

TWO-PARAMETER GROUPS OF FORMAL POWER SERIES

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Summary. By Ω^F we denote the group of the formal power series having the form $F(z) = \sum_{q=1}^{\infty} f_q z^q$, $f_1 \neq 0$, with respect to formal composition of power series.

The problem of analytic iteration leads to the study of subgroups of Ω^F , having the form

$$F(z, s) = \sum_{q=1}^{\infty} f_q(s) z^q$$

where the coefficients $f_q(s)$ are analytic functions of the complex parameter s , such that for any two complex numbers s and t the formal law of composition

$$F[F(z, s), t] = F(z, s+t)$$

is valid [6], [8].

The purpose of the present paper is to study similar two-parameter subgroups of Ω^F . In §1 r -parameter analytic subgroups of Ω^F are defined, as well as other concepts connected with the problem. In §2 the importance of two-parameter subgroups is emphasized. It is shown that the number of parameters of analytic subgroups of Ω^F can always be reduced to two at most. The existence of a countable number of classes of the two-parameter subgroups of Ω^F is shown. §3 gives the explicit form of the coefficients $f_q(a^1, a^2)$ of a two-parameter subgroup of Ω^F :

$$F(z, a^1, a^2) = \sum_{q=1}^{\infty} f_q(a^1, a^2) z^q.$$

In §4 the existence of canonical representations for two-parameter analytic subgroups of Ω^F is proven, and it is shown that every two-parameter analytic subgroup of Ω^F is globally isomorphic to one of the groups

$$H_n(z, a^1, a^2) = (1+a^1)z/(1+a^2z^n)^{1/n}, \quad n = 1, 2, \dots$$

(no two of which are globally isomorphic to each other).

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1. Introduction and definitions.

1.1. Let Σ^F denote the linear algebra of formal power series (over the field of complex numbers) having the form

$$(1) \quad F(z) = \sum_{q=0}^{\infty} f_q z^q.$$

(Operations on formal power series are defined in [2, Chapter 1].) Σ^F is equipped with the metric

$$(2) \quad \rho(F, G) = \sum_{q=0}^{\infty} 2^{-q} \frac{|f_q - g_q|}{1 + |f_q - g_q|}$$

for $F, G \in \Sigma^F$.

Let

$$(3) \quad F(z, s) = \sum_{q=0}^{\infty} f_q(s) z^q$$

be a family of formal power series depending on a complex parameter s , $s \in D$, D being a domain in the complex plane. We say, that $F(z, s)$ is analytically dependent on s , if the limit

$$\frac{\partial F(z, s)}{\partial s} = \lim_{h \rightarrow 0} \frac{F(z, s+h) - F(z, s)}{h} = \sum_{q=0}^{\infty} f'_q(s) z^q$$

exists in the metric of Σ^F for every $s \in D$. This limit exists if and only if all the coefficients $f'_q(s)$ are analytic functions of s in D .

Let C^r denote the r -dimensional complex space. $\mathbf{a} \in C^r$ denotes a vector with r complex components: $\mathbf{a} = (a^1, a^2, \dots, a^r)$. A family of formal power series

$$F(z, \mathbf{a}) = \sum_{q=0}^{\infty} f_q(\mathbf{a}) z^q$$

is said to be analytically dependent on \mathbf{a} in a domain D of C^r , if $F(z, \mathbf{a})$ is a continuous function from D into Σ^F , and is analytically dependent on each of the components of \mathbf{a} . $F(z, \mathbf{a})$ is thus analytically dependent on \mathbf{a} , if and only if all the coefficients $f_q(\mathbf{a})$, $q=0, 1, 2, \dots$ are analytic functions of the components of \mathbf{a} in D .

1.2. Let Ω^F denote the subset of Σ^F , which contains the power series of the form

$$F(z) = \sum_{q=1}^{\infty} f_q z^q, \quad f_1 \neq 0.$$

Ω^F forms a topological group with respect to formal substitution of formal power series, and the topology induced in Ω^F from Σ^F .

The group Ω^F can be represented by infinite matrices. Indeed, for every $F(z) \in \Omega^F$ and every integer $m > 0$ put

$$(4) \quad [F(z)]^m = \sum_{q=1}^{\infty} f_{m,q} z^q.$$

The matrix $\|f_{m,q}\|$, $1 \leq m, q < \infty$ represents the series $F(z)$ in the sense, that if the matrices $\|f_{m,q}\|$ and $\|g_{m,q}\|$ represent the series $F(z)$ and $G(z)$ respectively, then the series $F[G(z)]$ is represented by the matrix $\|f_{m,q}\| \times \|g_{m,q}\|$, where \times represents matrix multiplication ([5], [9]).

By Ω we denote the subgroup of Ω^F constituted by the power series of Ω^F with a nonzero radius of convergence.

1.3. A subgroup of Ω^F is said to be an *analytic r -parameter subgroup* of Ω^F , if its elements can be written as an r -parameter family of Ω^F : $F(z, \mathbf{a}) = F(z, a^1, \dots, a^r)$, where \mathbf{a} ranges in some domain D of C^r , so that:

(1) $F(z, \mathbf{a})$ is analytically dependent on \mathbf{a} .

(2) There is a vector function $\phi(\mathbf{a}, \mathbf{b})$, that is, r scalar functions of $2r$ variables

$$\phi^1(a^1, \dots, a^r, b^1, \dots, b^r), \dots, \phi^r(a^1, \dots, a^r, b^1, \dots, b^r)$$

analytic in $a^1, \dots, a^r, b^1, \dots, b^r$ for $\mathbf{a}, \mathbf{b} \in D$, which is called the "multiplication table", such that

$$(5) \quad F[F(z, \mathbf{a}), \mathbf{b}] = F[z, \phi(\mathbf{a}, \mathbf{b})]$$

holds for every $\mathbf{a}, \mathbf{b} \in D$.

An analytic r -parameter subgroup of Ω^F is thus a connected complex Lie group. The parametrization of such a group is not unique. We shall, for convenience, consider only parametrizations satisfying: $F(z, \mathbf{0}) = z$, which implies $\mathbf{0} \in D$. It follows, that $\phi(\mathbf{a}, \mathbf{b})$ should satisfy

$$(6) \quad \phi(\mathbf{0}, \mathbf{b}) = \mathbf{b}, \quad \phi(\mathbf{a}, \mathbf{0}) = \mathbf{a}.$$

By a *local r -parameter analytic subgroup* of Ω^F we call a family of elements of Ω^F , $F(z, \mathbf{a})$, analytically dependent on \mathbf{a} in some fixed connected neighborhood D of $\mathbf{a} = \mathbf{0}$ in C^r , with $F(z, \mathbf{0}) = z$, such that (5) is satisfied with respect to some multiplication table $\phi(\mathbf{a}, \mathbf{b})$, analytic in $a^1, \dots, a^r, b^1, \dots, b^r$, whenever \mathbf{a}, \mathbf{b} , and $\phi(\mathbf{a}, \mathbf{b})$ belong to D .

A local r -parameter analytic subgroup of Ω^F is thus a complex connected local Lie group. We note, that every connected neighborhood of the identity element of an analytic r -parameter subgroup of Ω^F forms a local subgroup.

1.4. Let $F(z, \mathbf{a})$ be an analytic r -parameter local or global subgroup of Ω^F , with the multiplication table $\phi(\mathbf{a}, \mathbf{b})$. We define

$$(7) \quad V_j^i(\mathbf{a}) = (\partial \phi^i(\mathbf{a}, \mathbf{b}) / \partial b^j)_{\mathbf{b}=\mathbf{0}}.$$

Because of (6) we have

$$(8) \quad V_j^i(\mathbf{0}) = \delta_j^i.$$

The *structure constants* of the subgroup are given by

$$(9) \quad c_{jk}^i = \left(\frac{\partial v_j^i(\mathbf{a})}{\partial a^k} \right)_{\mathbf{a}=\mathbf{0}} - \left(\frac{\partial v_k^i(\mathbf{a})}{\partial a^j} \right)_{\mathbf{a}=\mathbf{0}}$$

we note that for a commutative group $\phi(\mathbf{a}, \mathbf{b}) = \phi(\mathbf{b}, \mathbf{a})$ and from (7) and (9) follows $c_{jk}^i = 0, 1 \leq i, j, k \leq r$.

