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COUNTABLE PARACOMPACTNESS AND WEAK NORMALITY PROPERTIES

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In [4], Dowker proved that a normal space X is countably paracompact if and only if its product with the closed unit interval is normal. In this paper, we prove an analogue of Dowker's theorem. Specifically, we define the term *δ-normal* and then prove the following:

THEOREM 1. *A topological space is countably paracompact if and only if its product with the closed unit interval is δ-normal.*

After proving this theorem, we obtain similar results for the topological spaces studied in [7] and [11]. Also, cogent examples are given and the relation this note bears to the work of others is discussed.

We shall follow the terminology of [5] except we shall assume separation properties for a space only when these assumptions are explicitly stated.

For an infinite cardinal $\mathfrak{m}$, a set A in a topological space will be called a *$G_{\mathfrak{m}}$-set* (respectively, a *regular $G_{\mathfrak{m}}$-set*) provided it is the intersection of at most $\mathfrak{m}$ open sets (respectively, at most $\mathfrak{m}$ closed sets whose interiors contain A). If $\mathfrak{m}=\aleph_0$, we shall use the familiar terms G_δ-set and regular G_δ-set.

It is clear that the zero-set of any continuous real valued function is a regular G_δ-set and that the intersection of no more than $\mathfrak{m}$ such zero-sets is a regular $G_{\mathfrak{m}}$-set. In the remaining part of this paper, we shall use these facts without explicitly mentioning them.

DEFINITION. For an infinite cardinal $\mathfrak{m}$, a topological space is *$\mathfrak{m}$-normal* if each pair of disjoint closed sets, one of which is a regular $G_{\mathfrak{m}}$-set, have disjoint neighborhoods. For $\mathfrak{m}=\aleph_0$, we shall use the more suggestive term *δ-normal.*

Note that a normal space is $\mathfrak{m}$-normal and that a regular space is normal if and only if it is $\mathfrak{m}$-normal for every infinite cardinal $\mathfrak{m}$. On the other hand, a compact T_1-space that is not Hausdorff is $\mathfrak{m}$-normal for every infinite cardinal but yet it fails to be normal.

Recall that a space is *$\mathfrak{m}$-paracompact* if each open cover having cardinal less than or equal to $\mathfrak{m}$ has a locally finite open refinement. Characterizations of $\mathfrak{m}$-paracompact spaces may be found in [14] and [8].

THEOREM 2. *Each $\mathfrak{m}$-paracompact space is $\mathfrak{m}$-normal.*

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Proof. Suppose X is $\mathfrak{m}$-paracompact and let A and B be disjoint closed sets such that B is a regular $G_{\mathfrak{m}}$-set. Then there is a family $\mathscr{G}$, having cardinal less than or equal $\mathfrak{m}$, consisting of open neighborhoods of B such that B is the intersection of $\{\bar{G} : G \in \mathscr{G}\}$. Let Γ be the set of all finite nonempty subfamilies of $\mathscr{G}$. For $\alpha \in \Gamma$, define G_α to be the intersection of $\{G : G \in \alpha\}$. Then $B \subset G_\alpha$ and $B = \bigcap \bar{G}_\alpha$. The family of sets $X \backslash A \cap \bar{G}_\alpha$ is a directed open cover of X. By Theorem 5 in [8], there exists a locally finite open cover $\{V_\alpha\}$ such that $\bar{V}_\alpha \cap A \cap \bar{G}_\alpha = \varnothing$ for all α in Γ. Now let U be the union of all sets $V_\alpha \backslash \bar{G}_\alpha$ and V be the union of the sets $G_\alpha \backslash \bigcup \{\bar{V}_\beta : \beta \text{ is not a subset of } \alpha\}$. Clearly, U is open, contains A and is disjoint from V. Since $\{V_\alpha\}$ is locally finite it follows that V is open. For $x \in B$, let $\gamma = \bigcup \{\beta : x \in \bar{V}_\beta\}$. Since the cover $\{V_\alpha\}$ is locally finite, $\gamma \in \Gamma$ and $x \notin \bigcup \{\bar{V}_\beta : \beta \not\subset \gamma\}$; whence $x \in V$. Therefore $B \subset V$ and the proof is complete.

For the important special case where $\mathfrak{m} = \aleph_0$, we have:

THEOREM 3. *Each countably paracompact space is δ-normal.*

The *local weight* of a topological space is the least cardinal $\mathfrak{m}$ such that each point has a neighborhood base consisting of at most $\mathfrak{m}$ elements.

