A LOWER BOUND FOR THE Δ-NIELSEN NUMBER

BY

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Introduction. This paper is concerned with the number of solutions of three kinds of equations. Let $f, g: X \to Y$ and $h: X \to X$ be maps, and let $y_0 \in Y$. The equations we will study are (1) $f(x)=g(x)$, (2) $f(x)=y_0$, and (3) $h(x)=x$. In [2], the first author defined a lower bound N for the number of solutions of these equations which remains such a lower bound when f, g, and h are moved through homotopies. The number N is called the Δ-Nielsen number of the equation.

It is not, in general, possible to compute the Δ-Nielsen number for particular spaces and maps directly from its definition. An integer R which can be computed algebraically just from a knowledge of the maps induced on the fundamental groups was defined in [2], and it was proved that $N \leqq R$. In some cases it could be shown that $N=R$.

We will define a positive integer J, which is also easier to compute than N, and which has the property that $J \leqq N$. Bounding N between integers we know something about improves our chances of determining N. Furthermore, J is of interest in itself because it is also a lower bound (though a poorer one) for the number of solutions. We will further prove that, under mild additional hypotheses, J divides N, R, and an appropriately defined "Lefschetz number" for the equation we are considering.

In the case of the study of fixed points, that is, a solution to $h(x)=x$, there has been a lower bound of this kind, $N(h)$, the Nielsen number of h, for many years [6], [8]. Unfortunately, $N(h)$ is not, in general, the same as the Δ-Nielsen number N, in fact $N(h) \leqq N$. However, we are still able to prove that $J \leqq N(h) \leqq R$ and that, under an additional hypothesis, J divides both $N(h)$ and the classical Lefschetz number $L(h)$.

In §II, we will introduce our basic assumptions and will summarize those definitions and results that we will be using from [2]. The definition of J and the proofs of the results indicated above occupy §III. §IV is devoted to applications of these general theorems to the three kinds of equations.

II. **Outline of coincidence theory.** Throughout this paper $f, g: X \to Y$ will be maps of a compact, connected, and locally path connected Hausdorff space X into

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a path connected semilocally simply connected Hausdorff space Y. Under these hypotheses X is locally compact and regular so we may regard homotopies of maps from X into Y as paths in Map (X, Y) (with the compact-open topology), and every such path may be regarded as a homotopy [4, pp. 159–160]. We will assume that Δ is a class of ordered pairs of paths in Map (X, Y) that is closed under pairwise partitioning and multiplication, i.e., if $(F, G) \in \Delta$ then $(F_r^s, G_r^s) \in \Delta$ for every $r, s \in [0, 1]$—where $F_r^s(t) = F(r + t(s-r))$ for each $t \in [0, 1]$, and $(F, G), (F', G') \in \Delta$ implies $(FF', GG') \in \Delta$ whenever $(F(1), G(1)) = (F'(0), G'(0))$. The constant path at a point x will be denoted by x. We assume that the constant pair $(f, g) \in \Delta$. If C is a path, then $[C]$ denotes the set of all paths that are fixed-endpoint homotopic to C [4, p. 175]. For paths F in Map (X, Y) and C in X, let $\langle F, C\rangle$ be the path in Y defined by $\langle F, C\rangle(t) = F(t)(C(t))$—so that, in particular, $f \circ C = \langle f, C\rangle$. The operation $\langle \cdot, \cdot \rangle$ behaves well with respect to multiplication, inversion, and fixed-endpoint homotopy: $\langle F, C\rangle\langle F', C'\rangle = \langle FF', CC'\rangle$ when $(F(1), C(1)) = (F'(0), C'(0))$, $\langle F, C\rangle^{-1} = \langle F^{-1}, C^{-1}\rangle$, and $[\langle F, C\rangle] = [\langle F', C'\rangle]$ when $[F] = [F']$ and $[C] = [C']$. A point $x \in X$ is a *coincidence* of f and g iff $f(x) = g(x)$; the set of all such points is denoted by $\Gamma(f, g)$. If F and G are paths in Map (X, Y) then $x \in X$ is *F, G-related* to $x' \in X$ (write x—$F, G \rightarrow x'$) iff there is a path C in X from x to x' with $[\langle F, C\rangle] = [\langle G, C\rangle]$. If $x, x' \in \Gamma(f, g)$ they are *f, g-equivalent* iff they are f, g-related, i.e., iff there is a path C in X from x to x' with $[f \circ C] = [g \circ C]$. The f, g-equivalence is an equivalence relation on $\Gamma(f, g)$; the set of these equivalence classes is denoted by $\tilde{\Gamma}(f, g)$. An F, G-relation induces a one-to-one relation from $\tilde{\Gamma}(F(0), G(0))$ into $\tilde{\Gamma}(F(1), G(1))$, $a \in \tilde{\Gamma}(F(0), G(0))$ is F, G-related to $a' \in \tilde{\Gamma}(F(1), G(1))$ iff x—$F, G \rightarrow x'$ for at least one (and hence every) $x \in a$ and $x' \in a'$. In this case we also write a—$F, G \rightarrow a'$. A class $a \in \tilde{\Gamma}(f, g)$ is *Δ-essential* iff whenever $(F, G) \in \Delta$ and $(F(0), G(0)) = (f, g)$, there is an $a' \in \tilde{\Gamma}(F(1), G(1))$ such that a—$F, G \rightarrow a'$. An F, G-relation, $(F, G) \in \Delta$, then defines a one-to-one function from the Δ-essential elements of $\tilde{\Gamma}(F(0), G(0))$ onto those of $\tilde{\Gamma}(F(1), G(1))$. The number of Δ-essential elements of $\tilde{\Gamma}(f, g)$ is the *Δ-Nielsen number* of f and g and is denoted by $N(f, g, \Delta)$. $N(f, g, \Delta)$ is finite and $N(F(0), G(0), \Delta) = N(F(1), G(1), \Delta)$ for every $(F, G) \in \Delta$.