1.5. Let $F(z, \mathbf{a})$ be an analytic r -parameter local or global subgroup of Ω^F . Denote

$$(10) \quad L_j(z) = \left(\frac{\partial F(z, \mathbf{a})}{\partial a^j} \right)_{\mathbf{a}=\mathbf{0}} = \sum_{q=0}^{\infty} l_q^{(j)} z^{q+1}, \quad j = 1, \dots, r.$$

(Note that the $L_j(z)$ have no free terms.)

We define the *tangent space* to the subgroup $F(z, \mathbf{a})$ to be the linear subspace of Σ^F generated by $\{L_j(z)\}_{j=1}^r$. We note that the tangent space is invariant under regular analytic transformations of the group parameter \mathbf{a} . Indeed, let another r -dimensional parameter \mathbf{s} be defined by the analytic function $\mathbf{a} = \mathbf{a}(\mathbf{s})$ with $(\partial(a^1, \dots, a^r) / \partial(s^1, \dots, s^r))_{\mathbf{s}=\mathbf{0}} \neq 0$ and $\mathbf{0} = \mathbf{a}(\mathbf{0})$. Put

$$F^*(z, \mathbf{s}) = F[z, \mathbf{a}(\mathbf{s})].$$

Differentiating the last equation we get

$$L_j^*(z) = \left(\frac{\partial F^*(z, \mathbf{s})}{\partial s^j} \right)_{\mathbf{s}=\mathbf{0}} = \sum_{q=1}^r \left(\frac{\partial a^q}{\partial s^j} \right)_{\mathbf{s}=\mathbf{0}} \cdot \left(\frac{\partial F(z, \mathbf{a})}{\partial a^q} \right)_{\mathbf{a}=\mathbf{0}} = \sum_{q=1}^r \left(\frac{\partial a^q}{\partial s^j} \right)_{\mathbf{s}=\mathbf{0}} \cdot L_q(z)$$

and hence the $\{L_j^*(z)\}$ generate the same subspace of Σ^F as the $\{L_q(z)\}$.

1.6. We shall now establish a system of formal differential equations in Σ^F for a given r -parameter analytic subgroup of $\Omega^F, F(z, \mathbf{a})$, having the multiplication table $\phi(\mathbf{a}, \mathbf{b})$ and find the integrability conditions of this system.

Differentiating (5) with respect to b^j and putting $\mathbf{b} = \mathbf{0}$ we obtain

$$(11) \quad L_j[F(z, \mathbf{a})] = \sum_{k=1}^r \frac{\partial F(z, \mathbf{a})}{\partial a^k} v_j^k(\mathbf{a}), \quad 1 \leq j \leq r.$$

In order to obtain the integrability conditions of this system we differentiate (11) with respect to a^i and put $\mathbf{a} = \mathbf{0}$. We get, using (8)

$$(12) \quad L_j'(z)L_i(z) = \left(\frac{\partial^2 F(z, \mathbf{a})}{\partial a^i \partial a^j} \right)_{\mathbf{a}=\mathbf{0}} + \sum_{k=1}^r L_k(z) \left(\frac{\partial v_j^k(\mathbf{a})}{\partial a^i} \right)_{\mathbf{a}=\mathbf{0}}.$$

Interchanging the indices i and j , we get

$$(13) \quad L_i'(z)L_j(z) = \left(\frac{\partial^2 F(z, \mathbf{a})}{\partial a^j \partial a^i} \right)_{\mathbf{a}=\mathbf{0}} + \sum_{k=1}^r L_k(z) \left(\frac{\partial v_i^k(\mathbf{a})}{\partial a^j} \right)_{\mathbf{a}=\mathbf{0}}.$$

Subtracting (13) from (12), using (9), we get

$$(14) \quad L_j'(z)L_i(z) - L_i'(z)L_j(z) = \sum_{k=1}^r c_{ji}^k L_k(z), \quad 1 \leq i, j \leq r.$$

We shall refer to the system (14) as the *integrability conditions of the group* $F(z, \mathbf{a})$.

2. Essential parameters and classification of analytic subgroups of Ω^F .

2.1. Let $F(z, \mathbf{a}) = F(z, a^1, \dots, a^r)$ be an analytic r -parameter local or global subgroup of Ω^F . The parameters a^1, \dots, a^r are said to be *essential*, if the group $F(z, \mathbf{a})$ cannot be represented as an $(r-1)$ -parameter group; that is, if it is impossible to find $r-1$ analytic functions: $A^j = A^j(a^1, \dots, a^r)$, $1 \leq j \leq r-1$ such that

$$F(z, a^1, \dots, a^r) \equiv F(z, A^1, \dots, A^{r-1}).$$

We cite the following criterion for the parameters a^1, \dots, a^r to be essential [3, Chapter 1, §3]:

Let $F(z, \mathbf{a}) = F(z, a^1, \dots, a^r)$ be a family of series of Ω^F , analytically dependent on a^1, \dots, a^r . A necessary and sufficient condition for the parameters a^1, \dots, a^r to be essential is that the equation

$$(15) \quad \sum_{k=1}^r \psi_k(\mathbf{a}) \frac{\partial F(z, \mathbf{a})}{\partial a^k} \equiv 0$$

where $\psi_k(\mathbf{a})$ are analytic functions of a^i , $1 \leq j \leq r$, be satisfied identically for all \mathbf{a} if and only if $\psi_k(\mathbf{a}) \equiv 0$ for $k=1, \dots, r$ for all \mathbf{a} .

As a direct consequence we have:

LEMMA 1. Let $F(z, \mathbf{a})$ be an analytic r -parameter local or global subgroup of Ω^F . If for some j ,

$$(16) \quad L_j(z) = \left(\frac{\partial F(z, \mathbf{a})}{\partial a^j} \right)_{\mathbf{a}=\mathbf{0}} \equiv 0,$$

then the group parameters a^1, \dots, a^r are not essential.

Proof. From (16) and (11) it follows that:

$$\sum_{k=1}^r V_j^k(\mathbf{a}) \frac{\partial F(z, \mathbf{a})}{\partial a^k} \equiv 0,$$

while from (8) we know that $V_j^j(\mathbf{0}) = 1 \neq 0$; hence from the cited criterion follows that the parameters a^1, \dots, a^r are not essential.

2.2. We shall show now, that the maximal number of essential parameters in an analytic local or global subgroup of Ω^F is 2. This results from a theorem due to L. Bianchi [1]; we bring, though, an elementary proof due to E. Jabotinsky.

THEOREM 1. Let $F(z, a^1, \dots, a^r)$ be an analytic r -parameter local or global subgroup of Ω^F . If the parameters a^1, \dots, a^r are essential, then $r \leq 2$.

