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THE C^k-CLASSIFICATION OF CERTAIN OPERATORS IN L_p. II

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Introduction. We study the one-parameter family of operators

$$T_\alpha = M+\alpha J$$

acting in $L_p(0, 1)$, $1<p<\infty$, where $\alpha \in \boldsymbol{C}$ (the complex field), $M: f(x) \to xf(x)$ and $J: f(x) \to \int_0^x f(t)\,dt$.

Our purpose is to bring the main results of [6] to the best possible form. This will be achieved by replacing Theorem 6, Proposition 13 and Proposition 15 of [6] by the following theorems.

THEOREM 1. *Let n be a nonnegative integer. Then T_α is of class C^n if and only if* $|\mathrm{Re}\,\alpha| \leqq n$.

THEOREM 2. *T_α is similar to T_β if and only if* $\mathrm{Re}\,\alpha = \mathrm{Re}\,\beta$.

THEOREM 3. *T_α is spectral if and only if* $\mathrm{Re}\,\alpha = 0$.

Theorem 2 was conjectured in [6].

The above results, along with some others, will follow from an interesting formula relating the holomorphic groups of operators $U_\alpha(z) = \exp(zT_\alpha)$ and $V_z(\alpha) = (I+zJ)^\alpha$ ($\alpha, z \in \boldsymbol{C}$).

1. **Preliminaries.** Let $\{J^{i\gamma}, \gamma \in \boldsymbol{R}\}$ be the boundary group of the Riemann-Liouville holomorphic semigroup acting in $L_p(0, 1)$, $1<p<\infty$ (cf. [4]). It is known that

$$\|J^{i\gamma}\| \leqq \exp(\pi|\gamma|/2) \qquad (\gamma \in \boldsymbol{R}) \tag{0}$$

and

$$T_{\beta+i\gamma} = J^{-i\gamma} T_\beta J^{i\gamma} \qquad (\beta, \gamma \in \boldsymbol{R}) \tag{1}$$

(cf. [4] and [6, Lemma 2]).

For $n = 0, 1, 2, \ldots$, let $C^n[0, 1]$ denote the Banach algebra of all complex functions of class C^n on $[0, 1]$ with the norm

$$\|\phi\|_n = \sum_{j=0}^{n} \sup_{[0,1]} |\phi^{(j)}|/j'!.$$

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Let T be a bounded operator acting on a Banach space X, with spectrum in $[0, 1]$. We say that T is of class C^n if there exists a continuous representation τ of $C^n[0, 1]$ on X which sends the functions $\phi(t) \equiv 1$ and $\phi(t) \equiv t$ to the identity operator I and to T, respectively. The representation τ is unique (when it exists), and is called the C^n-operational calculus for T (cf. [5]). For example, it follows from [6, Lemma 3] that the operator $T_n = M + nJ$ acting in $L_p(0, 1)$, $1 \leqq p < \infty$, is of class C^n, and its C^n-operational calculus is given by

$$\tau_n(\phi) = \sum_{j=0}^{n} \binom{n}{j} M(\phi^{(j)})J^j, \qquad \phi \in C^n[0, 1], \tag{2}$$

where $M(\psi)$ denotes the operator of multiplication by the function ψ.

Since J is quasi-nilpotent, the operator $(I+zJ)^\alpha$ ($\alpha, z \in \boldsymbol{C}$) is well defined by means of the analytic operational calculus:

$$(I+zJ)^\alpha = \frac{1}{2\pi i} \int_\Gamma \lambda^\alpha[(\lambda-1)I - zJ]^{-1}\, d\lambda, \tag{3}$$

where, to fix the ideas, Γ is the circle $|\lambda - 1| = 1/2$. For each fixed z, $\{(I+zJ)^\alpha; \alpha \in \boldsymbol{C}\}$ is a holomorphic group of operators. We shall need a simple estimate on its norm. We have

$$[(\lambda-1)I - zJ]^{-1} = (\lambda-1)^{-1} \sum_{n=0}^{\infty} [z/(\lambda-1)]^n J^n, \qquad \lambda \neq 1.$$

Since $\|J^n\| \leqq 1/n!$, we see that $\|[(\lambda-1)I - zJ]^{-1}\| \leqq 2 \exp(2|z|)$ ($\lambda \in \Gamma$).

