

INFINITE DEFICIENCY IN FRÉCHET MANIFOLDS

BY
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Abstract. Denote the countable infinite product of lines by s , let X be a separable metric manifold modeled on s , and let K be a closed subset of X having Property Z in X , i.e. for each nonnull, homotopically trivial, open subset U of X , it is true that $U \setminus K$ is nonnull and homotopically trivial. We prove that there is a homeomorphism h of X onto $X \times s$ such that $h(K)$ projects onto a single point in each of infinitely many different coordinate directions in s . Using this we prove that there is an embedding of X as an open subset of s such that K is carried onto a closed subset of s having Property Z in s . We also establish stronger versions of these results.

1. Introduction. Let s denote the countable infinite product of open intervals and let I^∞ denote the Hilbert cube, i.e. the countable infinite product of closed intervals. A *Fréchet manifold (or F-manifold)* is defined to be a separable metric space having an open cover by sets each homeomorphic to s .

Let α be a set of positive integers and for the remainder of the paper let T denote s or I^∞ . A subset K of T is said to be *deficient with respect to α in T* provided that for each of the coordinate intervals which is indexed by an element of α , K projects onto a single interior point of the interval. If α is infinite, then we say that K has *infinite deficiency in T* . More generally let X be a space and let α be as given above. A subset K of $X \times T$ is said to be *deficient with respect to α in $X \times T$* provided that $\pi_T(K)$ is deficient with respect to α in T , where π_T is the projection of $X \times T$ onto T . We say that K has *infinite deficiency in $X \times T$* provided that $\pi_T(K)$ has infinite deficiency in T . We remark that if X is any F -manifold, then in [4] it is proved that X , $X \times s$, and $X \times I^\infty$ are all homeomorphic.

A closed set K in a space X is said to have *Property Z in X* if, for each nonempty, homotopically trivial, open set U in X , it is true that $U \setminus K$ is nonempty and homotopically trivial. In [2] it is proved that a closed subset K of T has Property Z in T if and only if there exists a homeomorphism h of T onto itself such that $h(K)$ has infinite deficiency in T .

If X is any F -manifold and K is a closed subset of $X \times T$ having infinite deficiency in $X \times T$, then it is easy to show that K has Property Z in $X \times T$. This follows from the apparatus used in the proof of Theorem 9.1 of [2], which says that infinite deficiency in T implies Property Z in T . The question of Property Z implying

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infinite deficiency in $X \times s$ was raised at the conference on infinite-dimensional topology held at Cornell University, January 5–7, 1969, and it appears as Problem 11 in the report of that meeting, *Problems in the topology of infinite-dimensional manifolds*. In this paper we solve Problem 11 by proving the following theorem.

THEOREM 1. *Let X be any F -manifold and let $\{K_i\}_{i=1}^{\infty}$ be a collection of closed subsets of $X \times T$, with each K_i having Property Z in $X \times T$. Then there is a homeomorphism h of $X \times T$ onto itself and a collection $\{\alpha_i\}_{i=1}^{\infty}$, where the α_i 's are pairwise-disjoint, infinite sets of positive integers, such that for all $i > 0$, $h(K_i)$ is deficient with respect to α_i in $X \times T$.*

As an easy consequence of Theorem 1 and the preceding comments we obtain the following characterization of Property Z in F -manifolds.

COROLLARY. *Let X be any F -manifold and let K be a closed subset of X . A necessary and sufficient condition for K to have Property Z in X is that there exists a homeomorphism of X onto $X \times T$ carrying K onto a set having infinite deficiency in $X \times T$.*

In [8] David W. Henderson proved that any F -manifold can be embedded as an open subset of s . In Problem 10 of *Problems in the topology of infinite-dimensional manifolds* the following question was raised: If, for each $i > 0$, K_i is a closed subset of an F -manifold X and each K_i has Property Z in X , is there an open embedding $h: X \rightarrow s$ such that, for each $i > 0$, $h(K_i)$ is closed in s ? In our next theorem we use Henderson's open embedding theorem, modify it with our Theorem 1, and obtain a solution to Problem 10.

THEOREM 2. *Let X be any F -manifold and let $\{K_i\}_{i=1}^{\infty}$ be a collection of closed subsets of X , with each K_i having Property Z in X . Regard $s = s_1 \times s_2$, where s_1 and s_2 are copies of s , and let $f: X \rightarrow s_1$ be any open embedding. Then there exists an embedding $h: X \rightarrow s$ such that $h(X) = f(X) \times s_2$ and $h(K_i)$ is a closed subset of s having infinite deficiency in s , for all $i > 0$.*

We note that if X is any F -manifold not homeomorphic to s , then there is no open embedding $h: X \rightarrow s$ such that $h(X)$ is closed in s . Thus the assumption of Property Z in Theorem 2 cannot be omitted.