Proof. Let $F(z, a^1, \dots, a^r)$ be an analytic r -parameter local or global subgroup of Ω^F , where a^1, \dots, a^r are essential parameters. By a regular linear transformation of the parameters:

$$a^j = \sum_{i=1}^r \alpha_{ij} a^{*i}, \quad j = 1, \dots, r,$$

the group will have a new parametrization:

$$F^*(z, a^{*1}, \dots, a^{*r}) \equiv F(z, a^1, \dots, a^r).$$

The tangent vectors $L_j^*(z)$ are linear combinations of the $L_k(z)$:

$$L_j^*(z) = \left(\frac{\partial F^*(z, \mathbf{a}^*)}{\partial a^{*j}} \right)_{\mathbf{a}^* = \mathbf{0}} = \sum_{k=1}^r \left(\frac{\partial a^k}{\partial a^{*j}} \right)_{\mathbf{a}^* = \mathbf{0}} \left(\frac{\partial F(z, \mathbf{a})}{\partial a^k} \right)_{\mathbf{a} = \mathbf{0}} = \sum_{k=1}^r \alpha_{kj} L_k(z).$$

By a suitable choice of the matrix $\|\alpha_{i,j}\|$ we can obtain a system $\{L_j^*(z)\}$ in which the lowest power of z with a nonzero coefficient will be different in any two of the series $L_j^*(z)$, and that in every one of the series $L_j^*(z)$ the first nonzero coefficient will be equal to 1, that is, for every j we will have: $L_j^*(z) = z^{p_j} + \dots$, where $i \neq j$ implies $p_i \neq p_j$. The assumption that the parameters are essential implies, by Lemma 1, that $L_j^*(z) \neq 0$ for $1 \leq j \leq r$. Let i be the index for which $p_i = \max \{p_1, \dots, p_r\}$. The integrability condition of the group (14) for this i and any $j \neq i$ is:

$$(17) \quad L_i^*(z)L_j^{*'}(z) - L_i^{*'}(z)L_j^*(z) = \sum_{k=1}^r c_{ji}^{*k} L_k^*(z).$$

Equate now the lowest powers of z in both sides of (17). On the left-hand side the lowest power of z with a nonzero coefficient is:

$$p_j z^{p_i} z^{p_j-1} - p_i z^{p_j} z^{p_i-1} = (p_j - p_i) z^{p_i + p_j - 1}.$$

If we denote by z^{p_q} the lowest power of z with a nonzero coefficient on the right-hand side of (17) we get:

$$(18) \quad p_i + p_j - 1 = p_q.$$

From $p_i = \max \{p_1, \dots, p_r\}$ it follows that $p_j - 1 \leq 0$ or, as $L_j^*(z)$ has no constant term: $p_j = 1$. This should hold for every $j, j \neq i$; so all the $L_j^*(z), j \neq i$, should start with the term z . But as each of the $L_j^*(z)$ starts with a different power of z , there may exist only one $L_j^*(z)$ in addition to $L_i^*(z)$, and hence $r \leq 2$, completing the proof.

2.3. The one-parameter analytic subgroups of Ω^F were studied by E. Jabotinsky [6] and by the present author [8]. It turns out that although all the one-parameter analytic subgroups are locally isomorphic, there are two types of such groups, which are not globally isomorphic: The simply connected type and the nonsimply connected type.

In view of Theorem 1 there remains only the study of two-parameter analytic subgroups of Ω^F . There are only two types of two-dimensional Lie groups which are not locally isomorphic: The commutative groups and the noncommutative groups [1]. In a commutative two-parameter analytic subgroup of Ω^F the parameters are not essential. Indeed, as all the structure constants (9) of a commutative group are zero, the integrability condition (14) of the group takes the form:

$$L_1'(z)L_2(z) - L_2'(z)L_1(z) = 0$$

and this implies $L_1(z) = kL_2(z)$, where k is a complex number. After a linear transformation of the parameters:

$$a^1 = a^{*1}, \quad a^2 = -ka^{*1} + a^{*2}$$

we get $L_1^*(z) = 0$, and hence, by Lemma 1, the group parameters are not essential.

2.4. There thus remains to be studied only the case of the noncommutative two-parameter subgroups. All noncommutative two-dimensional Lie groups are locally isomorphic, and one can, by transformation of the parameters in some neighborhood of the identity element of a noncommutative two-parameter analytic local subgroup of Ω^F , bring its multiplication table to the form:

$$(19) \quad \phi_n^1(a^1, a^2, b^1, b^2) = a^1 + b^1 + a^1b^1, \quad \phi_n^2(a^1, a^2, b^1, b^2) = a^2 + b^2(1 + a^1)^n,$$

where n is a fixed positive integer. The multiplication table $\phi_n(\mathbf{a}, \mathbf{b})$ defined by (19) shall be called the *standard multiplication table of type n* . An example of a global two-parameter analytic subgroup of Ω^F having the multiplication table (19) is given by the series representing:

$$F(z, a^1, a^2) = (1 + a^1)z / (1 + a^2z^n)^{1/n}$$

where the parameters range in the domain: $D = \{(a^1, a^2) \mid a^1 \neq -1\}$. (This example shows that $\phi_n(\mathbf{a}, \mathbf{b})$ defined by (19) really define a multiplication table of a group.) An example of a local two-parameter analytic subgroup of Ω^F having the standard multiplication table of type 1 is given by

$$G(z, a^1, a^2) = (1 + a^1)^{1/2}z / (1 + a^2z^2)^{1/2}$$

where the parameters range in the domain $B = \{(a^1, a^2) \mid |1 + a^1| < 1\}$.

2.5. Let $F(z, a^1, a^2)$ be a two-parameter analytic subgroup of Ω^F , having the multiplication table $\phi_n(\mathbf{a}, \mathbf{b})$ given by (19). From (7) we get

$$V_1^1(\mathbf{a}) = 1 + a^1, \quad V_2^1(\mathbf{a}) = 0, \quad V_1^2(\mathbf{a}) = 0, \quad V_2^2(\mathbf{a}) = (1 + a^1)^n.$$

From here and from (9) we get the structure constants

$$c_{1,1}^1 = c_{2,2}^1 = c_{1,1}^2 = c_{2,2}^2 = c_{1,2}^1 = c_{2,1}^1 = 0, \quad c_{1,2}^2 = -n, \quad c_{2,1}^2 = n.$$

The integrability condition (14) becomes:

$$(20) \quad L_2'(z)L_1(z) - L_1'(z)L_2(z) = nL_2(z).$$

We shall use equation (20) to obtain some information about the $L_i(z)$, $i = 1, 2$. Using the notation (10) for the terms of $L_i(z)$, we shall compare the lowest powers of z with nonzero coefficients on both sides of (20). Suppose that the first nonzero coefficient of $L_1(z)$ is $l_{r-1}^{(1)}$ (the coefficient of z^r) and the first nonzero coefficient of $L_2(z)$ is $l_{s-1}^{(2)}$. The first nonzero term on the right in (20) is then $nl_{s-1}^{(2)}z^s$. The first nonzero term on the left in (20) is

$$(s-r)l_{s-1}^{(2)}l_{r-1}^{(1)}z^{s+r-1}.$$

We note that the coefficient of this term cannot vanish, because otherwise we get by comparing with the other side of (20): $s+r-1 < s$, that is, $r < 1$, which is impossible, as the $L_i(z)$ have no constant terms. Hence

$$(21) \quad (s-r)l_{s-1}^{(2)}l_{r-1}^{(1)}z^{s+r-1} = nl_{s-1}^{(2)}z^s.$$

Comparing the powers of z we get $r=1$, that is, $l_0^{(1)} \neq 0$. Putting $r=1$ into (21) and dividing by $l_{s-1}^{(2)}$ (which, by assumption is not 0) we get

$$(22) \quad (s-1)l_0^{(1)} = n$$

hence $s > 1$, that is, $l_0^{(2)} = 0$.