Write $\alpha = \beta + i\gamma$ ($\beta, \gamma \in \boldsymbol{R}$) and $\lambda = re^{i\theta}$. For $\lambda \in \Gamma$, we have $1/2 \leqq r \leqq 3/2$ and $|\theta| \leqq \pi/6$. Consequently

$$|\lambda^{\beta+i\gamma}| = r^\beta e^{-\theta\gamma} \leqq (1 \pm 1/2)^\beta \exp(\pi|\gamma|/6), \qquad \lambda \in \Gamma,$$

where the sign is $(+)$ if $\beta \geqq 0$ and $(-)$ if $\beta < 0$. By (3), it follows that

$$\|(I+zJ)^\alpha\| \leqq (1 \pm 1/2)^\beta \exp(\pi|\gamma|/6) \exp(2|z|) \qquad (\alpha = \beta + i\gamma). \tag{4}$$

2. The basic results.

THEOREM 4. *For all $\alpha, z \in \boldsymbol{C}$,*

$$\begin{aligned} \exp(zT_\alpha) &= e^{zM}(I+zJ)^\alpha \\ &= (I-zJ)^{-\alpha} e^{zM}. \end{aligned}$$

Proof. Note first that all the operator functions involved are entire functions of the complex variables α, z. Moreover, it suffices to prove the first identity (or the second), because once this is done, we get

$$\exp(zT_\alpha) = [\exp(-zT_\alpha)]^{-1} = [e^{-zM}(I-zJ)^\alpha]^{-1} = (I-zJ)^{-\alpha} e^{zM},$$

as wanted.

For z fixed, consider the operator-valued entire function

$$\Phi_z(\alpha) = e^{-zM}\exp(zT_\alpha)-(I+zJ)^\alpha.$$

We verify the hypothesis of [3, Theorem 3.13.7]. Since

$$\|e^{-zM}\exp(zT_\alpha)\| \leqq \exp(|z|\,\|M\|)\exp(|z|\ \|M+\alpha J\|) \leqq \exp(|z|(2+|\alpha|)),$$

it follows from (4) that

$$\|\Phi_z(\alpha)\| \leqq \exp(2|z|)\{\exp(|z|\ |\alpha|)+(3/2)^\beta\exp(\pi|\gamma|/6)\}$$

for $\alpha=\beta+i\gamma$, $\beta\geqq 0$.

Thus

$$\|\Phi_z(re^{i\theta})\| \leqq Ce^{r\lambda(\theta)}, \qquad -\pi/2 \leqq \theta \leqq \pi/2,$$

where $C=2e^{2|z|}$ and

$$\lambda(\theta) = \max\{|z|, \log(3/2)\cos\theta+(\pi/6)|\sin\theta|\}.$$

Clearly, $\lambda(\theta)$ is bounded, even, and

$$\lambda(\pm\pi/2) \leqq \pi \quad \text{for } |z| \leqq \pi.$$

Moreover, if $|z|<\pi$, we have

$$\limsup_{\delta\to 0+}\delta^{-1}\{\pi-\lambda(\pi/2-\delta)\} = \infty.$$

Using (2) with $\phi(t)=\phi_z(t)=e^{zt}$, we obtain

$$\begin{aligned}\exp(zT_n) &= \sum_{j=0}^{n}\binom{n}{j}z^jM(\phi_z)J^j\\ &= e^{zM}\sum_{j=0}^{n}\binom{n}{j}(zJ)^j\end{aligned}$$

i.e.,

$$\exp(zT_n) = e^{zM}(I+zJ)^n, \qquad n=0,1,2,\ldots. \tag{5}$$

Thus $\Phi_z(n)=0$, $n=0,1,2,\ldots$. By Theorem 3.13.7 in [3], it follows that

$$\Phi_z(\alpha) = 0, \qquad \operatorname{Re}\alpha \geqq 0,\ |z| < \pi.$$

Since $\Phi_z(\alpha)$ is entire in both variables, we conclude that $\Phi_z(\alpha)=0$ for all $\alpha, z\in \boldsymbol{C}$. Q.E.D.

