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AN ANALOGUE OF A PROBLEM OF J. BALÁZS AND P. TURÁN. III

BY
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1. One of the most important properties of ordinary polynomials is that any such polynomial of degree $\leqq n-1$ is uniquely determined by its values at an arbitrarily given system of n distinct points. This was observed by Lagrange and the formula for the determination of any such polynomial is known as Lagrange interpolation formula. A more general procedure in interpolation is that of Hermite [6], in which certain consecutive derivatives of the interpolating polynomial are required to agree with prescribed values at the nodes of interpolation. In 1906 G. D. Birkhoff [1] considered the most general interpolation problem where consecutive derivatives of the interpolation polynomial are not necessarily prescribed. As far as the author knows, this was the first paper concerning the so called Lacunary interpolation. Remarking on this paper of Birkhoff, J. Suranyi and P. Turán [13] mentioned that the paper is so general that one cannot expect better formulae than those of Hermite; one cannot even see from this paper the new feature of this general interpolation, viz. for each n by the suitable choice of the x_i's and the indices of the prescribed derivatives the problem can be solvable or can have an infinity of solutions. On this account in a series of papers [2], [3] Turán and his associates have initiated the problem of existence, uniqueness, explicit representation and the problem of convergence of interpolatory polynomials of degree $\leqq 2n-1$ when the values and second derivatives are prescribed on n given nodes. They called it (0, 2) interpolation. Here we wish to remark that the problem of explicit representation and convergence was settled only on the zeros of $\pi_n(x)=(1-x^2)P'_{n-1}(x)$, where $P_{n-1}(x)$ denotes the Legendre polynomial of degree $\leqq n-1$. Other interesting results of this type are due to O. Kis [7], Saxena and Sharma [10] and the author [9], [18]. It is important to mention that even in a very special case of Birkhoff interpolation G. Polya [8] obtained interesting results concerning the solution of the problem of bending of beam. Motivated by this result Turán and his associates observed a possibility of applying (0, 2) interpolation for finding out approximate solution of a differential equation of the form $y''+A(x)y=0$.

The object in this paper is to consider another case of Birkhoff interpolation. Thus my aim here is to consider the problem of existence, uniqueness, explicit

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representation and the problem of convergence of the polynomials $R_n(x, f)$ which satisfy the properties

$$(1.1)\qquad \begin{array}{lll} R_n(x_i) = f(x_i), & R_n'(x_i) = \alpha_i; & i = 1, 2, \ldots, n+2, \\ R_n''(x_i) = \beta_i, & R_n^{(iv)}(x_i) = \gamma_i; & i = 2, 3, \ldots, n+1, \end{array}$$

where the x_i's are $(n+2)$ zeros of $(1-x^2)P_n(x)$ given by

$$(1.2)\qquad 1 = x_1 > x_2 > \cdots > x_{n+1} > x_{n+2} = -1$$

and $P_n(x)$ denotes the Legendre polynomial of degree $\leqq n$. Following the terminology introduced by P. Turán we call this case as modified (0, 1, 2, 4) interpolation. This is modified in the sense that second and fourth derivatives are not prescribed at ± 1. It will be shown that $R_n(x,f)$ is of degree $\leqq 4n+3$ and exists uniquely only when n is even. It turns out that the fundamental polynomials are very closely connected with the polynomials introduced by Egervary and Turán [4] and as well as with the fundamental polynomials of (0, 2) interpolation [9]. In view of the relation

$$(1.3)\qquad [(1-x^2)P_n^2(x)]'''_{x=x_j} = 0, \qquad j = 2, 3, \ldots, n+1,$$

the fundamental polynomials turn out to be simple and this is one of the reasons why we have introduced ± 1 as nodes of interpolation. Another important remark to be mentioned here is that this sequence of polynomials converges uniformly to $f(x)$ in $[-1, +1]$ provided $f(x)$ satisfies the Zygmund condition. This came to the author as somewhat surprising, in view of the results by Saxena and Sharma [10] and Saxena [11], [12] where they required even the existence of first and second derivative respectively. Details of convergence theorem are given in [5]. The conditions of convergence theorem are very similar to G. Freud [5].

2. **Existence problem.** Here we shall prove the following:

THEOREM 2.1. *Let n be even and the points $x_1, x_2, \ldots, x_{n+2}$ satisfy* (1.2); *then there exists a unique polynomial $f(x)$ of degree $\leqq 4n+3$ such that for given y_{i0}, y_{i1}, y_{i2}, and y_{i4}*

$$(2.1)\qquad f^{(p)}(x_i) = y_{ip} \quad \text{if } p = 0, 1, \text{ then } i = 1, 2, \ldots, n+2, \text{ if } p = 2, 4, i = 2, \ldots, n+1$$

but if n is odd there does not exist in general a polynomial $f(x)$ of degree $\leqq 4n+3$ satisfying (2.1) *and if there exists such then there are infinitely many.*

Proof. First we will prove the result for n even. We will prove that when the system (2.1) is homogenous, i.e.

