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COMPACT IMBEDDING THEOREMS FOR QUASIBOUNDED DOMAINS

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1. **Introduction.** Let G be an unbounded open set in Euclidean n-space E_n. In this paper we investigate (for a large class of such domains) the problem of determining for which values of m, p, j and r the Sobolev space imbedding

$$W_0^{m,p}(G) \to W_0^{j,r}(G) \tag{1}$$

is or is not compact. Provided $j<m$ continuous imbeddings of this type are known to exist for $p \leqq r \leqq np(n-mp+jp)^{-1}$ if $n>mp-jp$ or for $p \leqq r<\infty$ if $n \leqq mp-jp$ (the Sobolev Imbedding Theorem, e.g. [5, Lemma 5]). If G were bounded Kondrašov's compactness theorem [9] would yield the complete continuity of these imbeddings except in the extreme case $r=np(n-mp+jp)^{-1}$. Such compactness theorems are useful for studying existence and spectral theory for partial differential operators on G.

In a sequence of recent papers the writer [1]–[4] and C. W. Clark [6], [7], [8] have studied such compactness problems for various unbounded domains. It is clear that the imbedding (1) cannot be compact if G contains infinitely many disjoint congruent balls, for if a fixed C^∞ function has support in one of these balls then the set of its translates with supports in the other balls is bounded in any space $W_0^{m,p}(G)$ but is not precompact in any such space. Thus a necessary condition for the compactness of imbedding (1) is that G should be *quasibounded*, i.e. that dist $(x, \text{bdry } G) \to 0$ whenever $|x| \to \infty$, $x \in G$. In [1] the writer has shown that if $n>1$ then quasiboundedness is not sufficient for compactness.

The dimension of the boundary of G is a critical factor in determining whether or not (1) is compact. If G is quasibounded and bounded by smooth "reasonably unbroken" $(n-1)$-dimensional manifolds then (1) is compact [3, Theorem 1] for any m and p and for the same values of j and r as in the case of bounded G. However if G has discrete (0-dimensional) boundary then [2, Theorem 1] no such imbedding can be compact unless $mp>n$.

Our purpose in this paper is to study the compactness of imbedding (1) for quasibounded domains G whose boundaries are comprised of smooth manifolds

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of various dimensions. Roughly speaking our results are as follows. If k is the smallest integer for which those boundary manifolds of G having dimension not less than $n-k$ bound a quasibounded domain then no imbedding of type (1) can be compact when $mp<k$. On the other hand, if, in addition, the boundary manifolds are "reasonably unbroken" and if $mp>n+p-np/k$ then (1) is compact for the same values of j and r as in the case of bounded G. Our results thus interpolate between the extreme cases mentioned above. We consider first domains G with flat (planar) boundaries, establishing in §2 a necessary condition for the compactness of (1) for such G, and in §3 a slightly stronger sufficient condition. If $m=1$ these conditions are equivalent for certain domains. In §4 similar results are obtained for nonflatly-bounded domains G.

As usual, in this paper $W_0^{m,p}(G)$ denotes, for $p \geqq 1$ and $m=0, 1, 2, \ldots$, the Sobolev space obtained by completing with respect to the norm

$$\|u\|_{m,p,G} = \left\{ \sum_{j=0}^{m} |u|_{j,p,G}^p \right\}^{1/p} \tag{2}$$

the space $C_0^\infty(G)$ of all infinitely differentiable, complex functions having compact support in G where

$$|u|_{j,p,G}^p = \sum_{|\alpha|=j} \int_G |D^\alpha u(x)|^p \, dx.$$

α denotes an n-tuple of nonnegative integers $(\alpha_1, \ldots, \alpha_n)$; $|\alpha| = \alpha_1 + \cdots + \alpha_n$; $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$; $D_j = \partial/\partial x_j$. Note that $|u|_{0,p,G} = \|u\|_{0,p,G}$ is the norm of u in $L^p(G)$. $W^{m,p}(G)$ represents the completion with respect to the norm (2) of the space of all infinitely differentiable functions on G for which (2) is finite. Provided the boundary of G satisfies certain mild regularity conditions [5, Lemma 5] the Sobolev Imbedding Theorem referred to above, and also (provided G is bounded) the Kondrašov Compactness Theorem, remain valid for imbeddings of $W^{m,p}(G)$. No compactness theorems of this sort are yet known if G is unbounded.