REMARKS. 1. Theorem 4 is also valid for $p=1$, since we used only Lemma 3 of [6], which is true in this case as well.

2. Let n be a nonnegative integer. The second identity in Theorem 4 shows that

$$\exp(zT_{-n}) = (I-zJ)^ne^{zM}. \tag{6}$$

This formula follows also from Lemma 5 in [6], and could be used instead of (5) to prove Theorem 4, thus relying on Lemma 5 in [6] rather than on Lemma 3 there. As a matter of fact, the proof of Theorem 4 can be used to show that the two lemmas are consequences of each other (cf. [5, proof of Lemma 2.11]).

3. It follows from Theorem 4 that the holomorphic groups $U_\alpha(z)=\exp(zT_\alpha)$ satisfy the "cocycle" identity:

$$U_{\alpha+\beta}(z) = U_\alpha(z)e^{-zM}U_\beta(z) \qquad (\alpha, \beta, z \in \boldsymbol{C}).$$

By Theorem 4 and the spectral mapping theorem, the spectrum of the operator $e^{-zM}\exp(zT_\alpha)=(I+zJ)^\alpha$ consists of the single point $\lambda=1$. Therefore, the analytic operational calculus may be used to define powers $[e^{-zM}\exp(zT_\alpha)]^\beta$ for $\beta \in \boldsymbol{C}$, and by Theorem VII.3.12 in [2], one has

$$[e^{-zM}\exp(zT_\alpha)]^\beta = (I+zJ)^{\alpha\beta} = e^{-zM}\exp(zT_{\alpha\beta}).$$

A similar relation follows from the second identity in Theorem 4:

COROLLARY 5. *For all* $\alpha, \beta, z \in \boldsymbol{C}$,

$$\begin{aligned}\exp(zT_{\alpha\beta}) &= e^{zM}[e^{-zM}\exp(zT_\alpha)]^\beta \\ &= [\exp(zT_\alpha)e^{-zM}]^\beta e^{zM}.\end{aligned}$$

We consider now the one-parameter groups of operators

$$G_\alpha(t) = \exp(itT_\alpha) \qquad (t \in \boldsymbol{R}).$$

THEOREM 6. *For each* $\beta \in \boldsymbol{R}$, *there exists a constant* $C_\beta>0$ *such that*

$$C_\beta e^{-\pi|\gamma|} \leqq (1+|t|)^{-|\beta|}\|G_{\beta+i\gamma}(t)\| \leqq e^{\pi/2}e^{\pi|\gamma|}$$

for all $\gamma, t \in \boldsymbol{R}$.

Proof. By (0) and (1), it suffices to prove the theorem for $\gamma=0$.

Fix $t \in \boldsymbol{R}$, and consider the operator-valued entire function

$$\psi_t(\alpha) = \exp(\pi\alpha^2)G_\alpha(t) \qquad (\alpha \in \boldsymbol{C}). \tag{7}$$

By (1)

$$\|\psi_t(\alpha)\| \leqq \exp(\pi(\beta^2+\tfrac{1}{4}))\|G_\beta(t)\| \qquad (\alpha = \beta+i\gamma). \tag{8}$$

In particular, $\psi_t(\beta+i\gamma)$ is bounded in the strip $n-1\leqq\beta\leqq n$, for any integer n. By (5), (6) and (8),

$$\begin{aligned}\|\psi_t(n+i\gamma)\| &\leqq \exp(\pi(n^2+\tfrac{1}{4}))\|(I\pm itJ)^{|n|}\| \\ &\leqq \exp(\pi(n^2+\tfrac{1}{4}))(1+|t|)^{|n|}.\end{aligned}$$