THEOREM 4. (i) *A Hausdorff* $\mathfrak{m}$-*normal space having local weight* $\leqq \mathfrak{m}$, *is regular.*
(ii) *A Hausdorff* $\mathfrak{m}$-*normal space having cardinal* $\leqq \mathfrak{m}$, *is regular.*

Proof. Under the hypotheses in each case, a singleton is a regular $G_{\mathfrak{m}}$-set.

For emphasis, we state the following special case:

THEOREM 5. *If a δ-normal Hausdorff space is either countable or satisfies the first axiom of countability, then it is regular.*

COROLLARY 6 (AULL [1]). *Each countably paracompact, first countable, Hausdorff space is regular.*

EXAMPLE. For each infinite cardinal $\mathfrak{m}$, there is an $\mathfrak{m}$-normal, Hausdorff space which is not regular. Given $\mathfrak{m}$, let w_α be the least ordinal having cardinal greater than $\mathfrak{m}$. Denote by W^* the set of ordinals less than or equal to w_α and by W, the set $W^* \backslash \{w_\alpha\}$. In $W^* \times W^* \backslash \{(w_\alpha, w_\alpha)\}$, identify all points of $W \times \{w_\alpha\}$. The quotient space X is Hausdorff but not regular (the images in X of the upper edge and the diagonal are not separated by disjoint open sets). Nonetheless, X is $\mathfrak{m}$-compact; hence it is $\mathfrak{m}$-paracompact and $\mathfrak{m}$-normal.

Throughout this paper I will denote the closed unit interval. In the proof of Theorem 1, we shall use the following lemma.

LEMMA 7. *For any topological space X, the following are equivalent*:

(a) *X is countably paracompact.*

(b) *If g is a strictly positive lower semicontinuous function on X, then there exist real valued functions l and u with l lower semicontinuous and u upper semicontinuous such that* $0 < l(x) \leqq u(x) \leqq g(x)$ *for all* $x \in X$.

(c) *If A is a closed subset of $X\times I$ and K is closed in I such that A and $X\times K$ are disjoint, then A and $X\times K$ have disjoint neighborhoods.*

Proof. The equivalence of (a) and (c) is due to Tamano (Theorem 3.9 in [16]) while the equivalence of (a) and (b) is an easy consequence of Theorem 10 in [7].

Proof of Theorem 1. If X is countably paracompact, then $X\times I$ is countably paracompact (Theorem 1 in [4]). By Theorem 3 above $X\times I$ is δ-normal. Conversely, suppose K is closed in I. Since I is metrizable, K is a regular G_δ-set. Therefore $X\times K$ is a regular G_δ-set in $X\times I$. In view of Lemma 7, the δ-normality of $X\times I$ will imply that X is countably paracompact.

THEOREM 8. *A closed continuous image of an $\mathfrak{m}$-normal space is $\mathfrak{m}$-normal.*

Proof. Observe that a continuous inverse image of a regular $G_{\mathfrak{m}}$-set is regular $G_{\mathfrak{m}}$. Using this fact, the standard proof that a closed continuous image of a normal space is normal, becomes applicable here.

REMARK. In general, it is not true that preimages of $\mathfrak{m}$-normal spaces are $\mathfrak{m}$-normal even for perfect maps (i.e., continuous closed maps for which the preimages of compact sets are compact). Note that in view of Theorem 1 and the fact that a space X is always the perfect image of $X\times I$, it follows that if every perfect preimage of X is δ-normal, then X is countably paracompact.

Many of the standard examples of nonnormal spaces, also, fail to be δ-normal. Here we give a partial list of such examples. (i) The space $S\times S$ where S is the reals with the half-open interval topology [15]. (ii) The space $X\times Y$ constructed by Michael [12]. (iii) The space R^R where R is the space of reals. (iv) The spaces constructed in problems 3K, 5I, 6P, 6Q of [5]. We shall use the space $S\times S$ to illustrate a technique that can be used to verify that these spaces are not δ-normal. In $S\times S$ let A be the set of points $(x, -x)$ where x is rational and B be the set of such points for irrational x. Then A is closed and B is a regular G_δ-set while these sets do not have disjoint neighborhoods. To show this, one can exploit the fact that the irrationals are not an F_σ-set in the reals (cf. [12]).