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2. Preliminaries. We will regard the Hilbert cube I^{∞} as a canonical compactification of s in which $I^{\infty} = \prod_{i=1}^{\infty} I_i$ and $s = \prod_{i=1}^{\infty} I_i^0$, where for each $i > 0$ we have $I_i = [-1, 1]$ and $I_i^0 = (-1, 1)$. The metric we use for I^{∞} and s is given by

$$d((x_i), (y_i)) = \left(\sum_{i=1}^{\infty} 2^{-i} \cdot (x_i - y_i)^2 \right)^{1/2},$$

where $(x_i), (y_i) \in I^\infty$. Whenever no confusion arises we will use d to denote the metric of any space under consideration.

Let N denote the set of positive integers and for each $i \in N$ let τ_i be the projection of I^∞ onto I_i . If $\alpha \subset N$ define $I^\alpha = \prod_{i \in \alpha} I_i$, $s^\alpha = \prod_{i \in \alpha} I_i^0$, and let τ_α be the projection of I^∞ onto I^α . For each $i > 0$ let $W_i^+ = \tau_i^{-1}(1)$, $W_i^- = \tau_i^{-1}(-1)$, and $W_i = W_i^+ \cup W_i^-$. We call W_i^+ and W_i^- the *endslices* of I^∞ in the i -direction.

A subset of I^∞ of the form $\prod_{i=1}^\infty J_i$ is called a *closed basic set in I^∞* provided that J_i is a closed subinterval of I_i , for each i , and $J_i = I_i$ for all but finitely many i . An *open basic set in s* is the intersection of a closed basic set in I^∞ with s .

A homeomorphism h of I^∞ onto itself is said to be a β^* -homeomorphism provided that $h(s) = s$. The *pseudo-boundary* of I^∞ is $B(I^\infty) = I^\infty \setminus s$ and the *pseudo-interior* of I^∞ is s .

If $\{f_i\}_{i=1}^\infty$ is a sequence of homeomorphisms of a space X onto itself for which the sequence $\{f_i \circ f_{i-1} \circ \dots \circ f_1\}_{i=1}^\infty$ converges pointwise to a homeomorphism f of X onto itself, then we call f the *infinite left product* of $\{f_i\}_{i=1}^\infty$ and write $f = L \prod_{i=1}^\infty f_i$.

We list below three convergence procedures that we will need to insure the existence of an infinite left product of homeomorphisms. The first of these is Lemma 2.1 of [2] and the second is Theorem 2 of [5]. The third is an easy consequence of the apparatus used in [5] to establish the second.

CONVERGENCE PROCEDURE A. For each homeomorphism g of a compact metric space X onto itself and each $\varepsilon > 0$ let

$$\eta(g, \varepsilon) = \text{g.l.b. } \{d(g(x), g(y)) \mid d(x, y) \geq \varepsilon\}.$$

If $\{f_i\}_{i=1}^\infty$ is a sequence of homeomorphisms of X onto itself such that

$$d(f_i, \text{id}) < \min((3^{-i}), (3^{-i}) \cdot \eta(f_{i-1} \circ \dots \circ f_1, 2^{-i})),$$

for all $i > 1$, then $f = L \prod_{i=1}^\infty f_i$ exists.

CONVERGENCE PROCEDURE B. Let \mathcal{U} be a countable star-finite open cover of any space X . (By star-finite cover we mean a cover such that the closure of each member of the cover intersects the closures of only finitely many other members of the cover.) There exists an ordering $\{U_i\}_{i=1}^\infty$ of the elements of \mathcal{U} such that for any sequence $\{f_i\}_{i=1}^\infty$ of homeomorphisms of X onto itself, where f_i is the identity on $X \setminus U_i$ for all $i > 0$, $f = L \prod_{i=1}^\infty f_i$ exists. Moreover, we can assign a positive integer n_i to each U_i , independent of the choice of $\{f_i\}_{i=1}^\infty$, such that $n_i \leq n_{i+1}$ and

$$f(U_i) = (f_{n_i} \circ f_{n_i-1} \circ \dots \circ f_1)(U_i),$$

for all $i > 0$.