Write β as the convex combination $bn+c(n-1)=n-c$; $|\beta|=b|n|+c|n-1|$. Then, by the "three lines theorem" [2, VI.10.3] and the preceding inequalities,

$$\begin{aligned}\|\psi_t(\beta+i\gamma)\| &\leqq \exp\pi[b(n^2+\tfrac{1}{4})+c((n-1)^2+\tfrac{1}{4})](1+|t|)^{|\beta|} \\ &= \exp\pi[n^2-2cn+c+\tfrac{1}{4}](1+|t|)^{|\beta|} \\ &= \exp\pi[(n-c)^2+c(1-c)+\tfrac{1}{4}](1+|t|)^{|\beta|} \\ &\leqq \exp\pi[\beta^2+\tfrac{1}{2}](1+|t|)^{|\beta|}.\end{aligned}$$

Thus

$$\|G_\beta(t)\| = \exp(-\pi\beta^2)\|\psi_t(\beta)\| \leqq \exp(\pi/2)(1+|t|)^{|\beta|}.$$

Next, fix $\beta \in \boldsymbol{R}$ and let

$$C_\beta = \inf_{t\in \boldsymbol{R}} (1+|t|)^{-|\beta|}\|G_\beta(t)\| \geqq 0.$$

We must show that $C_\beta > 0$. This is obvious for $\beta = 0$, since $\|G_0(t)\| = 1$. So consider $\beta \neq 0$, and *fix* an integer $n \geqq 1$ such that $n|\beta| > 1$. Trivially, $(1+|t|)^{-|\beta|}\|G_\beta(t)\| > 0$ for each $t \in \boldsymbol{R}$. Assume $C_\beta = 0$. There exists then a sequence $\{t_k\}$ in $\boldsymbol{R}$ such that $|t_k| \to \infty$ and $(1+|t_k|)^{-|\beta|}\|G_\beta(t_k)\| \to 0$ as $k \to \infty$. Fix $\varepsilon > 0$ and choose k_0 such that

$$(1+|t_k|)^{-|\beta|}\|G_\beta(t_k)\| < \varepsilon^{|\beta|} \quad \text{for } k \geqq k_0. \tag{9}$$

For $k \geqq k_0$ fixed, consider the entire functions

$$F_k^{\pm}(\zeta) = (1+|t_k|)^{\mp\zeta}\psi_{t_k}(\zeta), \qquad \zeta \in \boldsymbol{C}.$$

It follows from (8) that $F_k^{\pm}(\zeta)$ is bounded in each vertical strip $a \leqq \operatorname{Re}\zeta \leqq b$ ($a, b \in \boldsymbol{R}$) and

$$\|F_k^{\pm}(i\eta)\| \leqq \exp(\pi/4) \qquad (\eta \in \boldsymbol{R}).$$

By Corollary 5,

$$\|G_{n\beta}(t_k)\| = \|[\exp(-it_kM)G_\beta(t_k)]^n\| \leqq \|G_\beta(t_k)\|^n.$$

Therefore, by (9),

$$(1+|t_k|)^{-n|\beta|}\|G_{n\beta}(t_k)\| \leqq [(1+|t_k|)^{-|\beta|}\|G_\beta(t_k)\|]^n \leqq \varepsilon^{n|\beta|}.$$

By (1), we then have

$$\|F_k^{\pm}(n\beta+i\eta)\| \leqq \exp(\pi(n^2\beta^2+\tfrac{1}{4}))\varepsilon^{n|\beta|}$$

where the superscript of F_k is $(+)$ if $\beta > 0$ and $(-)$ if $\beta < 0$. By the "three lines theorem" applied to F_k^+ (resp. F_k^-) in the strip $0 \leqq \xi \leqq n\beta$ (resp. $n\beta \leqq \xi \leqq 0$), we obtain

$$\|F_k^{\pm}(\xi+i\eta)\| \leqq \exp(\pi(n^2\beta^2+\tfrac{1}{4}))\varepsilon^{|\xi|}$$

in the respective strips.

This is true in particular for $\zeta = \xi = 1$ (resp. -1), since $n|\beta| > 1$. Thus

$$(1+|t_k|)^{-1}\|G_{\pm 1}(t_k)\| = e^{-\pi}\|F_k^{\pm}(\pm 1)\| \leqq C\varepsilon$$

where C does *not* depend on k.