2. **Flatly-bounded domains—noncompact imbeddings.** Let H be a k-dimensional plane $(0 \leqq k \leqq n-1)$ in E_n and let a be a point on H. With respect to a new system of rectangular coordinates z in E_n having origin at a and obtained from the usual coordinates by an affine transformation, H has equations $z_1 = z_2 = \cdots = z_{n-k} = 0$, or more simply $r=0$ where $r = \sum_{i=1}^{n-k} z_i^2$. The coordinate r, together with $n-k-1$ angle coordinates collectively denoted σ and the coordinates $z' = (z_{n-k+1}, \ldots, z_n)$ form a system of cylindrical polar coordinates in E_n with origin at a and cylindrical axis H.

The *k-tube* $T_\delta(H)$ of radius δ and axis H is the set $\{x \in E_n : \operatorname{dist}(x, H) = r < \delta\}$. By a *tube function* for the tube $T_\delta(H)$ we mean a C^∞ function $\theta: E_n \to [0, 1]$ whose value at x depends only on $r = \operatorname{dist}(x, H)$ and which vanishes identically near H and is identically unity outside $T_\delta(H)$.

LEMMA 1. *Let H be a k-plane in E_n and $a \in H$. Let $1 \leqq p < \infty$. If $u(x) = v(r)$ where $r = \operatorname{dist}(x, H)$ and $v \in C^{|\alpha|}((0, \infty))$ then for all $x \notin H$*

$$|D^\alpha u(x)|^p \leqq \text{const} \sum_{j=1}^{|\alpha|} |v^{(j)}(r)|^p r^{pj - p|\alpha|}$$

where the constant depends only on α, p and k.

Proof. Since $z_i = \sum_{j=1}^n c_{ij}(x_j - a_j)$ and so $\partial/\partial x_i = \sum_{j=1}^n c_{ji}\, \partial/\partial z_j$ we may assume with no loss of generality that H is a coordinate plane and $z = x$. We show that there exist homogeneous polynomials $P_{\alpha,j}(x)$ of degree $|\alpha|$ (possibly the zero polynomial) such that for $r > 0$

$$D^\alpha u(x) = \sum_{j=1}^{|\alpha|} P_{\alpha,j}(x) v^{(j)}(r) r^{j - 2|\alpha|}. \tag{3}$$

Since $|P_{\alpha,j}(x)| \leqq \text{const}\, r^{|\alpha|}$ the conclusion of the lemma for $p = 1$ follows at once. The result for general p then follows from the well-known inequality

$$\left| \sum_{j=1}^N A_j \right|^p \leqq \text{const} \sum_{j=1}^N |A_j|^p \tag{4}$$

where the constant depends only on p and N.

Note that $D^\alpha u(x) = 0$ unless $\alpha_{n-k+1} = \cdots = \alpha_n = 0$. If $1 \leqq i \leqq n-k$ then $D_i u(x) = v'(r) x_i / r$ which is of the required form. Assume (3) holds for all α with $|\alpha| \leqq m$. If $|\beta| = m+1$ then $D^\beta = D_i D^\alpha$ for some i, α where $|\alpha| = m$. Applying the induction hypothesis and the chain rule we obtain

$$\begin{aligned} D^\beta u(x) &= \sum_{j=1}^m \{D_i P_{\alpha,j}(x) v^{(j)}(r) r^{j-2m} + P_{\alpha,j}(x) v^{(j+1)}(r) x_i r^{j-2m-1} \\ &\qquad - (2m - j) P_{\alpha,j}(x) v^{(j)}(r) x_i r^{j-2m-2}\} \\ &= \sum_{j=1}^{m+1} P_{\beta,j}(x) v^{(j)}(r) r^{j - 2(m+1)} \end{aligned}$$

where $P_{\beta,j}$ is given by

$$\begin{aligned} P_{\beta,1}(x) &= r^2 D_i P_{\alpha,1}(x) - (2m-1) x_i P_{\alpha,1}(x), \\ P_{\beta,j}(x) &= r^2 D_i P_{\alpha,j}(x) - (2m-j) x_i P_{\alpha,j}(x) + x_i P_{\alpha,j-1}(x) \quad \text{if } 2 \leqq j \leqq m, \\ P_{\beta,m+1}(x) &= x_i P_{\alpha,m}(x). \end{aligned}$$