CONVERGENCE PROCEDURE B'. Let \mathcal{U} be a countable star-finite open cover of any space X and let $\{U_i\}_{i=1}^\infty, \{n_i\}_{i=1}^\infty$ be as in Convergence Procedure B. If Y is any space and $\{f_i\}_{i=1}^\infty$ is any sequence of homeomorphisms of $X \times Y$ onto itself such that f_i is

the identity on $(X \setminus U_i) \times Y$, for all $i > 0$, then $f = L \prod_{i=1}^{\infty} f_i$ exists. Moreover we have

$$f(U_i \times Y) = (f_{n_i} \circ \cdots \circ f_1)(U_i \times Y),$$

for all $i > 0$.

3. Two technical lemmas. We will need a result on extensions of homeomorphisms in I^∞ , where the given homeomorphism and its extension are required to lie in the same neighborhood of the identity.

LEMMA 1. *Let K_1, K_2 be compact subsets of I^∞ having Property Z in I^∞ and let $\varepsilon > 0$ be given. If $h: K_1 \rightarrow K_2$ is an onto homeomorphism such that $h(K_1 \cap s) = K_2 \cap s$ and $d(h, \text{id}) < \varepsilon$, then h can be extended to a β^* -homeomorphism H such that $d(H, \text{id}) < \varepsilon$.*

A homeomorphism extension theorem for I^∞ was first established by Anderson in [2], and modified with an “ ε -condition” in [6] by Barit and in [7] by Bessaga and Pełczyński. If the “ ε -condition” is omitted from the statement of our Lemma 1, then we obtain a statement which is equivalent to a homeomorphism extension theorem of Toruńczyk, which is given as Theorem 6 of [9]. If the extension theorem of Barit or that of Bessaga and Pełczyński (loc. cit.) is used, then our Lemma 1 follows routinely from the proof of Toruńczyk’s extension theorem.

LEMMA 2. *Let $\{F_i\}_{i=1}^{\infty}$ be a collection of relatively closed subsets of s , with each F_i having Property Z in s , let W be a finite union of endslices of I^∞ , and let $\{\alpha_i\}_{i=1}^{\infty}$ be a collection of pairwise-disjoint, infinite sets of positive integers. Then there is a β^* -homeomorphism h and an infinite, increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers such that h is the identity on W and $h(\bigcup_{i=1}^{\infty} F_i) = \bigcup_{i=1}^{\infty} K_i$, where each K_i is a relatively closed subset of s having deficiency with respect to α_{n_i} .*

Proof. We will construct h as an infinite left product of homeomorphisms $\{h_n\}_{n=1}^{\infty}$. Convergence will be assured by Convergence Procedure A by noting that whenever h_1, \dots, h_n have been constructed, h_{n+1} can be constructed arbitrarily close to the identity. We will only explicitly construct h_1 and h_2 , as the construction of h_2 will essentially constitute the inductive step.

Let $\beta_1 = \{i_1, i_2, \dots\}$ be an infinite, increasing sequence of positive integers such that $N \setminus \beta_1$ is infinite. We will construct a β^* -homeomorphism h_1 such that $h_1(F_1) = \bigcup_{n=1}^{\infty} A_n$, where each A_n is a relatively closed subset of s having deficiency with respect to α_{i_n} , and $h_1|_W = \text{id}$.

From Theorem 8.2 of [2] we know that \bar{F}_1 , the closure of F_1 in I^∞ , has Property Z in I^∞ . Using Theorem II A of [3] and Theorem 8.5 of [2] there is a homeomorphism f_1 of I^∞ onto itself such that $f_1(s \cup \bar{F}_1) = s$. Since $f_1(W \cap \bar{F}_1)$ is a compact subset of s we can use Theorem II A of [3] to obtain a homeomorphism f_2 of I^∞ onto itself such that

$$f_2[f_1(W \cap \bar{F}_1) \cup B(I^\infty)] = B(I^\infty).$$

Using Lemma 1 we can clearly get a β^* -homeomorphism f_3 such that $f_3 \circ f_2 \circ f_1|_W = \text{id}$.