Throughout this paper we assume that $N(f, g, \Delta) > 0$, choose a Δ-essential $a_0 \in \tilde{\Gamma}(f, g)$, a coincidence $x_0 \in a_0$, and base the fundamental groups of X and Y at x_0 and $y_0 = f(x_0) = g(x_0)$ respectively. These groups are denoted by $\pi(X)$ and $\pi(Y)$; $f_\#, g_\#: \pi(X) \rightarrow \pi(Y)$ are the homomorphisms induced by f and g. Two elements $\alpha, \beta \in \pi(Y)$ are said to be *f, g-congruent* (write $\alpha \sim_R \beta$ if there is a $\gamma \in \pi(X)$ such that $f_\#(\gamma)\alpha = \beta g_\#(\gamma)$. The relation f, g-congruence is an equivalence relation on $\pi(Y)$; the f, g-congruence class containing $\alpha \in \pi(Y)$ is denoted by α_R; the set of such classes is denoted by $\pi(Y)/R$; the number of such classes is denoted by $R(f, g)$ and is called the *Reidemeister number* of f and g. If $\pi(Y)$ is abelian, then $\pi(Y)/R$ is simply the cokernel of $f_\# - g_\#(\pi(Y)/\text{image}(f_\# - g_\#))$, where $(f_\# - g_\#)(\gamma) = f_\#(\gamma) - g_\#(\gamma)$. There is a one-to-one function $\phi: \tilde{\Gamma}(f, g) \rightarrow \pi(Y)/R$ defined as follows: for $a \in \tilde{\Gamma}(f, g)$ let

C be a path in X from x_0 to an $x \in a$. The path $(f \circ C)(g \circ C)^{-1}$ represents an element $[(f \circ C)(g \circ C)^{-1}] \in \pi(Y)$ whose f, g-congruence class $[(f \circ C)(g \circ C)^{-1}]_R$ is independent of the choice of $x \in a$ and the path C from x_0 to x. Let $\phi(a) = [(f \circ C)(g \circ C)^{-1}]_R \in \pi(Y)/R$. Note that, since ϕ is one-to-one, then $N(f, g, \Delta) \leqq R(f, g)$.

We will let Δ' denote the set of all pairs $(F, G) \in \Delta$ such that F is a loop at f and G is a loop at g. If $(F, G) \in \Delta'$, then $[\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1} \in \pi(Y)$. The set of all such elements is denoted by $T(f, g, \Delta)$. The set of all $[\langle F, x_0\rangle] \in \pi(Y)$ such that F is a loop at f in Map (X, Y) is denoted by $T(f)$; the set of all $[\langle H, y_0\rangle] \in \pi(Y)$ such that H is a loop in Map (Y, Y) at the identity $1: Y \to Y$, is denoted by $T(Y)$. We always have $T(Y) \subset T(f)$.

A triple (f', g', A) is Δ-*admissible* if $f', g': X \to Y$ are maps, $(f', g') \in \Delta$, $A \subset X$, and there is a closed set $N \subset X$ with $\mathrm{Cl}\, A \subset \mathrm{Int}\, N$ and $\Gamma(f', g') \cap (N - A)$ empty. A function ω from the Δ-admissible triples into an abelian group A is a Δ-*index* if it satisfies the following two conditions:

1. (Additivity). If $A \subset X$, and $\{A_i\}$ is a finite indexed collection of subsets of A such that

(i) (f', g', A) is Δ-admissible, and (f', g', A_i) is Δ-admissible for each i, and

(ii) $(A - \bigcup_i A_i) \cap \Gamma(f', g')$ is empty,

then

$$\omega(f', g', A) = \sum_i \omega(f', g', A_i).$$

2. (Homotopy). If $(F, G) \in \Delta$, $A \subset X$ is open, and $(F(t), G(t), A)$ is Δ-admissible for each $t \in [0, 1]$, then

$$\omega(F(0), G(0), A) = \omega(F(1), G(1), A).$$

Throughout the rest of the paper we will assume that ω is a Δ-index with values in an abelian group A. If $a \in \tilde{\Gamma}(f, g)$, then (f, g, a) is Δ-admissible. Moreover, $\omega(F(0), G(0), a) = \omega(F(1), G(1), a')$ whenever $(F, G) \in \Delta$ and a—$F, G \to a'$. If $a \in \tilde{\Gamma}(f, g)$ and $\omega(f, g, a) \neq 0 \in A$, then a is Δ-essential.

Examples of Δ are the following: $\Delta_1(X, Y)$—the class of all pairs (F, G) of paths F and G in Map (X, Y); $\Delta_2(X, Y, y_0)$—the class of all pairs $(F, G) \in \Delta_1(X, Y)$ such that $G(t)(x) = y_0$ for all $t \in [0, 1]$ and all $x \in X$; and $\Delta_3(X)$—the class of all pairs $(F, G) \in \Delta_1(X, X)$ such that $G(t)(x) = x$ for all $t \in [0, 1]$ and all $x \in X$. We will denote these classes by Δ_1, Δ_2, and Δ_3 respectively. Δ_1 is appropriate for studying coincidences of f and g when both f and g are arbitrary; Δ_2 may be used for studying roots of the equation $f(x) = y_0$ for an arbitrary map $f: X \to Y$; Δ_3 is for studying fixed points of an arbitrary map $f: X \to X$. In these three cases, we have $T(f, g, \Delta_1) = T(f)T(g)$, $T(f, g, \Delta_2) = T(f)$, and $T(f, g, \Delta_3) = T(f)$. There is a Δ_1 index ω_1 with values in Hom $(H^*(Y \times Y, Y \times Y - D), H^*(X))$ where $H^*(Z)$ is the total cohomology (arbitrary coefficients) ring of Z, and D is the diagonal in $Y \times Y$. For this index, $\omega_1(f, g, X)$ is the cohomology homomorphism induced by the map