This proves that

$$\lim_{k\to\infty} (1+|t_k|)^{-1}\|G_{\pm 1}(t_k)\| = 0.$$

However, by (5) and (6), this limit is equal to

$$\lim_{k\to\infty} (1+|t_k|)^{-1}\|I \pm it_kJ\| = \|J\| \neq 0$$

(since $|t_k| \to \infty$ as $k \to \infty$). This contradiction shows that $C_\beta > 0$, and the proof is complete.

3. **Proofs of Theorems 1–3.**

Proof of Theorem 1. If $|\mathrm{Re}\,\alpha| \leqq n$, T_α is of class C^n by Theorem 6 in [6]. Suppose then that T_α is of class C^n for some $\alpha=\beta+i\gamma$ with $|\beta|>n$. It follows that (cf. [5, Lemma 2.11]) $\|G_\alpha(t)\| \leqq C(1+|t|)^n$, and therefore

$$(1+|t|)^{-|\beta|}\|G_{\beta+i\gamma}(t)\| \leqq C(1+|t|)^{n-|\beta|} \to 0$$

as $|t| \to \infty$, contradicting Theorem 6.

Proof of Theorem 2. By (1), T_α and T_λ are similar if $\mathrm{Re}\,\alpha = \mathrm{Re}\,\lambda$. It then remains to show that T_β and T_λ are *not* similar for distinct *real* numbers β and λ. Suppose $\beta, \lambda \in \boldsymbol{R}$, $\beta \neq \lambda$, and $T_\beta = Q^{-1}T_\lambda Q$ with Q nonsingular. First, assume $|\lambda|<|\beta|$. Then, by Theorem 6,

$$\begin{aligned}(1+|t|)^{-|\beta|}\|G_\beta(t)\| &= (1+|t|)^{-|\beta|}\|Q^{-1}G_\lambda(t)Q\| \\ &\leqq e^{\pi/2}\|Q\|\;\|Q^{-1}\|(1+|t|)^{|\lambda|-|\beta|} \to 0 \quad \text{as } |t| \to \infty,\end{aligned}$$

contradicting Theorem 6.

The following argument, which was kindly communicated to me by Professor G. K. Kalisch, disposes of the case $|\lambda|=|\beta|$. Suppose $T_\beta P = PT_{-\beta}$ for P nonsingular and $\beta>0$. By Lemma 1 in [6], it follows that the compact operator $J^\beta PJ^\beta$ commutes with M, and hence must be 0, a contradiction (cf. Lemma 2 in G. Kalisch, *On isometric equivalence of certain Volterra operators*, Proc. Amer. Math. Soc. **12** (1961), 93–98).

Proof of Theorem 3. By (1), T_α is trivially spectral (of scalar type) for $\mathrm{Re}\,\alpha=0$, and we already know that T_α is not spectral for $|\mathrm{Re}\,\alpha| \geqq 1$ [6, Proposition 15]. Suppose then that T_α is spectral for some $\alpha=\beta+i\gamma$ with $0<|\beta|<1$. By (1), it follows that T_β is spectral. Since T_β is of class C^1 (Theorem 1), it is necessarily of type $\leqq 1$, i.e., $T_\beta = S+N$ with S, N commuting, S spectral of scalar type and $N^2=0$ (cf. [1]). Thus $G_\beta(t)=e^{itS}e^{itN}=e^{itS}(I+itN)$. Since S has real spectrum (the spectrum of T_β), $\|e^{itS}\| \leqq M$, and therefore, by Theorem 6, we have as $|t| \to \infty$:

$$\begin{aligned}\|N\| &= \lim\,(1+|t|)^{-1}\|I+itN\| = \lim\,(1+|t|)^{-1}\|e^{-itS}G_\beta(t)\| \\ &\leqq M \limsup\,(1+|t|)^{-1}\|G_\beta(t)\| \leqq Me^{\pi/2}\limsup\,(1+|t|)^{|\beta|-1} = 0.\end{aligned}$$

Thus $T_\beta = S$ and $(1+|t|)^{-|\beta|}\|G_\beta(t)\| \leqq M(1+|t|)^{-|\beta|} \to 0$, contradicting Theorem 6.

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