Now we note that $f_3 \circ f_2 \circ f_1(\bar{F}_1 \setminus W) = \bigcup_{n=1}^\infty C_n$, a countable union of compact subsets of s . Let $\{\gamma_n\}_{n=1}^\infty$ be a collection of pairwise-disjoint, infinite sets of positive integers such that for each n , $\gamma_n \subset N \setminus \bigcup_{n=1}^\infty \alpha_{i_n}$. For each n write $\gamma_n = \gamma'_n \cup \gamma''_n$ such that $\gamma'_n \cap \gamma''_n = \emptyset$ and both are infinite, and where from Theorem 3.5 of [1] we can get a β^* -homeomorphism f_4 such that $f_4|W = \text{id}$ and $f_4(C_n)$ has deficiency with respect to $\alpha_{i_n} \cup \gamma'_n$.

Using Theorem 5.3 of [1] there is a homeomorphism f_5 of I^∞ onto itself such that

$$f_5[f_4 \circ f_3 \circ f_2 \circ f_1((\bar{F}_1 \cap B(I^\infty)) \setminus W) \cup B(I^\infty)] = B(I^\infty),$$

f_5 is the identity on W , and $\tau_i \circ f_5 = \tau_i$, for all $i \notin \bigcup_{n=1}^\infty \gamma'_n$. Then we put $h_1 = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$, which is obviously a β^* -homeomorphism satisfying $h_1|W = \text{id}$. It is also clear that $h_1(F_1) = \bigcup_{n=1}^\infty A_n$, where $A_n = f_5 \circ f_4(C_n) \cap s$ is a relatively closed subset of s which is deficient with respect to α_{i_n} , for all n .

We must be able to construct h_2 arbitrarily close to the identity. Thus let $\varepsilon > 0$ be given and let l be a positive integer such that $(\sum_{i=1}^\infty 2^{2^{-i}})^{1/2} < \varepsilon/6$. Let $\beta_2 = \{j_1, j_2, \dots\}$ be an infinite, increasing sequence of positive integers such that $\beta_2 \subset N \setminus \beta_1$, $N \setminus (\beta_1 \cup \beta_2)$ is infinite, and $i \in \bigcup_{n=1}^\infty \alpha_{j_n}$ implies that $i \geq l$. We will construct a β^* -homeomorphism h_2 such that $d(h_2, \text{id}) < \varepsilon$, $h_2 \circ h_1(F_2 \setminus F_1) = \bigcup_{n=1}^\infty B_n$, where each B_n is a relatively closed subset of s having deficiency with respect to α_{j_n} , and h_2 is the identity on $W_1 \cup h_1(W_1) \cup W \cup h_1(\bar{F}_1)$.

From Theorem IIA of [3] there is a homeomorphism g_1 of I^∞ onto itself such that $g_1(h_1(\bar{F}_1 \cup \bar{F}_2) \cup B(I^\infty)) = B(I^\infty)$ and $d(g_1, \text{id}) < \varepsilon/12$. Using Theorem 8.5 of [2] there is a homeomorphism g_2 of I^∞ onto itself such that $g_2(g_1 \circ h_1(\bar{F}_1 \cup \bar{F}_2) \cup s) = s$. Examining the apparatus used in [2] to obtain g_2 we find that we may additionally require that $d(g_2, \text{id}) < \varepsilon/12$. Put $A = W_1 \cup h_1(W_1) \cup W$ and note that

$$g_2 \circ g_1[h_1(\bar{F}_1 \cap B(I^\infty)) \cup (A \cap h_1(\bar{F}_2))]$$

is a σ -compact subset of s . Thus using Lemma IIA of [3] there is a homeomorphism g_3 of I^∞ onto itself such that

$$g_3\{g_2 \circ g_1[h_1(\bar{F}_1 \cap B(I^\infty)) \cup (A \cap h_1(\bar{F}_2))] \cup B(I^\infty)\} = B(I^\infty)$$

and $d(g_3, \text{id}) < \varepsilon/6$. Using Lemma 1 there is a β^* -homeomorphism g_4 such that $g_4 \circ g_3 \circ g_2 \circ g_1$ is the identity on $h_1(\bar{F}_1) \cup A$ and $d(g_4, \text{id}) < \varepsilon/3$.