$h\colon X \to (Y\times Y, Y\times Y-D)$ defined by $h(x)=(f(x), g(x))$. When Y is an orientable closed n-manifold there is a Δ_1 index λ with values in $H^n(X)$. If rational coefficients are used, then $\lambda(f, g, X)$ is the Lefschetz coincidence cocycle of f and g. If X is also a compact, orientable, closed n-manifold, then there is a Δ_1-index λ' with values in the integers. In this case $\lambda'(f, g, X)$ is the Lefschetz coincidence number of f and g. For the class Δ_2, we have a Δ_2-index ω_2 with values in $\operatorname{Hom}(H^*(Y, Y-y_0), H^*(X))$. For this index, $\omega_2(f, g, X)$ is the homomorphism induced by the composition $X \xrightarrow{f} Y \subset (Y, Y-y_0)$. Any Δ_1-index is automatically a Δ_2 index. When X and Y are both closed, orientable n-manifolds then $\lambda'(f, g, X)$ is simply the degree of f. Finally, when X is an ANR, there is a Δ_3 index ω_3 with values in the integers such that $\omega_3(f, g, X)=L(f)$, the Lefschetz number of f.

III. **The lower bound.** Let $T(f, g, \Delta)/R$ be the set of all classes $\tau_R \in \pi(Y)/R$ with a representative in $T(f, g, \Delta)$. We define the *Jiang number* of *f and g* to be the cardinality of $T(f, g, \Delta)/R$, and denote it by $J(f, g, \Delta)$. Our main result is the following

THEOREM 1. $J(f, g, \Delta) \leqq N(f, g, \Delta)$.

The proof is a consequence of Propositions 1–6 and Theorem 2 below.

Under somewhat stronger hypotheses than those introduced thus far, we will also establish the divisibility of $R(f, g)$, $N(f, g, \Delta)$, and $\omega(f, g, X)$ by $J(f, g, \Delta)$ (Theorem 4 below).

If $a, b \in \tilde{\Gamma}(f, g)$ and $a—F, G \to b$ for some (pairs of loops) $(F, G) \in \Delta'$, then we say that a and b are *Jiang equivalent* and write $a \sim_J b$.

PROPOSITION 1. *$\sim_J$ is an equivalence relation on $\tilde{\Gamma}(f, g)$.*

Proof. Let $a, b, c \in \tilde{\Gamma}(f, g)$ and $(F, G), (F', G') \in \Delta'$. Then $f, g \in \Delta'$, $(F^{-1}, G^{-1}) \in \Delta'$, and $(FF', GG') \in \Delta'$. Also [2, p. 20], $a—f, g \to a$, $a—F, G \to b$ implies $b—F^{-1}, G^{-1} \to a$, and $a—F, G \to b—F', G' \to c$ implies $a—FF', GG' \to c$. Thus $a \sim_J a$, $a \sim_J b$ implies $b \sim_J a$, and $a \sim_J b \sim_J c$ implies $a \sim_J c$.

The Jiang equivalence class containing $a \in \tilde{\Gamma}(f, g)$ is denoted by a_J. The set of all such classes is denoted by $\tilde{\Gamma}(f, g)/J$. From the discussion in the previous section we easily have

PROPOSITION 2. *If $a, b \in \tilde{\Gamma}(f, g)$ and $a \sim_J b$, then $\omega(f, g, a)=\omega(f, g, b)$, and a is Δ-essential iff b is.*

We define a relation on $\pi(Y)$ that is also denoted by $\sim_J$ and called *Jiang equivalence.* If $\alpha, \beta \in \pi(Y)$ then $\alpha \sim_J \beta$ if there is a loop C in X at x_0 and a pair $(F, G) \in \Delta'$ such that $[\langle F, C\rangle]\alpha=\beta[\langle G, C\rangle]$.

PROPOSITION 3. *$\sim_J$ is an equivalence relation on $\pi(y)$.*

Proof. Since $(f, g) \in \Delta'$ and $[\langle f, x_0\rangle]\alpha=\alpha=\alpha[\langle g, x_0\rangle]$ for all $\alpha \in \pi(Y)$, $\sim_J$ is reflexive. If $\alpha, \beta \in \pi(Y)$ and $[\langle F, C\rangle]\alpha=\beta[\langle G, C\rangle]$ for some $(F, G) \in \Delta'$ and some loop C at x_0, then $[\langle F^{-1}, C^{-1}\rangle]\beta=[\langle F, C\rangle]^{-1}\beta=\alpha[\langle G, C\rangle]^{-1}=\alpha[\langle G^{-1}, C^{-1}\rangle]$ so, since $(F^{-1}, G^{-1})\in\Delta'$, $\alpha\sim_J\beta$ implies $\beta\sim_J\alpha$. Similarly we use the fact that $[\langle F, C\rangle][\langle F', C'\rangle]=[\langle FF', CC'\rangle]$ to show that $\sim_J$ is transitive.

The Jiang equivalence class containing $\alpha\in\pi(Y)$ is denoted by α_J, the set of such classes by $\pi(Y)/J$. According to the next proposition, $\sim_J$ induces an equivalence on $\pi(Y)/R$.