We say, that a two-parameter analytic (local or global) subgroup of Ω^F , having the multiplication table (19) is of class m , if $l_0^{(2)} = l_1^{(2)} = \dots = l_{m-1}^{(2)} = 0$, $l_m^{(2)} \neq 0$. From (22) it follows that for an analytic two-parameter subgroup of Ω^F of class m , having the multiplication table of type n (9), we have: $l_0^{(1)} = n/m$. In particular we have:

LEMMA 2. *Let $F(z, a^1, a^2)$ be an analytic two-parameter (local or global) subgroup of Ω^F of class n , and having the standard multiplication table of type n . Then $l_0^{(1)} = 1$, $l_0^{(2)} = 0$.*

For every positive integer n there actually exists a two-parameter analytic subgroup of Ω^F of the class n ; for example:

$$F(z, a^1, a^2) = (1 + a^1)z / (1 + a^2z^n)^{1/n}.$$

Indeed, for this group (which has the standard multiplication table of type n) we have

$$L_1(z) = z, \quad L_2(z) = -z^{n+1}/n.$$

3. Explicit form of the coefficients of an analytic two-parameter subgroup of Ω^F .

3.1. Let

$$F(z, a^1, a^2) = \sum_{q=1}^{\infty} f_q(a^1, a^2)z^q$$

be an analytic two-parameter subgroup of Ω^F . We propose to find an explicit expression for the coefficients $f_q(a^1, a^2)$, or, more generally, to find an explicit expression for the elements of the matrix $\|f_{m,q}(a^1, a^2)\|$ representing $F(z, a^1, a^2)$ (which contains the coefficients $f_q(a^1, a^2)$ as its first row).

The corresponding coefficients for the one-parameter analytic subgroups of Ω^F were obtained by E. Jabotinsky [6] for the simply connected subgroups, and by the author [8] for the nonsimply connected subgroups. It turned out, that in order to study the nonsimply connected subgroups, it was convenient to use a different multiplication table than the one used for the study of the simply connected subgroups.

In studying two-parameter subgroups of Ω^F we shall be led to use a different

multiplication table for the study of each class of subgroups—namely, we shall use the standard multiplication table of type n to study analytic two-parameter subgroups of Ω^F of class n .

Remembering that in every two-parameter analytic subgroup of Ω^F with essential parameters we can introduce in some neighborhood of the identity element a parametrization with the standard multiplication table (19) of type n , we can state the theorem:

THEOREM 2. *Let G be an analytic two-parameter (local or global) subgroup of Ω^F with essential parameters, and of class n . Let D be a neighborhood of $\mathbf{a}=\mathbf{0}$ in C^2 where the standard multiplication table of type n can be introduced for some local subgroup $F(z, a^1, a^2)$ of G . Put*

$$\left(\frac{\partial F(z, a^1, a^2)}{\partial a^i}\right)_{\mathbf{a}=\mathbf{0}} = \sum_{q=1}^{\infty} l_{q-1}^{(i)} z^q, \quad i = 1, 2.$$

Then, for $(a^1, a^2) \in D$, the elements of the matrix $\|f_{m,p}(a^1, a^2)\|$ which represents the group are given by:

$$(23) \quad f_{m,p}(a^1, a^2) = (1 + a^1)^m \delta_{m,p}, \quad p \leq m$$

and for $m < p$:

$$(24) \quad f_{m,p}(a^1, a^2) = \sum \frac{(a^1)^s}{s!} \frac{(a^2)^r}{r!} T_{s,j} \prod_{v=1}^j q_{v-1} l_{d_v}^{(1)} \prod_{\sigma=j+1}^{j+r} q_{\sigma-1} l_{d_\sigma}^{(2)},$$

where the sum is taken over all integers $r, s: 0 \leq s \leq p, 0 \leq r \leq p - m, r + s > 0$, and for every integer $j: 0 \leq j \leq s$, and all integers q_i such that:

$$m = q_0 \leq q_1 \leq \dots \leq q_j < q_{j+1} < \dots < q_{j+r} = p$$

and where $d_i = q_i - q_{i-1}$.

The $T_{s,j}$ are the Stirling numbers of the first kind, defined by the recurrence formulas:

$$(25) \quad T_{0,j} = \delta_{0,j} \quad \text{and} \quad T_{s+1,j} = T_{s,j-1} - sT_{s,j}$$

If the parameters a^1, a^2 are permitted to range through the set $\{(a^1, a^2) \mid a^1 \neq -1\}$, formulas (23), (24) define a global two-parameter group H , which contains G if G was a local group, and coincides with G if G was global.

(Note. Further information about Stirling's numbers can be found in [7, Chapter IV].)

It is interesting to note, that formulas (23), (24) do not contain explicitly the number n , which is the class of G and the type of the standard multiplication table used.

The proof of our Theorem 2 will be carried out in several steps.

3.2. Let us introduce the following notation: given an analytic local or global two-parameter subgroup of Ω^F

$$F(z, \mathbf{a}) = \sum_{q=1}^{\infty} f_q(\mathbf{a})z^q,$$

we define, for every positive integer t :

$$(26) \quad [F(z, \mathbf{a})]^t = \sum_{q=0}^{\infty} f_q(\mathbf{a}, t)z^{q+t} = z^t \sum_{q=0}^{\infty} f_q(\mathbf{a}, t)z^q.$$

Note that, t being a positive integer, $f_q(\mathbf{a}, t) = f_{t, q+t}(\mathbf{a})$ where $f_{m,n}(\mathbf{a})$ is the appropriate element of the representation matrix. We also note, that if the series $F(z, \mathbf{a})$ has a nonzero radius of convergence for every \mathbf{a} (that is, if $F(z, \mathbf{a})$ is a subgroup of Ω), equation (26) is well defined for every complex t , with an appropriate choice of the branch. All the propositions that we are going to state about the coefficients $f_q(\mathbf{a}, t)$ remain valid for this case, too.

We first prove:

LEMMA 3. For an analytic two-parameter local subgroup of Ω^F , $F(z, a^1, a^2)$ having the multiplication table $\Phi(\mathbf{a}, \mathbf{b})$, and the tangent vectors:

$$\left(\frac{\partial F(z, a^1, a^2)}{\partial a^i}\right)_{\mathbf{a}=\mathbf{0}} = \sum_{q=1}^{\infty} l_{q-1}^{(i)}z^q, \quad i = 1, 2,$$

we have

$$(27) \quad f_p[\Phi(\mathbf{a}, \mathbf{b}), t] = \sum_{q=0}^p f_q(\mathbf{b}, t)f_{p-q}(\mathbf{a}, q+t),$$

and also

$$(28) \quad (\partial f_q(\mathbf{a}, t)/\partial a^i)_{\mathbf{a}=\mathbf{0}} = tl_q^{(i)}.$$

Proof. We rewrite the group property (5), raising both sides to the power t :

$$\{F[z, \Phi(\mathbf{a}, \mathbf{b})]\}^t = \{F[F(z, \mathbf{a}), \mathbf{b}]\}^t$$

or, using (26):

$$\sum_{p=0}^{\infty} f_p[\Phi(\mathbf{a}, \mathbf{b}), t]z^{p+t} = \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} f_q(\mathbf{b}, t)f_s(\mathbf{a}, q+t)z^{s+q+t}$$

from which, by equating coefficients of powers of z , (27) follows.

To get (28), we differentiate (26) with respect to a^i , and put $\mathbf{a}=\mathbf{0}$. As $F(z, \mathbf{0})=z$ we get:

$$tz^{t-1} \sum_{q=0}^{\infty} l_{q-1}^{(i)}z^q = \sum_{q=0}^{\infty} \left(\frac{\partial f_q(\mathbf{a}, t)}{\partial a^i}\right)_{\mathbf{a}=\mathbf{0}} z^{q+t}$$

from which (28) follows.

3.3. We now prove

LEMMA 4. Let the analytic two-parameter local or global subgroup of Ω^F , $F(z, a^1, a^2)$, be of class n and be written in the standard multiplication table of type n . Then the corresponding coefficients $f_q(a^1, a^2, t)$ have the form:

$$(29) \quad f_q(a^1, a^2, t) = (1 + a^1)^t \cdot P_q(a^1, a^2, t)$$

where $P_q(a^1, a^2, t)$ are polynomials of degree not exceeding q in each of the parameters a^1 and a^2 .