Let $g_4 \circ g_3 \circ g_2 \circ g_1(h_1(\bar{F}_2) \setminus (h_1(\bar{F}_1) \cup A)) = \bigcup_{n=1}^\infty D_n$, a countable union of compact subsets of s . Let $\{\delta_n\}_{n=1}^\infty$ be a collection of pairwise-disjoint, infinite sets of positive integers such that $\delta_n \subset N \setminus \bigcup_{n=1}^\infty (\alpha_{i_n} \cup \alpha_{j_n})$ and $m \in \delta_n$ implies that $m \geq l$, for all $n > 0$. Then write each $\delta_n = \delta'_n \cup \delta''_n$ such that $\delta'_n \cap \delta''_n = \emptyset$ and both are infinite, and where from Theorem 3.5 of [1] there is a β^* -homeomorphism g_5 such that $g_5(D_n)$ has deficiency with respect to $\alpha_{j_n} \cup \delta'_n$ and $d(g_5, \text{id}) < \varepsilon/6$. If we examine the apparatus used in [1] to obtain g_5 we find that we may additionally require that $g_5| (h_1(\bar{F}_1) \cup A) = \text{id}$.

Now let g_6 be a homeomorphism of I^∞ onto itself such that

$$g_6\{[g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(h_1(\bar{F}_2) \cap B(I^\infty)) \cup (h_1(\bar{F}_1) \cup A)] \cup B(I^\infty)\} = B(I^\infty),$$

$$g_6|_{(h_1(\bar{F}_1) \cup A)} = \text{id}, \quad d(g_6, \text{id}) < \epsilon/6,$$

and $\tau_i \circ g_6 = \tau_i$, for all $i \notin \bigcup_{n=1}^\infty \delta'_n$. Put $h_2 = g_6 \circ g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1$, which is obviously a β^* -homeomorphism satisfying $h_2|_{(h_1(\bar{F}_1) \cup A)} = \text{id}$. It is clear that $h_2 \circ h_1(F_2 \setminus F_1) = \bigcup_{n=1}^\infty B_n$, where $B_n = g_6 \circ g_5(D_n) \cap s$ is a relatively closed subset of s which is deficient with respect to α_{j_n} , for all n . This completes the construction of h_2 and the proof of the lemma.

4. Separating closed sets with Property Z in F -manifolds.

LEMMA 3. *Let $\{F_i\}_{i=1}^\infty, \{K_j\}_{j=1}^\infty$ be collections of relatively closed subsets of s , each having Property Z in s , and let W be a finite union of endslices of I^∞ . Then there exists a B^* -homeomorphism h such that $h|_W = \text{id}$ and $h(\bigcup_{j=1}^\infty K_j) \cap (\bigcup_{i=1}^\infty F_i) = \emptyset$.*

Proof. Using Lemma 2 there exists a β^* -homeomorphism f_1 , a collection $\{\alpha_i\}_{i=1}^\infty$ of pairwise-disjoint, infinite subsets of N , and relatively closed subsets $\{C_i\}_{i=1}^\infty$ of s , each C_i having deficiency with respect to α_i , such that $f_1|_W = \text{id}$ and $f_1(\bigcup_{i=1}^\infty F_i) = \bigcup_{i=1}^\infty C_i$.

Applying Lemma 2 to $\{K_j\}_{j=1}^\infty$ we obtain a β^* -homeomorphism f_2 , a collection $\{\beta_j\}_{j=1}^\infty$ of pairwise-disjoint, infinite sets of positive integers such that each β_j is constructed by choosing exactly one element from each α_i , and a collection $\{D_j\}_{j=1}^\infty$ of relatively closed subsets of s such that each D_j is deficient with respect to β_j , $f_2|_W = \text{id}$, and $f_2(\bigcup_{j=1}^\infty K_j) = \bigcup_{j=1}^\infty D_j$.

We will now construct a β^* -homeomorphism g such that $g|_W = \text{id}$ and $g(D_j) \cap C_i = \emptyset$, for all i and j . For each i and j let $g_{i,j}$ be a β^* -homeomorphism which satisfies the following conditions

- (1) $\tau_k \circ g_{i,j} = \tau_k$, for $k \notin \alpha_i \cap \beta_j$,
- (2) $\tau_{\alpha_i \cap \beta_j} \circ g_{i,j}(D_j) \neq \tau_{\alpha_i \cap \beta_j}(C_i)$,
- (3) $g_{i,j}|_W = \text{id}$, and
- (4) $g_{i,j}(W_n^+) = W_n^+$ and $g_{i,j}(W_n^-) = W_n^-$, for all $n > 0$.

It is clear that (3) can be achieved by phasing out the motion isotopically as we approach W . Then we define

$$g = L \prod_{n=1}^\infty (g_{1,n} \circ g_{2,n-1} \circ \dots \circ g_{n,1}),$$

which satisfies our requirements. It is clear that $h = f_1^{-1} \circ g \circ f_2$ satisfies the conditions of the lemma.