Proposition 4. *Suppose $\alpha, \alpha', \beta, \beta'\in\pi(Y)$ and $\alpha\sim_R\alpha'\sim_J\beta'\sim_R\beta$. Then $\alpha\sim_J\beta$.*

Proof. By hypothesis there are loops C, D, and E at x_0 in X and a pair $(F, G)\in\Delta'$ such that $[f\circ C]\alpha=\alpha'[g\circ C]$, $[\langle F, D\rangle]\alpha'=\beta'[\langle G, D\rangle]$, and $[f\circ E]\beta'=\beta[g\circ E]$. Thus $[\langle F, EDC\rangle]\alpha=[f\circ E][\langle F, D\rangle][f\circ C]\alpha=[f\circ E][\langle F, D\rangle]\alpha'[g\circ C]=[f\circ E]\beta'[\langle G, D\rangle][g\circ C]=\beta[g\circ E][\langle G, D\rangle][g\circ C]=\beta[\langle G, EDC\rangle]$, so $\alpha\sim_J\beta$.

From this proposition, if $\alpha\sim_J\beta$, for $\alpha, \beta\in\pi(Y)$, then $\alpha'\sim_J\beta'$ for every $\alpha'\in\alpha_R$ and $\beta'\in\beta_R$. In this event we write $\alpha_R\sim_J\beta_R$ and say that α_R is *Jiang-equivalent* to β_R. If $\alpha_R\in\pi(Y)/R$, its Jiang-equivalence class is denoted by α_{RJ}. The set of such equivalence classes is denoted by $\pi(Y)/RJ$. Note that for $\alpha\in\pi(Y)$, the union of all $\alpha'_R\in\alpha_{RJ}$ is simply α_J.

Proposition 5. $T(f, g, \Delta)/R=[y_0]_{RJ}$.

Proof. Suppose first that $\tau_R\in T(f, g, \Delta)/R$. Then for some pair $(F, G)\in\Delta'$ we have $[\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}\in\tau_R$. Now $[\langle F, x_0\rangle][y_0]=([\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1})[\langle G, x_0\rangle]$, so $[y_0]\sim_J[\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}$ whence $[y_0]_R\sim_J\tau_R$. Thus $T(f, g, \Delta)/R\subset[y_0]_{RJ}$. Conversely, suppose $[y_0]_R\sim_J\alpha_R$ for some $\alpha\in\pi(Y)$. Then there is a loop C in X at x_0 and a pair $(F, G)\in\Delta'$ such that $[\langle F, C\rangle][y_0]=\alpha[\langle G, C\rangle]$. Thus $\alpha=[\langle F, C\rangle]\cdot[\langle G, C\rangle]^{-1}=[f\circ C][\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}[g\circ C]^{-1}$, so $\alpha\sim_R[\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}\in T(f, g, \Delta)$. Therefore $\alpha_R\in T(f, g, \Delta)/R$.

The following theorem establishes the connection between the structures we have built on $\tilde{\Gamma}(f, g)$ and $\pi(Y)/R$. We refer to the one-to-one function ϕ of the previous section.

Theorem 2. *$\phi(a_J)\subset(\phi(a))_J$ for each $a\in\tilde{\Gamma}(f, g)$, and $\phi(a_J)=(\phi(a))_J$ when a is Δ-essential.*

Proof. To prove the inclusion suppose $a, a'\in\tilde{\Gamma}(f, g)$ and $a\sim_J a'$. We must show that $\phi(a)\sim_J\phi(a')$. Since $a\sim_J a'$, there is a path D in X from an x in a to an $x'\in a'$ and a pair $(F, G)\in\Delta'$ such that $[\langle F, D\rangle]=[\langle G, D\rangle]$. Let C be a path in X from x_0 to x. Then $[f\circ C][g\circ C]^{-1}\in\phi(a)$ and $[f\circ CD][g\circ CD]^{-1}\in\phi(a')$. Moreover $[\langle F, x_0\rangle][f\circ CD][g\circ CD]^{-1}=[\langle F, CD\rangle][g\circ CD]^{-1}=[f\circ C][\langle F, D\rangle][g\circ CD]^{-1}=[f\circ C][\langle G, D\rangle][g\circ CD]^{-1}=[f\circ C][\langle G, C^{-1}\rangle]=[f\circ C][g\circ C]^{-1}[\langle G, x_0\rangle]$, so $[f\circ CD][g\circ CD]^{-1}\sim_J[f\circ C][g\circ C]^{-1}$ and therefore $\phi(a')\sim_J(a)$. Conversely, suppose $a\in\tilde{\Gamma}(f, g)$ is Δ-essential, and $\alpha_R\sim_J\phi(a)$ for some $\alpha\in\pi(Y)$. We must show that $\alpha\in\phi(a')$ for some $a'\in\tilde{\Gamma}(f, g)$. Let C be a path in X from x_0 to an $x\in a$, so $[f\circ C][g\circ C]^{-1}\in\phi(a)$. Since $\alpha_R\sim_J\phi(a)$ we have $\alpha\sim_J[f\circ C][g\circ C]^{-1}$ so there is a loop D in X at x_0 and a pair $(F, G)\in\Delta'$ with $[\langle F, D\rangle]\alpha=[f\circ C][g\circ C]^{-1}[\langle G, D\rangle]$.