Proof. I. We first seek the explicit expression for $f_0(a^1, a^2, t)$. Equation (27), written for standard multiplication table of type n and for $p=0$ becomes

$$(30) \quad f_0[a^1 + b^1 + a^1 b^1, a^2 + b^2(1 + a^1)^n, t] = f_0(b^1, b^2, t) f_0(a^1, a^2, t).$$

Differentiating with respect to a^2 and putting $\mathbf{a} = \mathbf{0}$ we get, using (28):

$$(31) \quad \partial f_0(b^1, b^2, t) / \partial b^2 = f_0(b^1, b^2, t) l_0^{(2)} t.$$

As by Lemma 2: $l_0^{(2)} = 0$, it follows from (31) that $f_0(b^1, b^2, t)$ is actually independent of b^2 .

Differentiating now (30) with respect to b^1 , and putting $\mathbf{b} = \mathbf{0}$, we get, using (28):

$$(1 + a^1) \partial f_0(a^1, a^2, t) / \partial a^1 = t l_0^{(1)} f_0(a^1, a^2, t)$$

which implies, as $f_0(a^1, a^2, t)$ is independent of a^2 ,

$$(32) \quad f_0(a^1, a^2, t) = (1 + a^1)^{t l_0^{(1)}} K(t).$$

Putting $\mathbf{a} = \mathbf{0}$ and noting that $f_0(0, 0, t) = 1$ we find $k(t) \equiv 1$. By Lemma 2 we have $l_0^{(1)} = 1$, so that, finally:

$$(33) \quad f_0(a^1, a^2, t) = (1 + a^1)^t.$$

The lemma is thus valid for $q=0$.

II. We proceed, by induction on q , to show that $f_q(a^1, a^2, t)$ is a polynomial in a^2 of degree not exceeding q . The case $q=0$ is covered already; suppose that so is the case for $q=0, 1, \dots, p-1$.

Differentiate (27), where $\Phi(\mathbf{a}, \mathbf{b})$ is taken to be the standard multiplication table of type n , with respect to a^2 , and put $\mathbf{a} = \mathbf{0}$. Because of (28) we get

$$\frac{\partial f_p(b^1, b^2, t)}{\partial b^2} = \sum_{q=0}^{p-1} f_q(b^1, b^2, t) l_{p-q}^{(2)}(q+t).$$

(The summation goes only up to $q=p-1$, because $l_0^{(2)}=0$.) By the induction hypothesis, each term in the sum is a polynomial of degree not exceeding $p-1$ in b^2 ; hence $f_p(b^1, b^2, t)$ is a polynomial of degree not exceeding p in b^2 . Hence, in (29) $P_q(a^1, a^2, t)$, qua function of a^2 , is a polynomial of degree not exceeding q .

III. It remains to prove that $P_q(a^1, a^2, t)$, qua function of a^1 , is also a polynomial of degree not exceeding q . We shall complete the proof again by induction on q . Equation (33) proves the statement for $q=0$. Suppose that the statement

holds for $q=0, 1, \dots, p-1$. Differentiate (27), where $\Phi(\mathbf{a}, \mathbf{b})$ is taken to be the standard multiplication table of type n , with respect to b^1 and put $\mathbf{b}=\mathbf{0}$; because of (28) we get

$$(1+a^1) \frac{\partial f_p(a^1, a^2, t)}{\partial a^1} = \sum_{q=0}^p t l_q^{(1)} f_{p-q}(a^1, a^2, q+t).$$

Transfer to the left the term which contains $f_p(a^1, a^2, t)$, remembering that, by Lemma 2, $l_0^{(1)}=1$. Applying the induction hypothesis to the sum on the right, we get

$$\begin{aligned} (1+a^1) \frac{\partial f_p(a^1, a^2, t)}{\partial a^1} - t f_p(a^1, a^2, t) \\ = (1+a^1)^{t+1} \sum_{q=1}^p t l_q^{(1)} P_{p-q}(a^1, a^2, q+t) (1+a^1)^{q-1} \end{aligned}$$

where $P_{p-q}(a^1, a^2, q+t)$ is a polynomial in a^1 of degree not exceeding $p-q$. The last equation can be rewritten, after dividing both sides by $(1+a^1)^{t+1}$, in the form:

$$\frac{\partial}{\partial a^1} \left\{ \frac{f_p(a^1, a^2, t)}{(1+a^1)^t} \right\} = \sum_{q=1}^p t l_q^{(1)} P_{p-q}(a^1, a^2, q+t) (1+a^1)^{q-1}$$

hence, using the induction hypothesis, we find that

$$P_p(a^1, a^2, t) = f_p(a^1, a^2, t)/(1+a^1)^t$$

is a polynomial of degree not exceeding p in a^1 ; this completes the proof of the lemma.

3.4. We now state:

LEMMA 5. Let $F(z, a^1, a^2)$ be a local or global analytic two-parameter subgroup of Ω^F , with a standard multiplication table of arbitrary type. Then the derivatives of $f_p(a^1, a^2, t)$ with respect to a^1 are given by:

$$(34) \quad \frac{\partial^s f_p(a^1, a^2, t)}{\partial (a^1)^s} = \frac{1}{(1+a^1)^s} \sum T_{s,j} \prod_{v=1}^j (q_{v-1}+t) l_{d_v}^{(1)} f_{p-q_j}(a^1, a^2, q_j+t) \quad (s \geq 1)$$

where the sum is taken for every integer $j: 1 \leq j \leq s$ and for all integers q_i such that $0=q_0 \leq q_1 \leq \dots \leq q_j \leq p$ and where $d_v = q_v - q_{v-1}$; and the derivatives of $f_p(a^1, a^2, t)$ with respect to a^2 are given by

$$(35) \quad \frac{\partial^r f_p(a^1, a^2, t)}{\partial (a^2)^r} = \sum \prod_{\sigma=1}^r (k_\sigma+t) l_{e_\sigma}^{(2)} f_{k_r}(a^1, a^2, t) \quad (r \geq 1)$$

where the sum is taken for all integers k_i such that

$$0 \leq k_r < k_{r-1} < \dots < k_1 < k_0 = p,$$

and where $e_\sigma = k_{\sigma-1} - k_\sigma$.

Proof. (I) We shall first prove (34) by induction on s . Differentiate (27), where $\phi(\mathbf{a}, \mathbf{b})$ is taken to be any standard multiplication table of the form (19), with respect to b^1 and put $\mathbf{b} = \mathbf{0}$; because of (28) we get

$$(36) \quad (1 + a^1) \frac{\partial f_p(a^1, a^2, t)}{\partial a^1} = \sum_{q=0}^p t l_q^{(1)} f_{p-q}(a^1, a^2, q+t)$$

which is (34) for the case $s=1$ (because $T_{1,1} = 1$).

Assuming (34) to hold for the derivative of order s , we differentiate it to compute the derivative of order $s+1$:

$$\begin{aligned} \frac{\partial^{s+1} f_p(a^1, a^2, t)}{\partial (a^1)^{s+1}} &= \frac{1}{(1 + a^1)^s} \sum T_{s,j} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} \frac{\partial}{\partial a^1} [f_{p-q_j}(a^1, a^2, q_j + t)] \\ &\quad - \frac{s}{(1 + a^1)^{s+1}} \sum T_{s,j} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} f_{p-q_j}(a^1, a^2, q_j + t) \end{aligned}$$

where the sums are taken as in (34). By (36), we have

$$(37) \quad (1 + a^1) \frac{\partial}{\partial a^1} [f_{p-q_j}(a^1, a^2, q_j + t)] = \sum_{q=0}^{p-q_j} (q_j + t) l_q^{(1)} f_{p-q_j-q}(a^1, a^2, q + q_j + t).$$

In the last sum we put $q_{j+1} = q_j + q$, so that $q_j \leq q_{j+1} \leq p$. Substituting (37) with the new index q_{j+1} into the first sum of the expression for the $(s+1)$ th derivative, we get

$$\begin{aligned} \frac{\partial^{s+1} f_p(a^1, a^2, t)}{\partial (a^1)^{s+1}} &= \frac{1}{(1 + a^1)^{s+1}} \sum T_{s,j} \prod_{v=1}^{j+1} (q_{v-1} + t) l_{d_v}^{(1)} f_{p-q_{j+1}}(a^1, a^2, q_{j+1} + t) \\ &\quad - \frac{1}{(1 + a^1)^{s+1}} \sum s T_{s,j} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} f_{p-q_j}(a^1, a^2, q_j + t) \end{aligned}$$

where the first sum is taken for every $j: 1 \leq j \leq s$ and for all integers q_i such that: $0 \leq q_0 \leq q_1 \leq \dots \leq q_j \leq q_{j+1} \leq p$ and where $d_v = q_v - q_{v-1}$; the second sum is taken as in (34).