LEMMA 4. *Let $\{F_i\}_{i=1}^\infty$ and $\{K_j\}_{j=1}^\infty$ be collections of closed subsets of any F -manifold X , with each F_i and K_j having Property Z in X . Then there is a homeomorphism h of X onto itself such that $h(\bigcup_{j=1}^\infty K_j) \cap (\bigcup_{i=1}^\infty F_i) = \emptyset$.*

Proof. Using Henderson's open embedding theorem let $f: X \rightarrow s$ be an embedding of X as an open subset of s . From Theorem 1 of [5] there is a star-finite open cover $\{U_k\}_{k=1}^\infty$ of $f(X)$, where each U_k is a basic open subset of s contained in $f(X)$. Moreover let us assume that the U_k 's are indexed as in Convergence Procedure B.

Note that for each $k > 0$, \bar{U}_k is a Hilbert cube, where \bar{U}_k is the closure of U_k in I^∞ , and U_k may be regarded as the relative pseudo-interior of \bar{U}_k . It is obvious that the topological boundary of U_k in s is contained in a finite union of endslices of the relative pseudo-boundary of \bar{U}_k . We can then apply Lemma 3 to obtain a homeomorphism g_1 of $f(X)$ onto itself such that $g_1|_{f(X)\setminus U_1} = \text{id}$ and

$$g_1\left(\bigcup_{j=1}^\infty f(K_j)\right) \cap \left(\bigcup_{i=1}^\infty f(F_i)\right) \cap U_1 = \emptyset.$$

Suppose now that homeomorphisms $\{g_k\}_{k=1}^n$ of $f(X)$ onto itself have been defined such that $g_k|_{f(X)\setminus U_k} = \text{id}$ and

$$(g_k \circ \dots \circ g_1)\left(\bigcup_{j=1}^\infty f(K_j)\right) \cap \left(\bigcup_{i=1}^\infty f(F_i)\right) \cap U_k = \emptyset,$$

for $1 \leq k \leq n$. Once more apply Lemma 3 to obtain a homeomorphism g_{n+1} of $f(X)$ onto itself such that $g_{n+1}|_{f(X)\setminus U_{n+1}} = \text{id}$ and

$$(g_{n+1} \circ \dots \circ g_1)\left(\bigcup_{j=1}^\infty f(K_j)\right) \cap \left(\bigcup_{i=1}^\infty f(F_i)\right) \cap U_{n+1} = \emptyset.$$

It is clear that $g = L \prod_{k=1}^\infty g_k$ defines a homeomorphism of $f(X)$ onto itself such that $g(\bigcup_{j=1}^\infty f(K_j)) \cap (\bigcup_{i=1}^\infty f(F_i)) = \emptyset$. Then $h = f^{-1} \circ g \circ f$ satisfies the requirements of our lemma.

5. Proof of Theorem 1. We break Theorem 1 up into two lemmas, the first treating the case $T = I^\infty$ and the second treating the case $T = s$.

LEMMA 5. *Let X be any F -manifold and let $\{K_i\}_{i=1}^\infty$ be a collection of closed subsets of $X \times I^\infty$, with each K_i having Property Z in $X \times I^\infty$. Then there is a homeomorphism h of $X \times I^\infty$ onto itself and a collection $\{\alpha_i\}_{i=1}^\infty$, where the α_i 's are pairwise-disjoint, infinite sets of positive integers, such that for all $i > 0$, $h(K_i)$ is deficient with respect to α_i in $X \times I^\infty$.*

Proof. For each positive integer j let

$$F_j^+ = \{(x, y) \in X \times I^\infty \mid \tau_j(y) = 1\}, \quad F_j^- = \{(x, y) \in X \times I^\infty \mid \tau_j(y) = -1\},$$

and $F_j = F_j^+ \cup F_j^-$. A straightforward and standard argument shows that each F_j has Property Z in $X \times I^\infty$. Thus applying Lemma 4 to $X \times I^\infty$ there is a homeomorphism f_1 of $X \times I^\infty$ onto itself such that $f_1(\bigcup_{i=1}^\infty K_i) \cap (\bigcup_{j=1}^\infty F_j) = \emptyset$.