Thus $\alpha=[\langle F^{-1}, D^{-1}\rangle][f\circ C][g\circ C]^{-1}[\langle G, D\rangle]=[\langle F^{-1}, D^{-1}C\rangle][\langle G, C^{-1}D\rangle]$. Now a is Δ-essential and is therefore F, G related to an $a'\in\tilde{\Gamma}(f, g)$. Let E be a path from x to an $x'\in a'$ such that $[\langle F, E\rangle]=[\langle G, E\rangle]$. Then

$$\begin{aligned}[f\circ D]\alpha &= [f\circ D][\langle F^{-1},D^{-1}C\rangle][\langle G, C^{-1}D\rangle]\\ &= [\langle F^{-1}, C\rangle][\langle G, C^{-1}D\rangle]\\ &= [\langle F^{-1}, C\rangle][\langle G, E\rangle][\langle G, E\rangle]^{-1}[\langle G, C^{-1}D\rangle]\\ &= [\langle F^{-1}, C\rangle][\langle F, E\rangle][\langle G^{-1}, E^{-1}\rangle][\langle G, C^{-1}D\rangle]\\ &= [f\circ CE][g\circ E^{-1}C^{-1}][g\circ D]\\ &= ([f\circ CE][g\circ CE]^{-1})[g\circ D].\end{aligned}$$

Thus $\alpha\sim_R [f\circ CE][g\circ CE]^{-1}\in\phi(a')$, so $\alpha_R=\phi(a')$.

If in the definition of ϕ we let C be the constant path at x_0, we see that $\phi(a_0)=([f\circ C][g\circ C]^{-1})_R=[y_0]_R$. Now a_0 is Δ-essential; Theorem 2, therefore, gives us $\phi(a_{0J})=[y_0]_{RJ}$. Hence, since ϕ is one-to-one, Proposition 5 yields

PROPOSITION 6. $\phi(a_{0J})=T(f, g, \Delta)/R$ *and* Card $a_{0J}=J(f, g, \Delta)$.

From Proposition 2 it follows that a is Δ-essential for each $a\in a_{0J}$. Thus Card $a_{0J}\leq N(f, g, \Delta)$. Theorem 1 therefore follows from Proposition 6. We now turn to divisibility results.

THEOREM 3. *For every* $\alpha\in\pi(Y)$ *we have* Card $\alpha_{RJ}=J(f, g, \Delta)$, *provided that one of the following conditions is satisfied*:

1. $\pi(Y)$ *is abelian*,
2. $\Delta=\Delta_2$,
3. $\Delta=\Delta_3$, *and* $f_\#(\pi(X))$ *is abelian*.

Proof. Let $\alpha\in\pi(Y)$ and define a function $\Psi\colon T(f, g, \Delta)\to\pi(Y)$ by $\Psi(\tau)=\tau\alpha$. It suffices to show that Ψ induces a one-to-one function $\tilde{\Psi}\colon T(f, g, \Delta)/R\to\pi(Y)/R$ whose image is α_{RJ}. To see that Ψ induces $\tilde{\Psi}$, let $\tau, \tau'\in T(f, g, \Delta)$, and suppose that $\tau\sim_R\tau'$. We must show that $\Psi(\tau)\sim_R\Psi(\tau')$. Since $\tau\sim_R\tau'$ there is a $\gamma\in\pi(X)$ with $f_\#(\gamma)\tau=\tau' g_\#(\gamma)$. When $\pi(Y)$ is abelian (case 1), this yields $f_\#(\gamma)\Psi(\tau)=f_\#(\gamma)\tau\alpha=\tau' g_\#(\gamma)\alpha=\tau'\alpha g_\#(\gamma)=\Psi(\tau')g_\#(\gamma)$, so $\Psi(\tau)\sim_R\Psi(\tau')$. In case 2, $g_\#$ is trivial (g is the constant map into y_0), so $f_\#(\gamma)\Psi(\tau)=f_\#(\gamma)\tau\alpha=\tau' g_\#(\gamma)\alpha=\tau'\alpha=\Psi(\tau')=\Psi(\tau')g_\#(\alpha)$, so $\Psi(\tau)\sim_R\Psi(\tau')$. Finally in case 3, $f_\#(\pi(Y))$ is abelian and $g_\#$ is the identity so $f_\#(\alpha^{-1}\gamma\alpha)\Psi(\tau)=f_\#(\gamma)\Psi(\tau)=f_\#(\gamma)\tau\alpha=\tau' g_\#(\gamma)\alpha=\tau'\gamma\alpha=\tau'\alpha(\alpha^{-1}\gamma\alpha)=\Psi(\tau')g_\#(\alpha^{-1}\gamma\alpha)$, so $\Psi(\tau)\sim_R\phi(\tau')$. Thus Ψ induces a function $\tilde{\Psi}\colon T(f, g, \Delta)/R\to\pi(Y)/R$ defined by $\tilde{\Psi}(\tau_R)=(\Psi(\tau))_R$. To see that $\tilde{\Psi}$ is one-to-one suppose $\Psi(\tau)\sim_R\Psi(\tau')$, for some $\tau, \tau'\in T(f, g, \Delta)$. We must show that $\tau\sim_R\tau'$. Since $\Psi(\tau)\sim_R\Psi(\tau')$ there is a $\gamma\in\pi(X)$ with $f_\#(\gamma)\Psi(\tau)=\Psi(\tau')g_\#(\gamma)$. In case 1 we then have $f_\#(\gamma)\tau=f_\#(\gamma)\Psi(\tau)\alpha^{-1}=\Psi(\tau')g_\#(\gamma)\alpha^{-1}=\Psi(\tau')\alpha^{-1}g_\#(\gamma)=\tau' g_\#(\gamma)$, so $\tau\sim_R\tau'$. In case 2, $g_\#$ is trivial so $f_\#(\gamma)\tau=f_\#(\gamma)\Psi(\tau)\alpha^{-1}=\Psi(\tau')g_\#(\gamma)\alpha^{-1}=\Psi(\tau')\alpha^{-1}=\tau'=\tau' g_\#(\gamma)$, whence $\tau\sim_R\tau'$. Finally, in case 3, where $f_\#(\pi(X))$ is abelian and $g_\#$ the identity, $f_\#(\alpha\gamma\alpha^{-1})\tau=f_\#(\gamma)\Psi(\tau)\alpha^{-1}=\Psi(\tau')g_\#(\gamma)\alpha^{-1}=\Psi(\tau')\gamma\alpha^{-1}=\Psi(\tau')\alpha^{-1}(\alpha\gamma\alpha^{-1})=\tau' g_\#(\alpha\gamma\alpha^{-1})$, so $\tau\sim_R\tau'$. Thus $\tilde{\Psi}$ is one-to-one. To see that $\tilde{\Psi}(T(f, g, \Delta)/R)\subset\alpha_{RJ}$, it suffices to show that $\Psi(T(f, g, \Delta)) \subset \alpha_J$. Suppose therefore that $\tau \in T(f, g, \Delta)$, so that $\tau = [\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}$ for some $(F, G) \in \Delta'$. In case 1 $\pi(Y)$ is abelian, and in case 2 and 3 G is a constant homotopy so $[\langle G, x_0\rangle] = [y_0]$. Thus, in any case, $[\langle G, x_0\rangle]$ commutes with α, so $[\langle F^{-1}, x_0\rangle]\Psi(\tau) = [\langle F^{-1}, x_0\rangle][\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}\alpha = \alpha[\langle G^{-1}, x_0\rangle]$. Therefore, since $(F^{-1}, G^{-1}) \in \Delta'$, $\Psi(\tau) \sim_J \alpha$. Finally, to see that $\tilde{\Psi}(T(f, g, \Delta)) \supset \alpha_{RJ}$, suppose that $\alpha'_R \in \alpha_{RJ}$ for some $\alpha' \in \pi(Y)$. We must show that $\alpha'_R = \tilde{\Psi}(\tau'_R)$ for some $\tau' \in T(f, g, \Delta)$. Since $\alpha'_R \sim_J \alpha_R$, $\alpha' \sim_J \alpha$, so that there is a loop C in X at x_0 and a pair $(F, G) \in \Delta'$ with $[\langle F, C\rangle]\alpha = \alpha'[\langle G, C\rangle]$. Let $\tau' = [\langle F, x_0\rangle] \cdot [\langle G, x_0\rangle]^{-1}$, so $[f \circ C]\Psi(\tau') = [f \circ C][\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}\alpha = [\langle F, C\rangle][\langle G, x_0\rangle]^{-1}\alpha$. Again $[\langle G, x_0\rangle]^{-1}$ commutes with α, so $[f \circ C]\Psi(\tau') = [\langle F, C\rangle]\alpha[\langle G, x_0\rangle]^{-1} = \alpha'[\langle G, C\rangle][\langle G, x_0\rangle]^{-1} = \alpha'[g \circ C]$. Thus, $\Psi(\tau') \sim_R \alpha'$, so $\tilde{\Psi}(\tau_R) = \alpha'_R$.