In the first sum we replace j by $j-1$ and remember that $T_{s,0} = 0$. In the second sum we allow j to range from 1 to $s+1$, as $T_{s,s+1} = 0$. The expression for the $(s+1)$ th derivative becomes:

$$\begin{aligned} \frac{\partial^{s+1} f_p(a^1, a^2, t)}{\partial (a^1)^{s+1}} &= \frac{1}{(1 + a^1)^{s+1}} \sum T_{s,j-1} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} f_{p-q_j}(a^1, a^2, q_j + t) \\ &\quad - \frac{1}{(1 + a^1)^{s+1}} \sum s T_{s,j} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} f_{p-q_j}(a^1, a^2, q_j + t) \end{aligned}$$

where the sums are taken for all integers $j: 1 \leq j \leq s+1$ and all integers q_i such that: $0 \leq q_0 \leq q_1 \leq \dots \leq q_{j-1} \leq q_j \leq p$ and where $d_v = q_v - q_{v-1}$.

Finally, using (25), we get

$$\frac{\partial^{s+1} f_p(a^1, a^2, t)}{\partial (a^1)^{s+1}} = \frac{1}{(1 + a^1)^{s+1}} \sum T_{s+1,j} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} f_{p-q_j}(a^1, a^2, q_j + t)$$

where the sum is taken as before. This is the statement of the lemma for the $(s + 1)$ th derivative with respect to a^1 , and formula (34) is proved.

(II) We shall prove now (35) by induction on r . Differentiate (27), where $\Phi(\mathbf{a}, \mathbf{b})$ is taken to be any standard multiplication table of the form (19), with respect to a^2 and put $\mathbf{a} = \mathbf{0}$; we get, after replacing \mathbf{b} by \mathbf{a} :

$$(38) \quad \frac{\partial f_p(a^1, a^2, t)}{\partial a^2} = \sum_{q=0}^{p-1} (q+t) l_{p-q}^{(2)} f_q(a^1, a^2, t).$$

(The summation is only up to $q = p - 1$ because $l_0^{(2)} = 0$.) Formula (38) coincides with (35) for $r = 1$. Assume (35) to be valid for the r th derivative. Differentiation of (35) with respect to a^2 yields

$$\frac{\partial^{r+1} f_p(a^1, a^2, t)}{\partial (a^2)^{r+1}} = \prod_{\sigma=1}^r (k_\sigma + t) l_{e_\sigma}^{(2)} \frac{\partial}{\partial a^2} f_{k_\sigma}(a^1, a^2, t)$$

where the sum is taken as in (35). Applying (38) we get

$$\frac{\partial^{r+1} f_p(a^1, a^2, t)}{\partial (a^2)^{r+1}} = \sum \prod_{\sigma=1}^r (k_\sigma + t) l_{e_\sigma}^{(2)} \sum_{k_{r+1}=0}^{k_r-1} (k_{r+1} + t) l_{k_r-k_{r+1}}^{(2)} f_{k_{r+1}}(a^1, a^2, t)$$

where the sums are taken as in (35); the last equality may be rewritten in the form

$$\frac{\partial^{r+1} f_p(a^1, a^2, t)}{\partial (a^2)^{r+1}} = \sum \prod_{\sigma=1}^{r+1} (k_\sigma + t) l_{e_\sigma}^{(2)} f_{k_{r+1}}(a^1, a^2, t)$$

where the sum is taken for all positive integers k_i such that $0 \leq k_{r+1} < k_r < \dots < k_0 = p$ and where $e_\sigma = k_{\sigma-1} - k_\sigma$. As this is the statement of the lemma for the $(s + 1)$ th derivative with respect to a^2 , the proof is complete.

3.5. We can now proceed to obtain the Maclaurin expansion of $f_p(a^1, a^2, t)$. Combining the formulas (34) and (35), we obtain:

$$(39) \quad \frac{\partial^{s+r} f_p(a^1, a^2, t)}{\partial (a^1)^s \partial (a^2)^r} = \frac{1}{(1+a^1)^s} \sum T_{s,j} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} \prod_{\sigma=1}^r (k_\sigma + t) l_{e_\sigma}^{(2)} f_{k_r - q_j}(a^1, a^2, q_j + t) \quad (s+r > 0)$$

where the sum is taken for all integers $j: 0 \leq j \leq s$ and for all integers q_i and k_i such that:

$$0 = q_0 \leq q_1 \leq \dots \leq q_j \leq k_r < k_{r-1} < \dots < k_1 < k_0 = p$$

and where $d_v = q_v - q_{v-1}$, $e_\sigma = k_{\sigma-1} - k_\sigma$ (and $\prod_{v=1}^0 A_v = 1$).

In (39) put $\mathbf{a} = \mathbf{0}$. The only terms which do not vanish are those for which $k_r = q_j$. Noting that $f_0(0, 0, q_j + t) = 1$, we get

$$(40) \quad \frac{\partial^{s+r} f_p(0, 0, t)}{\partial (a^1)^s \partial (a^2)^r} = \sum T_{s,j} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} \prod_{\sigma=1}^r (k_\sigma + t) l_{e_\sigma}^{(2)}$$

where the sum is taken for all integers $j: 0 \leq j \leq s$ and for all integers q_i and k_i such that

$$0 \leq q_0 \leq q_1 \leq \dots \leq q_j = k_r < k_{r-1} < \dots < k_1 < k_0 = p$$

and where $d_v = q_v - q_{v-1}$, $e_\sigma = k_{\sigma-1} - k_\sigma$.

Renaming the integers k_α , $\alpha = 0, 1, \dots, r$ by writing $k_\alpha = q_{j+r-\alpha}$ we obtain (40) in the form

$$(41) \quad \frac{\partial^{s+r} f_p(0, 0, t)}{\partial (a^1)^s \partial (a^2)^r} = \sum T_{s,j} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} \prod_{\sigma=j+1}^{r+j} (q_{\sigma-1} + t) l_{e_\sigma}^{(2)}$$

where the sum is taken for every integer $j: 0 \leq j \leq s$, and for all integers q_i such that:

$$0 \leq q_0 \leq q_1 \leq \dots \leq q_j < q_{j+1} < \dots < q_{j+r} = p$$

and where $d_i = q_i - q_{i-1}$.