Let j and m be any positive integers and for each $x \in X \times \prod_{i \neq j} I_i$ let $U_x \times (\delta_x, 1]$

be an open subset of $(X \times \prod_{i \neq j} I_i) \times I_j$ containing $(x, 1)$ such that $(U_x \times (\delta_x, 1]) \cap f_1(K_m) = \emptyset$. Note that $\{U_x \mid x \in X \times \prod_{i \neq j} I_i\}$ is an open cover of $X \times \prod_{i \neq j} I_i$. Since we have already noted that $X \times \prod_{i \neq j} I_i$ is homeomorphic to X , and is hence an F -manifold, we can apply Lemma 5 of [5] to get a star-finite open refinement $\{U_i\}_{i=1}^\infty$ of $\{U_x \mid x \in X \times \prod_{i \neq j} I_i\}$ which covers $X \times \prod_{i \neq j} I_i$. Moreover we may assume that the U_i 's are indexed as in Convergence Procedure B. It is clear that for each $i > 0$ there is a number δ_i such that $1/2 \leq \delta_i \leq 1$ and $(U_i \times (\delta_i, 1]) \cap f_1(K_m) = \emptyset$.

Now construct a closed cover $\{C_i\}_{i=1}^\infty$ of $X \times \prod_{i \neq j} I_i$ such that $C_i \subset U_i$, for all $i > 0$. For each $i > 0$ let $\varphi_i: X \times \prod_{i \neq j} I_i \rightarrow [0, 1]$ be a continuous function such that $\varphi_i(x) = 1$ for $x \in C_i$ and $\varphi_i(x) = 0$ for $x \in (X \times \prod_{i \neq j} I_i) \setminus U_i$. For each $i > 0$ we now construct a homeomorphism g_i of $(X \times \prod_{i \neq j} I_i) \times I_j$ onto itself which slides points linearly in the I_j -direction as follows: if $x \in X \times \prod_{i \neq j} I_i$, then $\{x\} \times I_j$ is taken linearly onto itself such that $g_i(x, \delta_i) = (x, \delta_i(1 - \varphi_i(x)) + (1/2)\varphi_i(x))$. Applying Convergence Procedure B' to $\{g_i\}_{i=1}^\infty$ let $g = L \prod_{i=1}^\infty g_i$, which is a homeomorphism of $X \times I^\infty$ onto itself satisfying $g \circ f_1(K_m) \subset X \times \prod_{i \neq j} I_i \times [-1, 1/2]$. Using similar techniques we can construct a homeomorphism g' of $X \times I^\infty$ onto itself which slides points only in the I_j -direction such that $g' \circ g \circ f_1(K_m) \subset X \times \prod_{i \neq j} I_i \times [-1/2, 1/2]$.

Choose a collection $\{\beta_i\}_{i=1}^\infty$ of pairwise-disjoint, infinite sets of integers such that $N = \bigcup_{i=1}^\infty \beta_i$. Then we can successively use the above construction to obtain a homeomorphism f_2 of $X \times I^\infty$ onto itself such that $\tau_j \circ \pi_j^\infty \circ f_2 \circ f_1(K_i) \subset [-1/2, 1/2]$, for all $j \in \beta_i$ and all $i > 0$.

Using Theorem 3.5 of [1] there is a homeomorphism d_i of I^{β_i} onto itself such that $d_i(\prod_{j \in \beta_i} [-1/2, 1/2]_j)$ has deficiency with respect to an infinite subset α_i of β_i , for all $i > 0$. Grouping these together we obtain a homeomorphism d of $X \times I^\infty = X \times I^{\beta_1} \times I^{\beta_2} \times \dots$ onto itself defined by $d = (\text{id}, d_1, d_2, \dots)$. Then $h = d \circ f_2 \circ f_1$ is a homeomorphism of $X \times I^\infty$ onto itself such that $h(K_i)$ is deficient with respect to α_i , for all $i > 0$.

LEMMA 6. *Let X be any F -manifold and let $\{K_i\}_{i=1}^\infty$ be a collection of closed subsets of $X \times s$, with each K_i having Property Z in $X \times s$. Then there is a homeomorphism h of $X \times s$ onto itself and a collection $\{\alpha_i\}_{i=1}^\infty$, where the α_i 's are pairwise-disjoint, infinite sets of positive integers, such that for all $i > 0$, $h(K_i)$ is deficient with respect to α_i in $X \times s$.*

Proof. Using Theorem 9.3 of [1] there is a homeomorphism f of $s = \prod_{i=1}^\infty I_i^0$ onto $s^\alpha \times I^\beta$, where α is the set of even positive integers and β is the set of odd positive integers. Let \tilde{f} be the homeomorphism of $X \times s$ onto $X \times s^\alpha \times I^\beta$ defined by $\tilde{f}(x, y) = (x, f(y))$, where $(x, y) \in X \times s$.