An element ξ of an abelian group $\boldsymbol{G}$ is *divisible by* an integer n, if $\xi = n\zeta$ for some $\zeta \in \boldsymbol{G}$.

THEOREM 4. *Suppose $J(f, g, \Delta) = \text{Card } \alpha_{RJ}$ for every $\alpha \in \pi(Y)$—as it will if any of the conditions of Theorem 3 are met. Then*

1. *$J(f, g, \Delta)$ divides $R(f, g)$.*
2. *$J(f, g, \Delta)$ divides $N(f, g, \Delta)$.*
3. *$J(f, g, \Delta)$ divides* Card $\{\alpha \in \tilde{\Gamma}(f, g) \mid \omega(f, g, a) \in A\}$ *for any subset $A \subset \boldsymbol{A} - 0$.*
4. *$J(f, g, \Delta)$ divides $\omega(f, g, X)$.*

Proof. Since $\pi(Y)/RJ$ partitions $\pi(Y)/R$ into sets α_{RJ} each with cardinality $J(f, g, \Delta)$ we have $R(f, g) = \text{Card } \pi(Y)/R = J(f, g, \Delta) \cdot \text{Card } \pi(Y)/RJ$. This proves 1. According to Proposition 2, if $a \in \tilde{\Gamma}(f, g)$ is Δ-essential then a' is for every $a' \in a_J$. Thus $\{a_J \in \tilde{\Gamma}(f, g)/J \mid a \text{ is } \Delta\text{-essential}\}$ is a partition of the set of Δ-essential $a \in \tilde{\Gamma}(f, g)$. Since ϕ is one-to-one, Theorem 2 says that $\text{Card } a_J = \text{Card } \phi(a)_J = J(f, g, \Delta)$ whenever $a \in \tilde{\Gamma}(f, g)$ is Δ-essential. Thus $N(f, g, \Delta) = \text{Card }\{a \in \tilde{\Gamma}(f, g) \mid a \text{ is } \Delta\text{-essential}\} = J(f, g) \cdot \text{Card }\{a_J \in \tilde{\Gamma}(f, g)/J \mid a \text{ is } \Delta\text{-essential}\}$. This proves 2. Now suppose that $A \subset \boldsymbol{A} - 0$. Then

$$\text{Card }\{a \in \tilde{\Gamma}(f, g) \mid \omega(f, g, a) \in A\} = \sum_{\xi \in A} \text{Card }\{a \in \tilde{\Gamma}(f, g) \mid \omega(f, g, a) = \xi\}$$
$$= \sum_{\xi \in A} \sum_{aJ \in \tilde{\Gamma}(f,g)/J} \text{Card }\{a' \in a_J \mid \omega(f, g, a) = \xi\},$$