$$(2.2)\qquad f^{(p)}(x_i) = 0,\ p = 0, 1, \text{ then } i = 1, 2, \ldots, n+2,\ p = 2, 4,\ i = 2, 3, \ldots, n+1$$

the only solution is $f(x) \equiv 0$. Now this means that writing out (2.2) as a linear system, the determinant is $\neq 0$. Considering the general problem (2.1) this shows that the corresponding linear system is always uniquely soluble. In view of (2.2) we can write

$$f(x) = (1-x^2)^2P_n^3(x)q_{n-1}(x) \tag{2.3}$$

where $q_{n-1}(x)$ is a polynomial of degree $\leqq n-1$ in x. Using the fact that $f^{(\mathrm{iv})}(x_j)=0$, $j=2, 3, \ldots, n+1$, leads to

$$-x_jq_{n-1}(x_j)+(1-x_j^2)q'_{n-1}(x_j) = 0, \qquad j = 2, 3, \ldots, n+1.$$

But this implies that the zeros of $(1-x^2)q'_{n-1}(x)-xq_{n-1}(x)$ are the same as the zeros of $P_n(x)$. Thus we obtain

$$-xq_{n-1}(x)+(1-x^2)q'_{n-1}(x) = CP_n(x), \tag{2.4}$$

with C a numerical constant. Since every polynomial of degree $\leqq n-1$ can be written as a linear combination of $P_0(x), P_1(x), \ldots, P_{n-1}(x)$, we can write

$$q_{n-1}(x) = \sum_{k=0}^{n-1} a_kP_k(x) \tag{2.5}$$

using the known results

$$(k+1)P_{k+1}(x)-(2k+1)xP_k(x)+kP_{k-1}(x) = 0 \tag{2.6}$$

and

$$(1-x^2)P'_k(x) = kP_{k-1}(x)-kxP_k(x) \tag{2.7}$$

we can write (2.4) as

$$\sum_{k=0}^{n-2} \frac{(k+1)^2}{2k+3}\, a_{k+1}P_k(x)-\sum_{k=1}^{n} a_{k-1}\frac{k^2}{(2k-1)}P_k(x) = CP_n(x). \tag{2.8}$$

Now comparing the coefficients of $P_0(x), P_1(x), \ldots, P_{n-1}(x)$ we obtain

$$0 = a_{n-2} = a_{n-4} = \cdots = a_2 = a_0, \qquad n \text{ even},$$
$$0 = a_1 = a_3 = \cdots = a_{n-3} = a_{n-1}, \qquad n \text{ even},$$

from which it follows that $q_{n-1}(x)\equiv 0$ and using (2.3) we get $f(x)\equiv 0$, which it was required to prove. In a similar way it can be shown that for n odd $0=a_{n-2}=a_{n-4}=\cdots=a_3=a_1=0$ but $a_2, a_4, \ldots, a_{n-1}$ all cannot be determined. Therefore for n odd there does not exist a unique solution. For uniqueness theorem of general nature we refer to an interesting work [17].

3. **Preliminaries for explicit representation of interpolatory polynomials.** Egervary and Turán [4] gave a new characterization of $(1-x^2)P_n(x)$ while considering the

problem of most stable interpolation in finite interval $[-1, +1]$. Their polynomials of most stable interpolation turn out to be

$$\mu_i(x) = \frac{(1-x^2)}{1-x_i^2} l_i^2(x), \qquad l_i(x) = \frac{P_n(x)}{(x-x_i)P_n'(x_i)}, \tag{3.1}$$

$i=2, 3, \ldots, n+1$. Obviously

$$\begin{aligned} \mu_i(x_j) &= 1, \quad i = j, \\ &= 0, \quad i \neq j, \end{aligned} \quad j = 1, 2, \ldots, n+2, \tag{3.2}$$
$$\mu_i'(x_j) = 0, \quad j = 2, 3, \ldots, n+1.$$

We will denote

$$\lambda_i(x) = \frac{(1-x^2)}{1-x_i^2} l_i(x), \qquad i = 2, 3, \ldots, n+1. \tag{3.3}$$

Finally we shall require the fundamental polynomials of modified (0, 2) interpolation on the nodes (1.2) obtained in [9]. These fundamental polynomials $r_k(x)$ and $\rho_k(x)$ satisfy the following requirements:

$$\rho_i(x_j) = 0, \quad j = 1, 2, \ldots, n+2,$$
$$\begin{aligned} \rho_i''(x_j) &= 1, \quad i = j, \\ &= 0, \quad i \neq j, \end{aligned} \quad i, j = 2, 3, \ldots, n+1, \tag{3.4}$$

$$\begin{aligned} r_i(x_j) &= 1, \quad i = j, \\ &= 0, \quad i \neq j, \end{aligned} \quad i, j = 1, 2, \ldots, n+2, \tag{3.5}$$
$$r_i''(x_j) = 0, \quad j = 2, 3, \ldots, n+1$$

and are polynomials of degree $\leqq 2n+1$. Their explicit forms are given below. For $i=2, 3, \ldots, n+1$