Using this, equation (32), and, for $p > 0$ Maclaurin's formula:

$$f_p(a^1, a^2, t) = \sum_{s,r=0}^{\infty} \frac{(a^1)^s}{s!} \frac{(a^2)^r}{r!} \frac{\partial^{s+r} f_p(0, 0, t)}{\partial (a^1)^s \partial (a^2)^r}$$

(noting that $f_p(0, 0, t) = 0$ for $p > 0$) we obtain:

LEMMA 6. Let $F(z, a^1, a^2)$ be a local or global analytic two-parameter subgroup of Ω^F , with a standard multiplication table of arbitrary type. Then the coefficients $f_q(a^1, a^2, t)$ are given by

$$(42) \quad f_0(a^1, a^2, t) = (1 + a^1)^{(1)t}$$

and for $p \geq 1$:

$$(43) \quad f_p(a^1, a^2, t) = \sum \frac{(a^1)^s}{s!} \frac{(a^2)^r}{r!} T_{s,j} \prod_{v=1}^j (q_{v-1} + t) l_{d_v}^{(1)} \prod_{\sigma=j+1}^{r+j} (q_{\sigma-1} + t) l_{e_\sigma}^{(2)}$$

where the sum is taken over all integers $r, s: 0 \leq s, r < \infty, s+r > 0$, and for every integer $j: 0 \leq j \leq s$ and for all integers q_i such that

$$0 = q_0 \leq q_1 \leq \dots \leq q_j < q_{j+1} < \dots < q_{j+r} = p$$

and where $d_i = q_i - q_{i-1}$.

We note that Lemma 6 holds for all integer values of t , and when $F(z, a^1, a^2)$ is a subgroup of Ω —for all complex values of t , with the appropriate choice of the branch for $f_q(a^1, a^2, t)$ in (26). We shall use Lemma 6 only for the case where t is a positive integer, but it can be used to yield additional information about the subgroup, as the elements of the generalized representation matrix $\|f_{m,q}(a^1, a^2)\|$, $-\infty < m, q < \infty$, defined by:

$$[F(z, a^1, a^2)]^m = \sum_{q=-\infty}^{\infty} f_{m,q}(a^1, a^2) z^q$$

(see [5]).

3.6. **Proof of Theorem 2.** Using the relations $f_{m,p}(a^1, a^2) = f_{p-m}(a^1, a^2, m)$ for $p \geq m$ and $f_{m,p}(a^1, a^2) = 0$ for $p < m$ we get from Lemma 6:

$$(44) \quad f_{m,p}(a^1, a^2) = (1 + a^1)l_0^{(1)m} \cdot \delta_{m,p} \quad \text{for } p \leq m,$$

and for $p > m$:

$$(45) \quad f_{m,p}(a^1, a^2) = \sum \frac{(a^1)^s}{s!} \frac{(a^2)^r}{r!} T_{s,j} \prod_{v=1}^j (q_{v-1} + m)l_{d_v}^{(1)} \prod_{\sigma=j+1}^{r+j} (q_{\sigma-1} + m)l_{d_\sigma}^{(2)}$$

where the sum is taken over all integers $r, s: 0 \leq r, s < \infty, r + s > 0$, and over all integers $j: 0 \leq j \leq s$, and for all integers q_i such that

$$0 \leq q_0 \leq q_1 \leq \dots \leq q_j < q_{j+1} < \dots < q_{j+r} = p - m$$

and where $d_i = q_i - q_{i-1}$.

This is valid for a subgroup of any class, written in the standard multiplication table of any type. For groups of class n written in the standard multiplication table of type n we have, by Lemma 2, $l_0^{(1)} = 1$. Formula (23) in the statement of Theorem 2 now follows from (44). From Lemma 4 it follows, that for such a group, $f_{p-m}(a^1, a^2, m)$ which is $f_{m,p}(a^1, a^2)$, will be a polynomial of degree not exceeding p in a^1 , and of degree $p - m$ at most in a^2 ; hence, in (45), the summations over s and r only include terms for $0 \leq s \leq p$ and $0 \leq r \leq p - m$, yielding (24).

(It is interesting to remark that we have hereby shown that the summation figuring in (45) taken for a fixed value of s , such that $s > p$, vanish. This constitutes a combinatorial theorem already noted by the author [8], and for which a combinatorial proof was given by H. Hanani [4].)

It remains to prove that when the parameter \mathbf{a} is permitted to range in the domain $\{(a^1, a^2) \mid a^1 \neq -1\}$ the coefficients $f_{m,p}(a^1, a^2)$ given by (23) and (24) define a global analytic two-parameter subgroup of Ω^F . From (23) follows, that as long as $a^1 \neq -1, f_1(a^1, a^2) \neq 0$, and hence the formal power series:

$$F(z, a^1, a^2) = \sum_{q=1}^{\infty} f_q(a^1, a^2)z^q = \sum_{q=1}^{\infty} f_{1,q}(a^1, a^2)z^q$$

belongs to Ω^F . The coefficients $f_q(a^1, a^2)$ are, for small values of a^1 and a^2 , the coefficients of the local group from which we started in the statement of Theorem 2; hence $f_q(a^1, a^2)$ satisfy, in some neighborhood of $\mathbf{a} = \mathbf{0}$ the relations which follow from

$$(46) \quad F[F(z, \mathbf{a}), \mathbf{b}] = F[z, \Phi_n(\mathbf{a}, \mathbf{b})].$$

But $f_q(a^1, a^2)$, being polynomials satisfying a set of functional equations on an open set, satisfy these equations for all values of the parameters, and hence (46) holds whenever $\mathbf{a}, \mathbf{b} \in \{(a^1, a^2) \mid a^1 \neq -1\}$, and therefore $F(z, a^1, a^2)$ is a global group. This finishes the proof of Theorem 2.

4. Canonical representation of two-parameter analytic subgroups of Ω^F .

4.1. It is our purpose to prove:

THEOREM 3. *Every two-parameter analytic subgroup of Ω^F of class n has a representation of the form*

$$(47) \quad F(z, a^1, a^2) = \phi^{-1}[(1 + a^1) \cdot \phi(z)/(1 + a^2[\phi(z)]^n)^{1/n}]$$

where $\phi(z) = \sum_{q=1}^{\infty} \phi_q z^q \in \Omega^F$ and $\phi_1 = 1$.

Moreover,

$$(48) \quad g(z, a^{1*}, a^{2*}) = \psi^{-1}[(1 + a^{1*}) \cdot \psi(z)/(1 + a^{2*}[\psi(z)]^n)^{1/n}],$$

where $\psi(z) = \sum_{q=1}^{\infty} \psi_q z^q \in \Omega^F$ and $\psi_1 = 1$, is another representation of the subgroup (47) if and only if

$$(49) \quad \psi(z) = \phi(z)/(1 + k[\phi(z)]^n)^{1/n}$$

for some complex k .

4.2. In order to prove Theorem 3, we first prove:

LEMMA 7. *Let a formal power series $L_1(z) = z + \sum_{q=2}^{\infty} l_{q-1}^{(1)} z^q$ and a positive integer n be given.*

Define the formal power series $\Lambda(z)$ by

$$(50) \quad 1/L_1(z) = 1/z + \Lambda(z).$$

Further, define the formal power series $\phi(z) \in \Omega^F$ by

$$(51) \quad \phi(z) = z \exp \left[\int_0^z \Lambda(z) dz \right].$$

Then, the two-parameter analytic subgroup of Ω^F of class n (written in the standard multiplication table of type n), defined by

$$(52) \quad F(z, a^1, a^2) = \phi^{-1}[(1 + a^1) \cdot \phi(z)/(1 + a^2[\phi(z)]^n)^{1/n}]$$

satisfies

$$(53) \quad (\partial F(z, a^1, a^2)/\partial a^1)_{\mathbf{a}=\mathbf{0}} = L_1(z).$$

Proof. Differentiating (52) with respect to a^1 and putting $\mathbf{a} = \mathbf{0}$ we get

$$(54) \quad (\partial F(z, a^1, a^2)/\partial a^1)_{\mathbf{a}=\mathbf{0}} = \phi^{-1}'[\phi(z)] \cdot \phi(z).$$

Using the identity $\phi^{-1}'[\phi(z)] = 1/\phi'(z)$, and $\phi'(z) = \phi(z)/z + \phi(z) \cdot \Lambda(z)$ (obtained from (51)), (54) yields (53), by use of (50).