so it suffices to show that $\text{Card }\{a' \in a_J \mid \omega(f, g, a) = \xi\}$ is divisible by $J(f, g, \Delta)$ whenever $a \in \tilde{\Gamma}(f, g)$ and $\xi \in \boldsymbol{A}$, $\xi \neq 0$. By Proposition 2, $\omega(f, g, a') = \omega(f, g, a)$ for every $a' \in a_J$, hence $\text{Card }\{a' \in a_J \mid \omega(f, g, a) = \xi\}$ is either 0 or $\text{Card } a_J$. Of course, 0 is divisible by $J(f, g, \Delta)$. In the second case $\omega(f, g, a) = \xi \neq 0$ so a is Δ-essential, hence, by Theorem 2, $\text{Card } a_J = \text{Card } \phi(a)_J = J(f, g, \Delta)$, which completes the proof of 3. Finally, by additivity of ω we have

$$\omega(f, g, X) = \sum_{a \in \tilde{\Gamma}(f,g)} \omega(f, g, a) = \sum_{\xi \in \boldsymbol{A} - 0} (\text{Card }\{a \in \tilde{\Gamma}(f, g) \mid \omega(f, g, a) = \xi\}) \cdot \xi,$$

which together with 3 proves 4.

IV. Consequences of the main theorems. In this section, we will present some results based on Theorems 1, 3, and 4. We state the results first for the general coincidence problem (g arbitrary, $\Delta=\Delta_1$), and then give the analogous results for fixed points and for solutions to the equation $f(x)=y_0$.

If $f_\#=g_\#\colon \pi(X)\to\pi(Y)$, then for $(F, G), (F', G')\in\Delta_1'$, $[\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}\sim_R [\langle F', x_0\rangle][\langle G', x_0\rangle]^{-1}$ means that for some loop C in X at x_0 we have $[f\circ C]\cdot[\langle F', x_0\rangle][\langle G', x_0\rangle]^{-1}=[\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}[g\circ C]$, so $[\langle F', x_0\rangle][\langle G', x_0\rangle]^{-1}=[f\circ C]^{-1}[\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}[g\circ C]=[\langle F, C^{-1}\rangle][\langle G^{-1}, C\rangle]=[\langle F, x_0\rangle][f\circ C^{-1}]\cdot[g\circ C][\langle G, x_0\rangle]^{-1}=[\langle F, x_0\rangle][\langle G, x_0\rangle]^{-1}$, and so $f_\#=g_\#$ implies that $J(f, g, \Delta_1)$ is just the cardinality of the set $T(f, g, \Delta_1)=T(f)T(g)$. Thus, from Theorem 1

(1) *If $f_\#=g_\#\colon \pi(X)\to\pi(Y)$, then* Card $T(f)T(g)\leqq N(f, g, \Delta_1)$.

In other words, for any f' homotopic to f and g' homotopic to g, there are at least Card $T(f)T(g)$ solutions to the equation $f'(x)=g'(x)$.

We now assume that $\pi(Y)$ is abelian. Therefore $R(f, g)=$ Order Coker $(f_\#-g_\#)$. If $R(f, g)$ is prime and $J(f, g, \Delta_1)>1$, then $J(f, g, \Delta_1)=R(f, g)$ by Theorem 4. By Theorem 1, $J(f, g, \Delta_1)\leqq N(f, g, \Delta_1)\leqq R(f, g)$, so

(2) *If $J(f, g, \Delta_1)>1$ and $R(f, g)$ is prime, then* $N(f, g, \Delta_1)=$ Order Coker $(f_\#-g_\#)$.

Suppose ω is a Δ_1-index. If $\omega(f, g, X)\in A$ is divisible by only ± 1, then we can say quite a bit about the set $T(f)T(g)$. From Theorem 4 we conclude that $J(f, g, \Delta_1)=1$ and so $\tau\sim_R\tau'$ for all $\tau, \tau'\in T(f)T(g)$ and in particular $\tau\sim_R 0$. Therefore

(3) *If $\omega(f, g, X)$ is divisible by only ± 1, then $T(f)T(g)\subset$* Image $(f_\#-g_\#)$. *If, in addition, $f_\#=g_\#$, then $T(f)T(g)=0$.*

If X and Y are closed, orientable, n-manifolds, then we have the integer-valued Δ_1-index λ' for which $\lambda'(f, g, X)=L(f, g)$, the classical Lefschetz coincidence number of f and g. In this case, if $J(f, g, \Delta_1)>1$ and $L(f, g)$ is a prime, then $J(f, g, \Delta_1)=|L(f, g)|$ by Theorem 4, so by Theorem 1,

(4) *If X and Y are closed, orientable, n-manifolds, $J(f, g, \Delta_1)>1$, and $L(f, g)$ is prime, then $N(f, g, \Delta_1)\geqq|L(f, g)|$.*

Now let us turn to the fixed-point problem, where $\Delta=\Delta_3$, $Y=X$ is a compact ANR, and $g\colon X\to X$ is the identity map. Then we have the Δ_3-index ω_3 for which $\omega_3(f, g, X)=L(f)$, the Lefschetz number of f. To simplify notation, write $R(f)$ for $R(f, g)$, $J(f)$ for $J(f, g, \Delta_3)$. $T(f, g, \Delta_3)=T(f)$, which is a subgroup of $\pi(X)$. Let $N(f)$ be the classical Nielsen number of f, i.e., the number of $a\in\tilde{\Gamma}(f, g)$ such that $\omega_3(f, g, a)\neq 0$. Assume $L(f)\neq 0$, and that $a_0\in\tilde{\Gamma}(f, g)$ has been chosen so that $\omega_3(f, g, a_0)\neq 0$. Then by Propositions 2 and 5 $N(f)\geqq$ Card $a_{0J}=J(f)$. We also have $N(f)\leqq R(f)$. Write $1\colon \pi(X)\to\pi(X)$ for the identity isomorphism. When $f_\#=1$, then $J(f)$ is just the order of the group $T(f)$. Therefore, corresponding to (1) above, we have

(1′) *If $f_\#=1\colon \pi(X)\to\pi(X)$ and $L(f)\neq 0$, then* Order $T(f)\leqq N(f)$.