$$\rho_i(x) = \frac{(1-x^2)^{1/2} P_n(x)}{2P_n'(x_i)} \left[a_i \int_{-1}^{x} P_n(t) w(t)\, dt + \int_{-1}^{x} l_i(t) w(t)\, dt \right] \tag{3.6}$$

where

$$w(t) = (1-t^2)^{-1/2}, \qquad a_i \int_{-1}^{+1} P_n(t) w(t)\, dt + \int_{-1}^{+1} l_i(t) w(t)\, dt \tag{3.7}$$

$$r_1(x) = \frac{1+x}{2} P_n^2(x) - \frac{1}{2}(1-x^2) P_n(x) P_n'(x) - \frac{1}{2}(1-x^2)^{1/2} P_n(x) \int_{-1}^{x} P_n'(t) w(t)\, dt, \tag{3.8}$$

$$\begin{aligned} r_{n+2}(x) = {} & \frac{1-x}{2} P_n^2(x) + \frac{1}{2}(1-x^2) P_n(x) P_n'(x) \\ & - \frac{(1-x^2)^{1/2}}{2} P_n(x) \int_{-1}^{x} P_n'(t) w(t)\, dt. \end{aligned} \tag{3.9}$$

For $i=2, 3, \ldots, n+1$ we have

$$r_i(x) = \mu_i(x)+\tfrac{1}{2}P_n(x)\theta_i(x)+C_i\rho_i(x)+C_i' r_1(x)+C_i'' r_{n+2}(x) \tag{3.10}$$

where

$$\theta_i(x) = \frac{(1-x^2)l_i'(x)-xl_i(x)}{(1-x_i^2)P_n'(x_i)}, \tag{3.11}$$

$$C_i = \left[n(n+1)+\frac{1+x_i^2}{1-x_i^2}\right]\frac{1}{(1-x_i^2)}, \tag{3.12}$$

$$\begin{aligned} C_i' &= (2(1-x_i)(1-x_i^2)[P_n'(x_i)]^2)^{-1}, \\ C_i'' &= (2(1+x_i)(1-x_i^2)[P_n'(x_i)]^2)^{-1}. \end{aligned} \tag{3.13}$$

Here the fundamental polynomial $r_i(x)$ is put in much more simpler form than in [9].

4. **Problem of explicit representation of interpolatory polynomials.** For $R_n(x)$ satisfying (1.1) we have the representation

$$R_n(x) = \sum_{i=1}^{n+2} f(x_i)A_i(x)+\sum_{i=1}^{n+2}\alpha_i B_i(x)+\sum_{i=2}^{n+1}\beta_i C_i(x)+\sum_{i=2}^{n+1}\gamma_i D_i(x). \tag{4.1}$$

$A_i(x)$, $B_i(x)$, $C_i(x)$ and $D_i(x)$ are unique polynomials each of degree $\leqq 4n+3$ determined by the following requirements:

$$\begin{aligned} A_i(x_j) &= 1, \quad i=j, \\ &= 0, \quad i\neq j, \qquad A_i'(x_j) = 0, \quad i,j = 1,2,\ldots,n+1, \\ A_i''(x_j) &= A_i^{\mathrm{iv}}(x_j) = 0, \quad j = 2,3,\ldots,n+1, \quad i = 1,2,\ldots,n+2; \end{aligned} \tag{4.2}$$

$$\begin{aligned} B_i(x_j) &= 0, \quad B_i'(x_j) = 1, \quad i=j, \\ &\qquad\qquad = 0, \quad i\neq j, \qquad i,j = 1,2,\ldots,n+2, \\ B_i''(x_j) &= B_i^{\mathrm{iv}}(x_j) = 0, \quad j = 2,3,\ldots,n+1, \quad i = 1,2,\ldots,n+2; \end{aligned} \tag{4.3}$$

$$\begin{aligned} C_i(x_j) &= C_i'(x_j) = 0, \quad j = 1,2,\ldots,n+2, \quad i = 2,3,\ldots,n+1, \\ C_i''(x_j) &= 1, \quad i=j, \\ &= 0, \quad i\neq j, \qquad C_i^{\mathrm{iv}}(x_j) = 0, \quad i,j = 2,3,\ldots,n+1; \end{aligned} \tag{4.4}$$

$$\begin{aligned} D_i(x_j) &= D_i'(x_j) = 0, \quad j = 1,2,\ldots,n+2, \quad i = 2,3,\ldots,n+1, \\ D_i''(x_j) &= 0, \quad D_i^{\mathrm{iv}}(x_j) = 1, \quad i=j, \\ &\qquad\qquad = 0, \quad i\neq j, \qquad i,j = 2,3,\ldots,n+1. \end{aligned} \tag{4.5}$$