4.3. We have furthermore:

LEMMA 8. *Let a formal power series $L_1(z) = z + \sum_{q=2}^{\infty} l_{q-1}^{(1)} z^q$ and a positive integer n be given. Then there exists a unique two-parameter analytic subgroup of Ω^F of class n , $F(z, a^1, a^2)$, written in the standard multiplication table of type n , such that*

$$(55) \quad (\partial F(z, a^1, a^2)/\partial a^1)_{\mathbf{a}=\mathbf{0}} = L_1(z),$$

and $L_2(z)$ starts with the term $-z^{n+1}/n$, that is,

$$(56) \quad L_2(z) = \left(\frac{\partial F(z, a^1, a^2)}{\partial a^2} \right)_{\mathbf{a}=0} = -\frac{1}{n} z^{n+1} + \sum_{q=n+2}^{\infty} l_{q-1}^{(2)} z^q.$$

Proof. Noting that the group given by (52) satisfies (56), the existence of the required group is covered by Lemma 7. It remains to show that n , $L_1(z)$ and the condition (56) determine the group completely. The integrability condition (20) for an analytic two-parameter subgroup of Ω^F written in the standard multiplication table of type n shows us that $L_1(z)$, n , and the condition (56) determine $L_2(z)$ completely. As by Theorem 2 $F(z, a^1, a^2)$ is completely determined by $L_1(z)$ and $L_2(z)$, the lemma is proved.

4.4. Proof of Theorem 3. (I) Let an analytic two-parameter subgroup of Ω^F of class n , $F(z, a^1, a^2)$, be given. We may assume that the parametrization is written in the standard multiplication table of type n .

We note, that a transformation of the group parameters of the form $a^1 = a^{1*}$, $a^2 = ka^{2*}$, where $k \neq 0$ is some fixed complex number, leaves the standard multiplication table of type n unchanged, and we have

$$L_2^*(z) = \left(\frac{\partial F^*(z, a^{1*}, a^{2*})}{\partial a^{2*}} \right)_{\mathbf{a}^*=0} = \left(\frac{\partial F(z, a^{1*}, ka^{2*})}{\partial a^{2*}} \right)_{\mathbf{a}^*=0} = kL_2(z).$$

It follows, that k may be so chosen, that condition (56) is satisfied (that is, so that the coefficient of the first term in $L_2^*(z)$ will be $-1/n$). Hence we may assume that $F(z, a^1, a^2)$ is written in the standard multiplication table of type n and that condition (56) holds.

By Lemma 2 we have

$$L_1(z) = \left(\frac{\partial F(z, a^1, a^2)}{\partial a^1} \right)_{\mathbf{a}=0} = z + \sum_{q=2}^{\infty} l_{q-1}^{(1)} z^q$$

and, using Lemma 7, we can construct a group having the representation (52) and satisfying (53); by Lemma 8 the constructed group coincides with the subgroup from which we have started, and we have shown that the given subgroup, being of class n , has a representation of the form (47).

(II) Suppose now that a two-parameter analytic subgroup of Ω^F is represented by (47), and $\psi(z)$ is given by (49). We shall see, that $g(z, a^{1*}, a^{2*})$ given by (48) is another representation of the same group.

Write (49) in the form

$$(57) \quad \psi(z) = \frac{z}{(1+kz^n)^{1/n}} \circ \phi(z)$$

it follows that

$$(58) \quad \psi^{-1}(z) = \phi^{-1}(z) \circ z/(1-kz^n)^{1/n}.$$

Introducing (57) and (58) into (48) we get

$$g(z, a^{1*}, a^{2*}) = \phi^{-1}(z) \circ \frac{z}{(1-kz^n)^{1/n}} \circ \frac{(1+a^{1*})z}{(1+a^{2*}z^n)^{1/n}} \circ \frac{z}{(1+kz^n)^{1/n}} \circ \phi(z)$$

$$= \phi^{-1} \left[\frac{(1+a^{1*}) \cdot \phi(z)}{(1+\{a^{2*}+k-k(1+a^{1*})^n\}[\phi(z)]^n)^{1/n}} \right]$$

or

$$g(z, a^{1*}, a^{2*}) = \phi^{-1}[(1+a^1) \cdot \phi(z)/(1+a^2[\phi(z)]^n)^{1/n}]$$

where $a^1 = a^{1*}$ and $a^2 = a^{2*} + k - k(1+a^{1*})^n$.

When (a^{1*}, a^{2*}) ranges in the domain $\{(a^{1*}, a^{2*}) \mid a^{1*} \neq -1\}$ the elements of the subgroup $g(z, a^{1*}, a^{2*})$ are all the elements of $F(z, a^1, a^2)$, hence $g(z, a^{1*}, a^{2*})$ is another representation of the subgroup $F(z, a^1, a^2)$ given by (47).

(III) Suppose now that (47) and (48) are different representations of the same subgroup, that is, there exist functions $\theta_1(a^1, a^2)$ and $\theta_2(a^1, a^2)$ such that:

$$(59) \quad \phi^{-1} \left[\frac{(1+a^1) \cdot \phi(z)}{(1+a^2[\phi(z)]^n)^{1/n}} \right] = \psi^{-1} \left[\frac{\{1 + \theta_1(a^1, a^2)\} \psi(z)}{(1 + \theta_2(a^1, a^2)[\psi(z)]^n)^{1/n}} \right].$$

Comparing coefficients of z on both sides of (59) we get $\theta_1(a^1, a^2) = a^1$. Put $a^1 = 0$ in (59). We get

$$(60) \quad \phi^{-1} \left[\frac{\phi(z)}{(1+a^2[\phi(z)]^n)^{1/n}} \right] = \psi^{-1} \left[\frac{\psi(z)}{(1+\theta_2(0, a^2)[\psi(z)]^n)^{1/n}} \right].$$

Differentiate (60) with respect to a^2 and put $a^2 = 0$. Noting that $\theta_2(0, 0) = 0$ follows from (60), we get

$$(61) \quad -\frac{1}{n} \frac{[\phi(z)]^{n+1}}{\phi'(z)} = -\frac{1}{n} \frac{[\psi(z)]^{n+1}}{\psi'(z)} \cdot \frac{\partial \theta_2(0, 0)}{\partial a^2}.$$

Comparing the coefficients of z^{n+1} on both sides we get

$$\partial \theta_2(0, 0) / \partial a^2 = 1.$$

Formal integration of (61) yields

$$1/[\phi(z)]^n + k = 1/[\psi(z)]^n$$

from which (49) follows, and the proof of Theorem 3 is complete.

4.5. From Theorem 3 it follows, that every two-parameter analytic subgroup of Ω^F of class n is globally isomorphic to the group:

$$H_n(z, a^1, a^2) = (1+a^1)z/(1+a^2z^n)^{1/n}.$$

As $H_n(z, a^1, a^2)$ is not globally isomorphic to $H_m(z, a^1, a^2)$ when $n \neq m$, we conclude that two-parameter analytic subgroups of Ω^F of different classes, although being locally isomorphic, are not globally isomorphic.

REFERENCES

1. L. Bianchi, *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni*, Spoerri, Pisa, 1918.
2. H. Cartan, *Théorie élémentaire des fonctions analytique d'une ou plusieurs variables complexes*, Hermann, Paris, 1961.
3. L. P. Eisenhart, *Continuous groups of transformations*, Dover, New York, 1961.
4. H. Hanani, *A combinatorial identity*, Amer. Math. Monthly. (to appear).
5. E. Jabotinsky, *Sur la représentation de la composition de fonctions par un produit de matrices*, C. R. Acad. Sci. Paris **224** (1947), 323–324.
6. ———, *Analytic iteration*, Trans. Amer. Math. Soc. **108** (1963), 457–477.
7. C. Jordan, *Calculus of finite differences*, 2nd ed., Chelsea, New York, 1960.
8. A. Ran, *One parameter groups of formal power series*, Duke Math. J. (to appear).
9. I. Schur, *On Faber polynomials*, Amer. J. Math. **67** (1945), 33–41.

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