DEFINITION. A space will be called *δ-normally separated* if each closed set and each zero set disjoint from it are completely separated. A space will be termed *weakly δ-normally separated* if each regular closed set (i.e., the closure of an open set) and zero-set disjoint from it are completely separated.

REMARK. The properties of being δ-normal and δ-normally separated are, unfortunately, not comparable for arbitrary topological spaces. In a space where every regular G_δ-set is a zero-set, δ-normal separation implies δ-normality, but not conversely (see the example at the end of this paper). On the other hand, Hewitt's example [6] of an infinite regular Hausdorff space on which each continuous real-valued function is constant, is a δ-normally separated space which is not δ-normal (cf. Remark following Theorem 13). The author does not know whether among completely regular spaces, δ-normal separation implies δ-normality.

Clearly each normal space is δ-normally separated. Likewise, δ-normal separation implies weak δ-normal separation and the converse is true for δ-normal spaces.

The concept of δ-normal separation is not a new one. P. Zenor introduced this idea in [17] and used the term *Property Z.*

The δ-normal separation of a space X can be characterized in terms of properties of the ring $C(X)$ of a real-valued continuous function on X.

THEOREM 9. *A topological space X is δ-normally separated if and only if for each $f \in C(X)$ and each closed set A on which f is strictly positive, there exists a unit u of the ring $C(X)$ such that fu is identically one on A.*

Proof. Assume X is δ-normally separated and that f and A have the given properties. Then there exists a nonnegative element h of $C(X)$ which vanishes on A and assumes the value 1 everywhere on the zero-set of f. Then the ring inverse of $|f|+h$ is the desired unit. The converse is obvious.

We shall now proceed to state and prove the analogue of Theorem 1 for the δ-normal separation and weak δ-normal separation properties. To achieve this, we need to recall the definitions of cb-spaces and weak cb-spaces. A space X is a *cb-space* (respectively, *weak cb-space*) provided every locally bounded real valued function on X (respectively, every locally bounded lower semicontinuous function on X) is bounded above by a continuous function. Information concerning cb-spaces and weak cb-spaces may be found in [7] and [11], respectively. In comparing Theorem 1 with Theorem 11 below, it is useful to remember that a space is cb if and only if it is weak cb and countably paracompact.

LEMMA 10. (a) *Each cb-space is δ-normally separated.*
(b) *Each weak cb-space is weakly δ-normally separated.*

Proof. We shall prove (a) and make parenthetical comments to indicate the proof of (b). Let A be closed and Z be a zero-set disjoint from A. Given a nonnegative function h in $C(X)$ such that Z is the zero-set of h, define $g(x)=1+h(x)$ for x not in A and $g(x)=h(x)$ for x belonging to A. Then g is lower semicontinuous (normal lower semicontinuous if A is regular closed). Clearly g is strictly positive. By Theorem 1 in [7] (Theorem 3.1 in [11] for (b)) there is a strictly positive real valued continuous function f such that $f \leqq g$. Then the function h/f completely separates A and Z.

THEOREM 11. *Let X be a topological space. Then*
(a) *X is a cb-space if and only if $X \times I$ is δ-normally separated.*
(b) *X is a weak cb-space if and only if $X \times I$ is weakly δ-normally separated.*

Proof. The necessity follows from Lemma 10 above; the sufficiency from Corollary 12 and Theorem 13 in [9].

REMARK, In both Theorems 1 and 11, I may be replaced by any infinite compact metric space. Also, note that Theorem 10 in [9] implies that a variation of (a) in the above theorem is valid when I is replaced by an infinite product of intervals.

COROLLARY 12. (a) *Each countably compact space is both δ-normal and δ-normally separated.*

(b) *A completely regular, pseudocompact space is weakly δ-normally separated.*

Proof. Since countably compact spaces are *cb* (Corollary 3 in [7]) and completely regular pseudocompact spaces are weak *cb* (Corollary 3.8 in [11]), this theorem follows immediately from Lemma 10.

It is well known that a normal pseudocompact Hausdorff space is countably compact. In [17], Zenor shows that normality may be replaced by δ-normal separation. Here we show the condition can be further weakened to δ-normality.

THEOREM 13. *A completely regular space is countably compact if and only if it is δ-normal and pseudocompact.*

Proof. By Corollary 12 above, a pseudocompact δ-normal space is also δ-normally separated. This theorem now follows from Zenor's result (Theorem 3 in [17]).