The explicit representation of these fundamental polynomials is as follows. For n even we have

$$D_i(x) = \frac{S_{2n+2}(x)\rho_i(x)}{12(1-x_i^2)[P_n'(x_i)]^2}, \quad i = 2,3,\ldots,n+1, \quad S_{2n+2}(x) = (1-x^2)P_n^2(x), \tag{4.6}$$

$$C_i(x) = \frac{S_{2n+2}(x)r_i(x)}{2(1-x_i^2)[P_n'(x_i)]^2} + b_i D_i(x), \qquad i = 2, 3, \ldots, n+1, \tag{4.7}$$

where

$$b_i = -4\{n(n+1)/(1-x_i^2) + (1-x_i^2)^{-2}\}, \tag{4.8}$$

$$B_1(x) = -\tfrac{1}{2}S_{2n+2}(x)r_1(x), \qquad B_{n+2}(x) = \tfrac{1}{2}S_{2n+2}(x)r_{n+2}(x). \tag{4.9}$$

For $i=2, 3, \ldots, n+1$ we have

$$\begin{aligned} B_i(x) = {} & \frac{\mu_i(x)P_n(x)\lambda_i(x)}{P_n'(x_i)} + \frac{1}{3}\,\frac{S_{2n+2}(x)l_i(x)\theta_i(x)}{(1-x_i^2)P_n'(x_i)} \\ & + d_i B_1(x) + d_i' B_{n+2}(\alpha) + d_i'' D_i(x), \end{aligned} \tag{4.10}$$

where

$$d_i = -\frac{2}{3(1-x_i^2)^2(1-x_i)^2[P_n'(x_i)]^4}, \qquad d_i' = -\frac{2}{3(1-x_i^2)^2(1+x_i)^2[P_n'(x_i)]^4}, \tag{4.11}$$

$$d_i'' = \left[20n(n+1) + 32 - 48x_i^2 + \frac{64+24x_i^2}{1-x_i^2}\right] \frac{x_i}{(1-x_i^2)^2}, \tag{4.12}$$

$$\begin{aligned} A_1(x) = {} & \lambda_1^2(x)P_n^2(x) + \left(1 + \frac{7n(n+1)}{3}\right)B_1(x) - \frac{(1-x^2)P_n(x)\lambda_1(x)P_n'(x)}{3} \\ & + \frac{1}{6}\,S_{2n+2}(x)\left\{r_1(x) - \left(\frac{1+x}{2}\right)P_n^2(x)\right\}, \end{aligned} \tag{4.13}$$

with

$$\lambda_1(x) = \frac{1+x}{2}\,P_n(x), \qquad \lambda_{n+2}(x) = \frac{1-x}{2}\,P_n(x), \tag{4.14}$$

$$\begin{aligned} A_{n+2}(x) = {} & \lambda_{n+2}^2(x)P_n^2(x) + \left(1 + \frac{7n(n+1)}{3}\right)B_{n+2}(x) \\ & + \frac{(1-x^2)}{3}\,P_n(x)P_n'(x)\lambda_{n+2}^2(x) \\ & + \frac{1}{6}\,S_{2n+2}(x)\left\{r_{n+2}(x) - \frac{(1-x)}{2}\,P_n^2(x)\right\}. \end{aligned} \tag{4.15}$$

For $i=2, 3, \ldots, n+1$ we have

$$A_i(x) = \mu_i^2(x) + \tfrac{1}{4}\mu_i(x)P_n(x)\theta_i(x) + e_i B_i(x) + e_i' B_{n+2}(x) + e_i'' C_i(x) + e_i''' D_i(x), \tag{4.16}$$

where

$$e_i = \frac{1}{2(1-x_i^2)^2(1-x_i)^3[P_n'(x_i)]^4}, \qquad e_i' = \frac{1}{2(1+x_i)^3(1-x_i^2)^2[P_n'(x_i)]^4}, \tag{4.17}$$

$$e_i''' = \frac{3}{2}\left\{\frac{n(n+1)}{1-x_i^2} + \frac{1}{(1-x_i^2)^2}\right\} \tag{4.18}$$

and

$$e_i''' = -6[\mu_i''(x_i)]^2 - 2\mu_i^{\mathrm{iv}}(x_i) - \frac{3}{2}\,\theta_i''(x_i)P_n''(x_i) - \theta_i'(x_i)\{3\mu_i''(x_i)P_n'(x_i) + P_n'''(x_i)\}. \tag{4.19}$$

Verification of these fundamental polynomials are omitted.

5. **Convergence problem.** Consider the sequence of points

$$1 = x_{1,n} > x_{2n} > \cdots > x_{n+1,n} > x_{n+2,n} = -1 \tag{5.1}$$

where x_{in}'s stand for the zeros of $(1-x^2)P_n(x)$. [The notation (1.2) of the zeros $(1-x^2)P_n(x)$ was more suitable when n was fixed.] Forming the interpolatory polynomials (4.1) for each even n we shall write the fundamental functions as $A_{in}(x)$, $B_{in}(x)$ etc. Let $f(x)$ belong to $C[-1, +1]$ and we consider the sequence of polynomials

$$R_n[x,f] = \sum_{i=1}^{n+2} f(x_{in})A_{in}(x) + \sum_{i=1}^{n+2} \alpha_{in}B_{in}(x) + \sum_{i=2}^{n+1} \beta_{in}C_{in}(x) + \sum_{i=1}^{n+1} \gamma_{in}D_{in}(x). \tag{5.2}$$

Now we state the following convergence theorem.