When $f\colon X\to X$ is the identity map, then $L(f)=\chi(X)$, the Euler characteristic of X. Since all fixed points of the identity map are f, g-equivalent (see §II), then

$N(f) \leqq 1$. Since $f_{\#}=1$, then by (1′), Order $T(f)=1$. Thus we have a new proof of the following result

(5) (Gottlieb [3]). *If* $\chi(X) \neq 0$, *then* $T(X)$ *is trivial.*

Barnier [1] has worked out a number of examples that show that the inequalities $J(f) \leqq N(f) \leqq R(f)$ permit the computation of $N(f)$ for many maps for which such computation was previously impossible. We will outline one such example; the reader may refer to [1, pp. 83–84] for the details. The point of the example is that, previously, the best techniques for such computations (based on the results of Jiang [6]) required that $T(f)=\pi(X)$, while in this example $T(f) \neq \pi(X)$.

Let X be the closed orientable 2-manifold of genus 5, then $\pi(X)$ is presented by generators $\alpha_1, \alpha_2, \beta_1, \beta_2, \varepsilon$ and a single relation $\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\varepsilon^2=1$. It can be proved, using [5], that there exists a map $f: X \to X$ such that $f_{\#}(\alpha_1)=\alpha_1^n$, $f_{\#}(\alpha_2)=f_{\#}(\beta_1)=f_{\#}(\beta_2)=f_{\#}(\varepsilon)=1$. We will assume $n \neq 1$. By means of the Hurewicz homomorphism, one finds that $L(f)=1-n \neq 0$. By [1, p. 54], $T(f)$ is the centralizer of α_1^n which, by [7], is the proper subgroup of $\pi(X)$ generated by α_1. Direct computation proves that $\alpha_1^p \sim_R \alpha_1^q$ if and only if $p=q$ mod $(n-1)$, so $J(f)=|n-1|$. Furthermore; if $\alpha\gamma \in \pi(X)$, then $\gamma \sim_R \alpha_1^k$ for some k so $R(f)=|n-1|$ also. Therefore the inequalities tell us that $N(f)=|n-1|$.

Let us assume that $f_{\#}(\pi(X))$ is abelian. If $J(f)>1$ and $R(f)$ is prime, then, by Theorem 4, $J(f)=R(f)$ and we have:

(2′) *If* $L(f) \neq 0$, $J(f)>1$ *and* $R(f)$ *is prime, then* $N(f)=R(f)$.

Since $1 \leqq J(f) \leqq |L(f)|$ by Theorem 4, then $|L(f)|=1$ implies that all elements of $T(f)$ are f, g-congruent, which proves

(3′) *If* $|L(f)|=1$, *then* $T(f) \subseteq h(\pi(X))$, *where* $h: \pi(X) \to \pi(X)$ *is defined by* $h(\alpha)=f_{\#}(\alpha)\alpha^{-1}$.

By Theorem 4, $J(f)$ divides $L(f)$, so we have our final result concerning fixed points.

(4′) *If* $J(f)>1$ *and* $L(f)$ *is prime, then* $|L(f)| \leqq N(f)$.

We now turn briefly to the class Δ_2 where $N(f, g, \Delta_2)$ is a lower bound for the number of solutions to $f(x)=y_0$, g is the constant map of X to y_0, and $T(f, g, \Delta_2)=T(f)$.

By Theorem 3, we may apply the conclusions of Theorem 4 without further restrictions on f, X, or Y. The analogues of (1)–(4) hold by essentially the same arguments used above, so we just list them here.

(1″) *If* $f_{\#}=0$, *then* Order $T(f) \leqq N(f, g, \Delta_2)$.

(2″) *If* $J(f, g, \Delta_2)>1$ *and* $R(f, g)$ *is prime then* $N(f, g, \Delta_2)=R(f, g)$.

(3″) *If* $\omega_2(f, g, X)$ *is divisible only by* ± 1, *then* $T(f) \subset f_{\#}(\pi(X))$.

(4″) *If both* X *and* Y *are closed, orientable, n-manifolds,* $J(f, g, \Delta_2)>1$, *and the degree of* f *is prime, then* $|\deg(f)| \leqq N(f, g, \Delta_2)$.

One might well ask whether, for actual spaces and maps, it is possible to obtain enough information to apply the results of this section. The computation of $\lambda(f, g, X)$, $\lambda'(f, g, X)$, $\omega_2(f, g, X)$, and $\omega_3(f, g, X)$ is straightforward from a knowledge of the cohomology homomorphisms induced by f and g. In the remaining case, where $\Delta=\Delta_1$ and Y is not a manifold, nothing is known. For $R(f, g)$, the computation problem is purely algebraic. The extent of the difficulty depends upon how complicated $\pi(Y)$ is. The complete determination of $T(f, g, \Delta)$ seems quite difficult. Fortunately, (2) and (4) only require that we find a single element in $T(f, g, \Delta)$ that is not Reidemeister equivalent to the unit; the results of [1] suggest that this is a much easier problem. For the application of (1), we can only observe that the identification of any nontrivial subset of $T(f, g, \Delta)$ will produce a nontrivial lower bound for the number of solutions of $f(x)=g(x)$.

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