REMARK. In Theorem 13, it is essential that the space be completely regular; for there exist regular, countably paracompact, Hausdorff spaces, that are not countably compact, on which every real valued function is constant [10]. Such a space can be obtained by altering slightly the construction used by Hewitt in [6].

For a completely regular space X, let υX denote the Hewitt realcompactification. In [5, p. 120], it is noted that the normality of X and of υX are independent of each other. The same is true for δ-normality and δ-normal separation. To see this, first, let X be a completely regular pseudocompact space that is not countably compact (the Tychonoff plank will do nicely). Then υX is compact and hence is both δ-normal and δ-normally separated, but X has neither of these properties. On the other hand, let P be the product R^R of $\mathfrak{c}$ ($\mathfrak{c}=\text{card } R$) copies of the reals R and let X be an associated Σ-product. Then X is normal, and $\upsilon X=P$ (see [3]) but P is not countably paracompact. Whence it follows from Theorem 1 that P is not δ-normal and from Theorem 11 that P is not δ-normally separated.

The situation for weak δ-normal separation is entirely different as Theorems 14 and 17 below will show.

THEOREM 14. *If a completely regular Hausdorff space X is weakly δ-normally separated then υX is as well.*

Proof. If A is regular closed in υX and Z is a zero-set in υX, then $A \cap X$ is regular closed in X and $Z \cap X$ is a zero-set in X. Moreover, A and Z are the closures in υX of $A \cap X$ and $Z \cap X$ respectively (for the latter see 8.8(b) in [5]). If f is a continuous real valued function on X which completely separates $A \cap X$ and $Z \cap X$, then its extension to υX clearly separates A from Z.

COROLLARY 15. *Any product of complete separable metric spaces is weakly δ-normally separated.*

Proof. In [3], it is proved that any such product is vX for some normal space X.

In order to obtain a partial converse of Theorem 14, we prove the following lemma which seems to be of independent interest. A point x of a space X is a *q-point* [13] if it has a sequence $\{U_n\}$ of neighborhoods such that if $\{x_n\}$ is a sequence of distinct points with $x_n \in U_n$, then this sequence has an accumulation point.

LEMMA 16. *If every point of $vX\backslash X$ is a q-point of vX, then every pair of disjoint sets A, Z where A is regular closed in X and Z is a zero-set in X, have disjoint closures in vX.*

Proof. Suppose on the contrary that p belongs to the closure of both A and Z and let $\{U_n\}$ be a sequence of open neighborhoods of p given by the definition of q-points. Let G denote the interior of A in X and $f \in C(X)$ be a function whose zero-set is Z. By our assumption p belongs to the closure of $G \cap \{x : |f(x)| < 1/n\}$ (call this set H_n) for each positive integer n. Whence $U_n \cap H_n$ is nonempty for each n. Pick x_n from this set. Clearly, we may assume that the x_n are distinct. Since A and Z are disjoint, it follows that $\{H_n\}$ is locally finite. Thus the sequence $\{x_n\}$ has no accumulation point. But this is impossible since p is a q-point.

THEOREM 17. *If vX is locally compact (more generally, if each point in $vX\backslash X$ has a compact neighborhood in vX), then X is weakly δ-normally separated if and only if vX has the same property.*

A converse for Theorem 14 is not possible, without some sort of restriction on vX. This is shown in the example below.

It is natural to ask what relation the above results bear to the well known unanswered question [4, p. 221]: *Must the product of a normal Hausdorff space with the closed unit interval be normal*? In this regard, first, observe that if X is normal, then $X \times I$ is normal provided it is δ-normal. This fact suggests the following question: *If X is a regular, δ-normal space, must $X \times I$ be δ-normal*? Except for noting that without the assumption that the space is regular, the answer to this question is negative (see p. 221 in [4]), the author has not obtained any significant clues concerning the answer to this question. On the other hand, the answer to the corresponding question for δ-normal separation is negative. This is the substance of the following example.

EXAMPLE. Let X and X^* be the spaces constructed on pp. 240, 241 of [11]. There it is pointed out that X is locally compact, countably paracompact but not a cb-space while X^* is σ-compact but not locally compact and that $X^* = vX$. It is a simple matter (using Theorem 9 and the special relation that X bears to X^*) to show that X is also δ-normally separated. Since X is not a cb-space, it follows from Theorem 11 that $X \times I$ is not δ-normally separated (or even weakly δ-normally separated). It is, however, δ-normal.