THEOREM 5.1. *Let $f(x)$ satisfy the Zygmund condition*

$$|f(x+h)-2f(x)+f(x-h)| = o(h) \tag{5.3}$$

and for $i=1, 2, \dots, n+2$

$$\begin{aligned} &|\alpha_{in}| = o(n^{1/4}/\log n), \qquad i = 1, 2, \dots, n+2, \\ &|\beta_{in}| = o(n)/(1-x_{in}^2)^{1/2}, \qquad |\gamma_{in}| = o(n^3)/(1-x_{in}^2)^{3/2}, \qquad i = 2, \dots, n+1 \end{aligned} \tag{5.4}$$

then the sequence $R_n(x,f)$ converges uniformly to $f(x)$ in $[-1, +1]$.

6. **Preliminaries.** First we shall mention some known results concerning Legendre polynomials [15]. For $-1 \leqq x \leqq +1$ we have

$$(1-x^2)^{1/4}|P_n(x)| \leqq (2/n\pi)^{1/2}, \tag{6.1}$$

$$|(1-x^2)^{3/4}P_n'(x)| \leqq [2(n+1)]^{1/2}, \tag{6.2}$$

$$|(1-x^2)^{1/2}P_n'(x)| \leqq n, \tag{6.3}$$

$$|P_n'(x)| \leqq n(n+1)/2, \tag{6.4}$$

$$|P_n(x)| \leqq 1. \tag{6.5}$$

From the results of Egervary and Turán [4] we have

$$\sum_{i=2}^{n+1} \frac{(1-x^2)l_{in}^2(x)}{1-x_{in}^2} = 1-P_n^2(x) < 1, \tag{6.6}$$

we also need [15]

$$|P_n'(x_{in})| \geqq dn^2(i-1)^{-3/2}, \qquad i = 2, 3, \dots, n/2+1, \tag{6.7}$$

$$|P_n'(x_{in})| \geqq dn^2(n-i+2)^{-3/2}, \qquad i = n/2+2, \dots, n+1, \tag{6.8}$$

$$(1-x_{in}^2) \geqq (i-1)^2n^{-2}, \qquad i = 2, 3, \dots, n/2+1, \tag{6.9}$$

$$(1-x_{in}^2) \geqq (n-i+2)^2n^{-2}, \qquad i = n/2+2, \dots, n+1, \tag{6.10}$$

where $0<d\leqq 1$.

We will require some estimates concerning fundamental polynomials of Lagrange interpolation.

From Christoffel Darboux formula [15] we have

$$(6.11)\qquad l_{in}(x) = (1-x_{in}^2)^{-1}[P_n'(x_{in})]^{-2}\left[1+\sum_{r=1}^{n-1}(2r+1)P_r(x_{in})P_r(x)\right],$$

on using (6.5), (6.11) and (6.1) for $x=x_{in}$ we obtain

$$(6.12)\qquad |l_{in}(x)| \leqq n^{3/2}(1-x_{in}^2)^{-5/4}[P_n'(x_{in})]^{-2},$$

and similarly using (6.2)

$$(6.13)\qquad |(1-x^2)^{1/4}l_{in}(x)| \leqq n(1-x_{in}^2)^{-5/4}[P_n'(x_{in})]^{-2},$$

$$(6.14)\qquad |(1-x^2)^{3/4}l_{in}'(x)| \leqq 3n^2(1-x_{in}^2)^{-5/4}[P_n'(x_{in})]^{-2}.$$

From (3.14) and using above results we obtain

$$(6.15)\qquad |\theta_{in}(x)P_n(x)| \leqq 4n^{3/2}(1-x_{in}^2)^{-9/4}[P_n'(x_{in})]^{-3} \leqq 4d^{-3}.$$

7. Estimates of the fundamental polynomials of (0, 2) interpolation.

LEMMA 7.1. *For* $-1\leqq x\leqq +1$ *we have*

$$(7.1)\qquad |\rho_{in}(x)| \leqq 144n^{-1/2}(1-x_{in}^2)^{-13/8}[P_n'(x_{in})]^{-3},\qquad i=2,3,\ldots,n+1,$$

$$(7.2)\qquad |r_{1,n}(x)| \leqq 19n^{1/2},\qquad |r_{n+2,n}(x)| \leqq 19n^{1/2}.$$

And for $i=2,3,\ldots,n+1$ *we have*

$$(7.3)\qquad |r_{in}(x)| \leqq \mu_{in}(x)+578d^{-3}(1-x_{in}^2)^{-3/8}+38d^{-2}n^{-1/2}(1-x_{in}^2)^{-1/2},$$

$$(7.4)\qquad \sum_{i=2}^{n+1}|r_{in}(x)| \leqq 1773d^{-3}n,$$

and on using (6.7)–(6.10) *we obtain*

$$(7.5)\qquad \sum_{i=2}^{n+1}\frac{|r_{in}(x)|}{(1-x_{in}^2)^{3/2}[P_n'(x_{in})]^2} \leqq d^{-2}n^{-1}\sum_{i=2}^{n+1}|r_{in}(x)| \leqq 1773d^{-5}.$$

(7.4) *follows from* (7.3) *on using* (6.6)–(6.10). (7.3) *is an easy consequence of* (3.13), (6.15), (7.1), (7.2).

