{0,q}(B)).$$

From (16) we conclude that

(19) if $u \in C^p(\mathfrak{U}_2, \mathcal{E}^{0,q}(B))$ and the restriction of u to $\mathfrak{U}_2|_{U_2^{-r}(Z)} \cap X_2 - Z$ is zero for some $r \in N$, then the restriction of $\phi_{p,q}(u)$ to $\mathfrak{U}_2|_{U_2^{-r-1}(Z)} \cap X_2 - Z$ is zero.

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \Gamma(X_2 - Z, \mathcal{E}^{0,0}(B)) & \xrightarrow{\bar{\partial}} & \Gamma(X_2 - Z, \mathcal{E}^{0,1}(B)) & \xrightarrow{\bar{\partial}} & \dots \\
 & & \downarrow \mu & & \downarrow \mu & & \\
 0 & \longrightarrow & C^0(\mathfrak{U}_2, \mathcal{O}(B)) & \xrightarrow{\lambda} & C^0(\mathfrak{U}_2, \mathcal{E}^{0,0}(B)) & \xrightarrow{\bar{\partial}} & C^0(\mathfrak{U}_2, \mathcal{E}^{0,1}(B)) \xrightarrow{\bar{\partial}} \dots \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
 0 & \longrightarrow & C^1(\mathfrak{U}_2, \mathcal{O}(B)) & \xrightarrow{\lambda} & C^1(\mathfrak{U}_2, \mathcal{E}^{0,0}(B)) & \xrightarrow{\bar{\partial}} & C^1(\mathfrak{U}_2, \mathcal{E}^{0,1}(B)) \xrightarrow{\bar{\partial}} \dots \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

By induction on j we can define $h_j^i \in C^i(\mathfrak{U}_2, \mathcal{E}^{0,j}(B))$, $0 \leq i, j \leq n-1$, $i+j=n-1$, $n-2$ such that $h_0^{n-1} = \lambda h$, $h_j^i = \phi_{i+1,j} h_j^{i+1}$ and $h_{j+1}^i = \bar{\partial} h_j^i$ for $0 \leq i \leq n-2$ and $i+j=n-2$. It is easily verified by induction on j that $h_j^i \in Z^i(\mathfrak{U}_2, \mathcal{E}^{0,j}(B))$, $0 \leq i, j \leq n-1$, $i+j=n-1$. Hence we can find a unique $s \in \Gamma(X_2 - Z, \mathcal{E}^{0,n-1}(B))$ such that $\mu(s) = h_{n-1}^0$. $\bar{\partial}s = 0$. By (18) we have

$$\begin{aligned}
 h_0^{n-1} &= \lambda h, \\
 h_j^i &= \delta h_j^i \\
 h_{j+1}^i &= \bar{\partial} h_j^i \\
 h_{n-1}^0 &= \mu(s)
 \end{aligned} \quad (0 \leq i \leq n-2, i+j = n-2).$$

By (19) we have:

$$\begin{aligned}
 (21) \quad & \text{the restrictions of } h_j^i \text{ (} 0 \leq i, j \leq n-1, i+j = n-1, n-2 \text{) to} \\
 & \mathfrak{U}_2|_{U_2^{-k-n}(Z)} \cap X_2 - Z \text{ are zero and } d(\text{Supp } s, Z) \geq 2^{-k-n}.
 \end{aligned}$$

By Proposition 9 there exist $l \geq k+n$ and $\{s(m)\}_{m \in N} \subset \Gamma(X_3 - Z, \mathcal{E}^{0,n-1}(B))$ such that $\bar{\partial}s(m) = 0$, $d(\text{Supp } s(m), Z) \geq 2^{-l}$, $m \in N$, and $s(m)|_{X_1 - Z} \rightarrow s|_{X_1 - Z}$ in $\Gamma(X_1 - Z, \mathcal{E}^{0,n-1}(B))$.

Let $C_0^p(\mathfrak{U}_i, \mathcal{E}^{0,q}(B)) = \{a \in C^p(\mathfrak{U}_i, \mathcal{E}^{0,q}(B)) \mid \text{the restriction of } a \text{ to } \mathfrak{U}_i|_{U_2^{-i}(Z)} \cap X_i - Z \text{ is zero}\}$, $i=1, 3$. $C_0^p(\mathfrak{U}_i, \mathcal{E}^{0,q}(B))$ is a Fréchet space. Since the exact sequence