Also, in view of Theorem 2.8 in [2], note that $\upsilon(X\times I)=\upsilon X\times I$. Now since $\upsilon X\times I=X^*\times I$ is Lindelöf and regular, it is normal. Nonetheless $X\times I$ fails to be weakly δ-normally separated. This shows that the restriction on υX in Theorem 17 cannot be entirely suppressed.

In [14], Morita obtained the following generalization of Dowker's theorem [4]: A space X is $\mathfrak{m}$-paracompact and normal if and only if $X\times I^{\mathfrak{m}}$ is normal. In view of Morita's result, it is natural to ask: What condition on X is necessary and sufficient for $X\times I^{\mathfrak{m}}$ to be $\mathfrak{m}$-normal? Theorem 2 implies that $\mathfrak{m}$-paracompactness of X is a sufficient condition; however the author has been unable to determine whether $\mathfrak{m}$-paracompactness is also necessary. The chief stumbling block is the lack of a characterization of $\mathfrak{m}$-paracompactness similar to that for countable paracompactness given by Tamano (Lemma 7 above).

In contrast to the obstacles encountered in attempting to obtain an analogue of Theorem 1 for uncountable cardinals, Theorem 11 (as pointed out in the Remark following that theorem) can be extended by merely giving an appropriate meaning to the term $\mathfrak{m}$-normal separation. Specifically, define a space to be $\mathfrak{m}$-*normally separated* provided the intersection of any family consisting of at most $\mathfrak{m}$ zero-sets is completely separated from any closed set disjoint from it. Then for any space X, $X\times I^{\mathfrak{m}}$ is $\mathfrak{m}$-normally separated if and only if X is an $H(\mathfrak{m})$-space in the sense of [9].

References

1. C. E. Aull, *A note on countably paracompact spaces and metrization*, Proc. Amer. Math. Soc. **16** (1965), 1316–1317. MR **32** #3039.

2. W. W. Comfort and S. Negrepontis, *Extending continuous functions on $X\times Y$ to subsets of $\beta X\times\beta Y$*, Fund. Math. **59** (1966), 1–12. MR **34** #782.

3. H. H. Corson, *Normality in subsets of product spaces*, Amer. J. Math. **81** (1959), 785–796. MR **21** #5947.

4. C. H. Dowker, *On countably paracompact spaces*, Canad. J. Math. **3** (1951), 219–224. MR **13**, 264.

5. L. Gillman and M. Jerison, *Rings of continuous functions*, The University Series in Higher Math., Van Nostrand, Princeton, N. J., 1960. MR **22** #6994.

6. E. Hewitt, *On two problems of Urysohn*, Ann. of Math. (2) **47** (1946), 503–509. MR **8**, 165.

7. John Mack, *On a class of countably paracompact spaces*, Proc. Amer. Math. Soc. **16** (1965), 467–472. MR **31** #1651.

8. ———, *Directed covers and paracompact spaces*, Canad. J. Math. **19** (1967), 649–654. MR **35** #2263.

9. ———, *Product spaces and paracompactness*, J. London Math. Soc. (2) **1** (1969), 90–94.

10. ———, *Countably paracompact spaces on which every real-valued continuous function is constant*, Proc. Amer. Math. Soc. (to appear).

11. John Mack and D. G. Johnson, *The Dedekind completion of $C(\mathscr{X})$*, Pacific J. Math. **20** (1967), 231–243. MR **35** #2150.

12. E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. **69** (1963), 375–376. MR **27** #2956.

13. E. Michael, *A note on closed maps and compact sets*, Israel J. Math. **2** (1964), 173–176. MR **31** #1659.

14. K. Morita, *Paracompactness and product spaces*, Fund Math. **50** (1961/62), 223–236. MR **24** #A2365.

15. R. H. Sorgenfrey, *On the topological product of paracompact spaces*, Bull. Amer. Math. Soc. **53** (1947), 631–632. MR **8**, 594.

16. H. Tamano, *On compactifications*, J. Math. Kyoto Univ. **1** (1961/62), 161–193. MR **25** #5489.

17. P. Zenor, *A note on Z-mapping and WZ-mappings*, Proc. Amer. Math. Soc. **23** (1969), 273–275.

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