8. Estimation of fundamental polynomials $D_{in}(x)$ and $C_{in}(x)$.

LEMMA 8.1. *For* $-1\leqq x\leqq +1$ *we have*

$$(8.1)\qquad |D_{in}(x)| \leqq 8d^{-3}(1-x_{in}^2)^{-3/8}[P_n'(x_{in})]^{-2}n^{-3},\qquad i=2,3,\ldots,n+1,$$

and

$$(8.2)\qquad \sum_{i=2}^{n+1}\frac{|D_{in}(x)|}{(1-x_{2}^{in})^{3/2}} \leqq 20n^{-3}d^{-5}.$$

Proof. (8.1) is an immediate consequence of (4.6), (6.1) and (7.1). (8.2) follows from (8.1) and (6.7)–(6.10).

LEMMA 8.2. *For* $-1 \leqq x \leqq +1$ *we have*

$$\sum_{i=2}^{n+1} \frac{|C_{in}(x)|}{(1-x_{in}^2)^{1/2}} \leqq 900n^{-1}d^{-5}, \tag{8.3}$$

where d is a constant stated in (6.7).

Proof. From (4.8) it follows that

$$|b_{in}| \leqq 12n^2(1-x_{in}^2)^{-1}, \qquad i = 2, 3, \ldots, n+1. \tag{8.4}$$

Therefore, on using (8.2) it follows that

$$\sum_{i=2}^{n+1} \frac{|b_{in}D_{in}(x)|}{(1-x_{in}^2)^{1/2}} \leqq 240n^{-1}d^{-5}. \tag{8.5}$$

Since

$$|S_{2n+2}(x)| \leqq 2/n\pi, \qquad -1 \leqq x \leqq +1, \tag{8.6}$$

on using (7.5), (8.6) we obtain

$$\sum_{i=2}^{n+1} \frac{S_{2n+2}(x)r_{in}(x)}{2(1-x_{in}^2)^{3/2}[P_n'(x_{in})]^2} \leqq 591n^{-1}d^{-5}. \tag{8.7}$$

Now, using (4.7), (8.5) and (8.7), (8.3) follows.

9. Estimation of fundamental polynomials $B_{in}(x)$.

LEMMA 9.1. *For* $-1 \leqq x \leqq +1$ *we have*

$$|B_{1n}(x)| \leqq 7n^{-1/2}, \qquad |B_{n+2,n}(x)| \leqq 7n^{-1/2}, \tag{9.1}$$

$$\sum_{i=2}^{n+1} |B_{in}(x)| \leqq 1610d^{-5}n^{-1/4}. \tag{9.2}$$

Proof. From (4.9) and (7.2) we obtain (9.1). Since

$$l_{in}'(x) = P_n'(x)l_{in}(x) - l_{in}^2(x)P_n'(x_{in}). \tag{9.3}$$

Therefore, from (3.13) we obtain

$$\theta_{in}(x)P_n(x) = \frac{(1-x^2)P_n'(x)l_{in}(x)}{(1-x_{in}^2)P_n'(x_{in})} - \mu_{in}(x) - \frac{xl_{in}(x)P_n(x)}{(1-x_{in}^2)P_n'(x_{in})}.$$

Hence, on using (6.1), (6.2), (6.5)–(6.10) and (6.13) we obtain

$$\left|\frac{\theta_{in}(x)(1-x^2)P_n^2(x)l_{in}(x)}{3(1-x_{in}^2)P_n'(x_{in})}\right| \leqq 2\mu_{in}(x)n^{-1}d^{-2}. \tag{9.4}$$

From (4.14), (3.11), (6.13) and (6.1) we obtain

$$\left|\frac{\mu_{in}(x)\lambda_{in}(x)P_n(x)}{P_n'(x_{in})}\right| \leqq n^{-1}d^{-3}\mu_{in}(x). \tag{9.5}$$

Therefore, on using (6.6)

$$(9.6)\qquad \sum_{i=2}^{n+1}\left|\frac{\lambda_{in}(x)P_n(x)\mu_{in}(x)}{P_n'(x_{in})}+\frac{1}{3}\,\frac{\theta_{in}(x)(1-x^2)P_n^2(x)l_{in}(x)}{(1-x_{in}^2)P_n'(x_{in})}\right| \leqq 3n^{-1}d^{-3}.$$