$$C_0^p(\mathfrak{U}_3, \mathcal{E}^{0,q-1}(B)) \xrightarrow{\bar{\partial}} C_0^p(\mathfrak{U}_3, \mathcal{E}^{0,q}(B)) \xrightarrow{\bar{\partial}} C_0^p(\mathfrak{U}_3, \mathcal{E}^{0,q+1}(B))$$

is the direct sum of

$$C_0^p(\mathfrak{U}_1, \mathcal{E}^{0,q-1}(B)) \xrightarrow{\bar{\partial}} C_0^p(\mathfrak{U}_1, \mathcal{E}^{0,q}(B)) \xrightarrow{\bar{\partial}} C_0^p(\mathfrak{U}_1, \mathcal{E}^{0,q+1}(B))$$

and another exact sequence, by Lemma 4 we have the following:

$$(22) \quad \begin{aligned} & \text{if } \{a_m\}_{m \in N} \subset C_0^p(\mathfrak{U}_3, \mathcal{E}^{0,q}(B)), \quad a \in C_0^p(\mathfrak{U}_1, \mathcal{E}^{0,q}(B)), \text{ and} \\ & b \in C_0^p(\mathfrak{U}_1, \mathcal{E}^{0,q-1}(B)) \text{ satisfy } \bar{\partial}b = a, \bar{\partial}a_m = 0, m \in N, \\ & \text{and } a_m|_{\mathfrak{U}_1} \rightarrow a, \text{ then there exists } \{b_m\}_{m \in N} \subset C_0^p(\mathfrak{U}_3, \mathcal{E}^{0,q-1}(B)) \\ & \text{such that } \bar{\partial}b_m = a_m, m \in N, \text{ and } b_m|_{\mathfrak{U}_1} \rightarrow b. \end{aligned}$$

Let $A = X_3 - U_{2^{-1}}(Z)$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} & & E_A^{0,0}(B|_{X_3-Z}) & \xrightarrow{\bar{\partial}} & E_A^{0,1}(B|_{X_3-Z}) & \xrightarrow{\bar{\partial}} & E_A^{0,2}(B|_{X_3-Z}) \xrightarrow{\bar{\partial}} \dots \\ & & \downarrow \mu & & \downarrow \mu & & \downarrow \mu \\ 0 \longrightarrow & C_0^0(\mathfrak{U}_3, \mathcal{O}(B)) & \xrightarrow{\lambda} & C_0^0(\mathfrak{U}_3, \mathcal{E}^{0,0}(B)) & \xrightarrow{\bar{\partial}} & C_0^0(\mathfrak{U}_3, \mathcal{E}^{0,1}(B)) & \xrightarrow{\bar{\partial}} C_0^0(\mathfrak{U}_3, \mathcal{E}^{0,2}(B)) \xrightarrow{\bar{\partial}} \dots \\ & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ 0 \longrightarrow & C_0^1(\mathfrak{U}_3, \mathcal{O}(B)) & \xrightarrow{\lambda} & C_0^1(\mathfrak{U}_3, \mathcal{E}^{0,0}(B)) & \xrightarrow{\bar{\partial}} & C_0^1(\mathfrak{U}_3, \mathcal{E}^{0,1}(B)) & \xrightarrow{\bar{\partial}} C_0^1(\mathfrak{U}_3, \mathcal{E}^{0,2}(B)) \xrightarrow{\bar{\partial}} \dots \\ & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ 0 \longrightarrow & C_0^2(\mathfrak{U}_3, \mathcal{O}(B)) & \xrightarrow{\lambda} & C_0^2(\mathfrak{U}_3, \mathcal{E}^{0,0}(B)) & \xrightarrow{\bar{\partial}} & C_0^2(\mathfrak{U}_3, \mathcal{E}^{0,1}(B)) & \xrightarrow{\bar{\partial}} C_0^2(\mathfrak{U}_3, \mathcal{E}^{0,2}(B)) \xrightarrow{\bar{\partial}} \dots \\ & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

The composites of any two consecutive horizontal maps or any two consecutive vertical maps are zero. All the rows except the first are exact. By (20), (21), and (22) we can find $\{s(m)_{jj}^i\}_{m \in N} \subset C_0^i(\mathfrak{U}_3, \mathcal{E}^{0,j}(B))$, $0 \leq i, j \leq n-1$, $i+j=n-1$, $n-2$, and $\{g_m\}_{m \in N} \subset C_0^{n-1}(\mathfrak{U}_3, \mathcal{O}(B))$ such that

$$(i) \quad \begin{aligned} s(m)_{n-1}^0 &= \mu s(m) \\ s(m)_{j+1}^i &= \bar{\partial} s(m)_j^i \\ s(m)_{j+1}^{i+1} &= \delta s(m)_j^i \quad (0 \leq i \leq n-2, i+j=n-2) \quad m \in N, \\ s(m)_0^{n-1} &= \lambda g_m, \end{aligned}$$