From Lemma 5 there is a homeomorphism g of $(X \times s^\alpha) \times I^\beta$ onto itself such that $\pi_j^\beta \circ g \circ \tilde{f}(K_i)$ is deficient with respect to β_i , where the β_i 's are pairwise-disjoint, infinite, and $\beta = \bigcup_{i=1}^\infty \beta_i$. Now write $\alpha = \bigcup_{i=1}^\infty \alpha'_i$, where the α'_i 's are pairwise-disjoint and infinite. Furthermore write $\alpha'_i = \bigcup_{j=1}^\infty \alpha_{i,j}$, where for each $i > 0$ the $\alpha_{i,j}$'s are

pairwise-disjoint and infinite, and write $\beta_i = \{m_{i,1}, m_{i,2}, \dots\}$, an increasing sequence of integers.

We note that for each i and j we have

$$\tau_k \circ \pi_{s^\alpha \times s^\beta} \circ g \circ \tilde{f}(K_i) \subset s^{\alpha_{i,j}} \times \{a_{i,j}\},$$

where $a_{i,j} \in I_{m_{i,j}}^0$ and $k = \alpha_{i,j} \cup \{m_{i,j}\}$. Theorem 9.3 of [1] gives a homeomorphism of $\prod_{i=1}^\infty I_i^0 \times [-1, 1]$ onto $\prod_{i=1}^\infty I_i^0 \times (-1, 1)$. If we examine the apparatus used to prove this we find that for each i and j we can get a homeomorphism $d_{i,j}$ of $s^{\alpha_{i,j}} \times I_{m_{i,j}}^0$ onto $s^{\alpha_{i,j}} \times I_{m_{i,j}}^0$ such that $d_{i,j}(s^{\alpha_{i,j}} \times \{a_{i,j}\}) = s^{\alpha_{i,j}} \times \{a_{i,j}\}$. This then gives a homeomorphism d of $X \times s^\alpha \times I^\beta$ onto $X \times s^\alpha \times s^\beta$ such that $d \circ g \circ \tilde{f}(K_i)$ is deficient with respect to β_i , for all $i > 0$. Thus with $h = d \circ g \circ \tilde{f}$ and $\{\alpha_i\}_{i=1}^\infty = \{\beta_i\}_{i=1}^\infty$ the proof of the lemma is complete.

6. Proof of Theorem 2. Using Theorem 1 there is a homeomorphism f_1 of X onto $X \times s$ and a collection $\{\alpha_i\}_{i=1}^\infty$ of pairwise-disjoint, infinite sets of positive integers such that for each $i > 0$, $f_1(K_i)$ is deficient with respect to α_i in $X \times s$.

Applying Henderson's open embedding theorem let $f_2: X \rightarrow s_1$ be an open embedding, where $s_1 = \prod \{I_i^0 \mid i \text{ odd}\}$. If $s_2 = \prod \{I_i^0 \mid i \text{ even}\}$ then there is a homeomorphism f_3 of s onto s_2 which is defined coordinatewise. Thus $f_4 = f_2 \times f_3$ is an open embedding of $X \times s$ into $s = s_1 \times s_2$ and there is a collection $\{\beta_i\}_{i=1}^\infty$ of pairwise-disjoint, infinite sets of positive even integers such that $f_4 \circ f_1(K_i)$ is deficient with respect to β_i , for all $i > 0$.

If $\text{Cl}(f_4 \circ f_1(K_i))$ is the closure of $f_4 \circ f_1(K_i)$ in I^∞ , then $\text{Cl}(f_4 \circ f_1(K_i)) \setminus f_4 \circ f_1(K_i) = \bigcup_{j=1}^\infty C_{i,j}$, a countable union of compact subsets of I^∞ , with each $C_{i,j}$ having deficiency with respect to β_i , for all $i > 0$. Using Theorem 5.3 of [1] there is a homeomorphism g of I^∞ onto itself such that $g[(\bigcup_{i=1}^\infty \bigcup_{j=1}^\infty C_{i,j}) \cup B(I^\infty)] = B(I^\infty)$ $\tau_i \circ g = \tau_i$, for i odd, and $g \circ f_4 \circ f_1(K_i)$ has infinite deficiency, for all $i > 0$. Clearly $h = g \circ f_4 \circ f_1$ is the required embedding.

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