From (4.11) we obtain

$$(9.7)\qquad |d_{in}| \leqq 3(1-x_{in}^2)^{-4}[P_n'(x_{in})]^{-4}, \qquad |d'_{in}| \leqq 3(1-x_{in}^2)^{-4}[P_n'(x_{in})]^{-4}.$$

From (9.7), (6.7)–(6.10) and (9.1) we obtain

$$(9.8)\qquad \sum_{i=2}^{n+1} |d_{in}B_{1n}(x)+d'_{in}B_{n+2,n}(x)| \leqq 84d^{-4}n^{-1/2}.$$

Lastly, using (4.12) we obtain

$$(9.9)\qquad |d''_{in}| \leqq 190n^2(1-x_{in}^2)^{-2}, \qquad i = 2, 3, \ldots, n+1.$$

Therefore from (9.9) and (8.1) we obtain

$$(9.10)\qquad \sum_{i=2}^{n+1} |d''_{in}D_{in}(x)| \leqq 1520n^{-1/4}d^{-5}.$$

(9.6), (9.8) and (9.10) proves the lemma.

10. Estimation of the fundamental polynomials in $A_{in}(x)$.

LEMMA 10.1. *For* $-1 \leqq x \leqq +1$ *we have*

$$(10.1)\qquad |A_{1n}(x)| \leqq 33n^{3/2}, \qquad |A_{n+2,n}(x)| \leqq 33n^{3/2},$$

$$(10.2)\qquad \sum_{i=2}^{n+1} (1-x_{in}^2)^{1/2}|A_{in}(x)| \leqq f_1 n, \qquad f_1 = 11000d^{-5},$$

$$(10.3)\qquad \sum_{i=2}^{n+1} |A_{in}(x)| \leqq f_1 n^2.$$

Proof. (10.1) follows very easily from the corresponding estimates. Since $|\mu_{in}(x)| \leqq 1$ we have from (6.6)

$$(10.4)\qquad \sum_{i=2}^{n+1} |\mu_{in}^2(x)| \leqq 1.$$

Again, using (6.6) and (6.15) we obtain

$$(10.5)\qquad \sum_{i=2}^{n+1} \tfrac{1}{4}|\mu_{in}(x)P_n(x)\theta_{in}(x)| \leqq d^{-3}.$$

From (4.17) we obtain

$$(10.6)\qquad |e_{in}| \leqq 4(1-x_{in}^2)^{-5}|[P_n'(x_{in})]^{-4}|, \qquad |e'_{in}| \leqq 4\,\frac{(1-x_{in}^2)^{-5}}{|[P_n'(x_{in})]^4|}.$$

Now, using (10.6), (6.7)–(6.10) and (9.1) we obtain

$$\sum_{i=2}^{n+1} |e_{in}B_{1,n}(x)+e_{in}B_{n+2,n}(x)| \leqq 112n^{3/2}d^{-4}. \tag{10.7}$$

Similarly

$$\sum_{i=2}^{n+1} |e_{in}B_{1,n}(x)+e'_{in}B_{n+2,n}(x)|(1-x_{in}^2)^{1/2} \leqq 112n^{1/2}d^{-4}. \tag{10.8}$$

From (4.18) we have

$$|e''_{in}| \leqq 5n^2(1-x_{in}^2)^{-1} \leqq 5n^3(1-x_{in}^2)^{-1/2}, \qquad i = 2, 3, \dots, n+1. \tag{10.9}$$

Therefore, using (10.9), and (8.3) we obtain

$$\sum_{i=2}^{n+1} |e''_i C_{i,n}(x)| \leqq 4500n^2d^{-5}, \tag{10.10}$$

and

$$\sum_{i=2}^{n+1} |e''_i C_{i,n}(x)|(1-x_{in}^2)^{1/2} \leqq 4500nd^{-5}. \tag{10.11}$$

From (4.19) it follows on simplifying and using the estimates

$$|e'''_i| \leqq 3000n^4/(1-x_{in}^2). \tag{10.12}$$

Therefore,

$$\sum_{i=2}^{n+1} |e'''_i D_{in}(x)| \leqq 24000nd^{-5}. \tag{10.13}$$

On using (10.4), (10.5), (10.8), (10.11) and (10.13) we obtain (10.2). Instead of using (1.8), (10.11), if we use (10.7) and (10.10) we obtain (10.3). This completes the proof of the above lemma.