and (ii) $s(m)_i^j|_{\mathfrak{U}_1} \rightarrow h_j^i|_{\mathfrak{U}_1}$ for $0 \leq i, j \leq n-1$, $i+j=n-1$, $n-2$. It follows that $\delta g_m = 0$, $m \in N$ and $g_m|_{\mathfrak{U}_1} \rightarrow h|_{\mathfrak{U}_1}$ in $C_0^{n-1}(\mathfrak{U}_1, \mathcal{O}(B))$. Let

$$\mathfrak{B}_i = \mathfrak{U}_i \cup (\mathfrak{U}|_{U_{2^{-1-1}}(Z)} \cap X_i), \quad i = 1, 3.$$

By (17) \mathfrak{B}_i is a Stein refinement for $\mathfrak{U}|_{X_i}$, $i=1, 3$. Define $\{f_m\}_{m \in N} \subset Z^{n-1}(\mathfrak{B}_3, \mathcal{F})$ by setting $f_m = g_m$ on \mathfrak{U}_3 and $f_m = 0$ on $\mathfrak{U}|_{U_{2^{-1-1}}(Z)} \cap X_3$. Then $f_m|_{\mathfrak{B}_1} \rightarrow f|_{\mathfrak{B}_1}$. The result follows from Proposition 11. Q.E.D.

V. Proof of the Main Theorem. By Proposition 4 we need only prove (A) $_n^\#$ for $n \in N$. The case $n=1$ is well known [5, p. 270, IX.B.10]. Suppose \mathcal{F} is a torsion-free coherent analytic sheaf on a connected, reduced, normal, noncompact complex space X with $2 \leq \dim X = n$. We have to show that $H^n(X, \mathcal{F}) = 0$.

Take a sequence of relatively compact open subsets $\{X_m\}_{m \in N}$ of X such that $\text{Env}_X(X_m^-) \subset X_{m+1}$ for $m \in N$ and $\bigcup_{m \in N} X_m = X$. Take a Stein covering \mathfrak{U} of X such that $\mathfrak{U}_m = \mathfrak{U}|_{X_m}$ covers X_m for $m \in N$ and $\mathfrak{U} = \bigcup_{m \in N} \mathfrak{U}_m$.

Take $f \in Z^n(\mathfrak{U}, \mathcal{F})$. By Proposition 8 there exists $g_m \in C^{n-1}(\mathfrak{U}_m, \mathcal{F})$ such that $f|_{\mathfrak{U}_m} = \delta g_m$, $m \in N$.

Take a sequence of seminorms $\{\|\cdot\|_p^{(m)}\}_{p \in N}$ on $C^{n-1}(\mathfrak{U}_m, \mathcal{F})$, $m \in N$, such that (i) $\{\|\cdot\|_p^{(m)}\}_{p \in N}$ defines the Fréchet space topology of $C^{n-1}(\mathfrak{U}_m, \mathcal{F})$, $m \in N$, (ii) $\|\cdot\|_p^{(m)} \leq \|\cdot\|_{p+1}^{(m)}$ and $\|\cdot\|_p^{(m)} \leq \|\cdot\|_p^{(m+1)}$, $p, m \in N$.

We are going to construct by induction on m $h_m \in C^{n-1}(\mathfrak{U}_m, \mathcal{F})$ such that $\delta h_m = f|_{\mathfrak{U}_m}$ and $\|(h_{m+2} - h_{m+1})|_{\mathfrak{U}_m}\|_m^{(m)} < 2^{-m}$, $m \in N$. Set $h_m = g_m$ for $m = 1, 2$. Suppose h_1, \dots, h_m are found for some $m \geq 2$. Since $\delta(h_m - g_{m+1}) = 0$ on \mathfrak{U}_m , by Proposition 12 there exists $s \in Z^{n-1}(\mathfrak{U}_{m+1}, \mathcal{F})$ such that

$$\|(h_m - g_{m+1} - s)|_{\mathfrak{U}_{m-1}}\|_{m-1}^{(m-1)} < 2^{-m+1}.$$

Set $h_{m+1} = g_{m+1} + s$. The construction is complete. Define $h \in C^{n-1}(\mathfrak{U}, \mathcal{F})$ by setting $h|_{\mathfrak{U}_m} = \lim_{v \geq m} h_v|_{\mathfrak{U}_m}$. h is well defined and $\delta h = f$. Q.E.D.

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