11. **The approximating polynomial.** We shall need the following

LEMMA 11.1. *Let $f(x)$ satisfy the Zygmund condition.*

$$|f(x+h)-2f(x)+f(x-h)| \leqq \varepsilon(h), \qquad (x-h, x+h) \in [-1, +1], \tag{11.1}$$

with

$$\lim_{h\to 0} \frac{\varepsilon(h)}{h} = 0, \tag{11.2}$$

then there exist a sequence of polynomials $\{\phi_n(x)\}$ having the properties

$$f(x)-\phi_n(x) = o(n^{-1})((1-x^2)^{1/2}+n^{-1}), \tag{11.3}$$

$$|\phi'_n(x)| = O(\log n), \tag{11.4}$$

$$|\phi''_n(x)| = o(n) \min\{(1-x^2)^{-1/2}, n\}, \tag{11.5}$$

which hold in $[-1, +1]$ *and also*

$$|\phi_n^{(iv)}(x)| = \frac{o(n^3)}{(1-x^2)^{3/2}} \quad in\ -1 < x < 1. \tag{11.6}$$

Proof. Existence of $\{\phi_n(x)\}$ under the assumption (11.1) and (11.2) has been independently obtained by V. K. Dzyadyk (Dokl. Akad. Nauk SSSR **121** (1958), 903–906) and G. Freud (Math. Ann. **137** (1959), 17–25). These details are incorporated in the book by A. F. Timan [14]. (11.5) follows from the lemma of G. Freud on page 339 of [5]. (11.6) follows from (11.5) on using Bernstein inequality in $-1<x<1$. Proof of (11.4) is on the same lines as of (11.5).

12. **Proof of Theorem 5.1.** Let $f(x)$ satisfy the Zygmund condition. Then from Art 11 it follows that there exist a sequence $\{\phi_n(x)\}$ of polynomials of degree at most n satisfying (11.3)–(11.6). Hence in view of uniqueness theorem

$$\begin{aligned}\phi_n(x) = {} & \sum_{i=1}^{n+2} \phi_n(x_{in})A_{in}(x) + \sum_{i=1}^{n+2} \phi_n'(x_{in})B_{in}(x) + \sum_{i=2}^{n+1} \phi_n''(x_{in})C_{in}(x) \\ & + \sum_{i=2}^{n+1} \phi_n^{(iv)}(x_{in})D_{in}(x),\end{aligned} \tag{12.1}$$

$$R_n(x,f)-f(x) = R_n(x,f)-\phi_n(x)+\phi_n(x)-f(x). \tag{12.2}$$

From (12.1) and (5.2) we obtain

$$\begin{aligned}R_n(x,f)-\phi_n(x) = {} & \sum_{i=1}^{n+2} [f(x_{i,n})-\phi_n(x_{in})]A_{in}(x) + \sum_{i=1}^{n+2} [\alpha_{i,n}-\phi_n'(x_{in})]B_{in}(x) \\ & + \sum_{i=2}^{n+1} [\beta_{in}-\phi_n''(x_{i,n})]C_{i,n}(x) + \sum_{i=2}^{n+1} [\gamma_{i,n}-\phi_n^{(iv)}(x_{in})]D_{i,n}(x) \\ = {} & J_1+J_2+J_3+J_4 \qquad \text{(say)}.\end{aligned} \tag{12.3}$$

From (11.3), (10.1), (10.2) and (10.3) we obtain

$$\begin{aligned}|J_1| &= o(n^{-2})33n^{3/2} + \sum_{i=2}^{n+1} o(n^{-1})((1-x_{in}^2)^{1/2}+n^{-1})|A_{in}(x)| \\ &= o(n^{-1/2})+o(1) = o(1).\end{aligned} \tag{12.4}$$

From (11.4), (9.1), (9.2) and (5.4) we obtain

$$|J_2| = o(1). \tag{12.5}$$

From (11.5) and (8.3)

$$\sum_{i=2}^{n+1} \phi_n''(x_{in})|C_{in}(x)| = o(n)\sum_{i=2}^{n+1} \frac{|C_{in}(x)|}{(1-x_{in}^2)^{1/2}} = n^{-1}d^{-5}o(n) = o(1). \tag{12.6}$$

From (8.3) and (5.4)

$$\sum_{i=2}^{n+1} |\beta_{in}|\ |C_{in}(x)| \leq \sum_{i=2}^{n+1} o(n)\frac{|C_{in}(x)|}{(1-x_{in}^2)^{1/2}} = o(1). \tag{12.7}$$

From (12.6) and (12.7) it follows that

$$|J_3| = o(1). \tag{12.8}$$

Finally, using (5.4) and (8.2)

$$\sum_{i=2}^{n+1} |\gamma_{in} D_{in}(x)| \leqq \sum_{i=2}^{n+1} \frac{o(n^{+3})}{(1-x_{in}^2)^{3/2}} \; |D_{in}(x)| = o(1). \tag{12.9}$$

Similarly on using (8.2) and (11.6) we obtain

$$\sum_{i=2}^{n+1} |\phi_n^{(\mathrm{iv})}(x_{in}) D_{in}(x)| \leqq \sum_{i=2}^{n+1} \frac{o(n^3)}{(1-x_{in}^2)^{3/2}} \, |D_{in}(x)| = o(1). \tag{12.10}$$

Therefore from (12.9) and (12.10) we obtain

$$|J_4| = o(1). \tag{12.11}$$

Hence, on using (12.4), (12.5), (12.8), (12.11) we obtain

$$R_n(x,f) - \phi_n(x) = o(1). \tag{12.12}$$

From (12.2) and (12.12) we finally obtain

$$|R_n(x,f) - f(x)| = o(1).$$

This completes the proof of our theorem.

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