Clearly $P_{\alpha,j}(x)$ is a polynomial of the desired type and the proof is complete.

LEMMA 2. *Let λ be a positive integer and let $r = s^\lambda$, $s > 0$. If $f \in C^j((0, \infty))$ and $1 \leqq p < \infty$ then*

$$|(d/dr)^j f(r^{1/\lambda})|^p \leqq \text{const} \sum_{i=1}^{j} \lambda^{-ip} s^{ip - j\lambda p} |f^{(i)}(s)|^p \tag{5}$$

where the constant depends only on j and p.

Proof. Again the case of general p follows from the special case $p=1$ via (4). For $p=1$ (5) is an immediate consequence of the formula

$$(d/dr)^j = \lambda^{-j} \sum_{i=1}^{j} P_{j-i,j}(\lambda) s^{i-\lambda j} (d/ds)^i \tag{6}$$

where $P_{i,j}$ is a polynomial of degree i depending on j. We prove (6) by induction on j. Note that $d/dr = \lambda^{-1} s^{1-\lambda} d/ds$ which is of the required form. Assuming (6) we have

$$\begin{aligned}(d/dr)^{j+1} &= \lambda^{-j} \sum_{i=1}^{j} P_{j-i,j}(\lambda) \lambda^{-1} s^{1-\lambda} d/ds[s^{i-\lambda j}(d/ds)^i] \\ &= \lambda^{-(j+1)} \sum_{i=1}^{j} P_{j-i,j}(\lambda)\{s^{i+1-\lambda-\lambda j}(d/ds)^{i+1} + (i-\lambda j)s^{i-\lambda j-\lambda}(d/ds)^i\} \\ &= \lambda^{-(j+1)} \sum_{i=1}^{j+1} P_{j+1-i,j+1}(\lambda) s^{i-\lambda(j+1)}(d/ds)^i\end{aligned}$$

where the polynomials $P_{i,j+1}$ are given by

$$\begin{aligned}P_{0,j+1}(\lambda) &= P_{0,j}(\lambda), \\ P_{i,j+1}(\lambda) &= P_{i,j}(\lambda) + (j+1-i-\lambda j)P_{i-1,j}(\lambda) \quad \text{for } 1 \leqq i \leqq j-1, \\ P_{j,j+1}(\lambda) &= (1-\lambda j)P_{j-1,j}(\lambda),\end{aligned}$$

which are of the desired form.

LEMMA 3. *Let T be a k-tube in E_n with axis H and radius $\delta \leqq 1$. Let $1 \leqq p < \infty$ and let λ be a positive integer. Then there exists a tube function θ for T satisfying for $|\alpha| > 0$,*

$$|D^\alpha \theta(x)|^p \leqq \text{const } \lambda^{-p} s^{p-\lambda p|\alpha|}$$

where $s^\lambda = r = \text{dist}(x, H)$ and the constant depends only on α, n, p and k and not on λ.

Proof. Let f: $[0, \infty) \to [0, 1]$ be a fixed C^∞ function such that $f(s)=0$ near $s=0$ and $f(s)=1$ for $s^\lambda \geqq \delta$. Define θ by $\theta(x)=v(r)=f(s)$. Clearly θ is a tube function for T. By Lemmas 1 and 2 we have

$$\begin{aligned}|D^\alpha \theta(x)|^p &\leqq \text{const} \sum_{j=1}^{|\alpha|} r^{pj-p|\alpha|} |v^{(j)}(r)|^p \\ &\leqq \text{const} \sum_{j=1}^{|\alpha|} \sum_{i=1}^{j} \lambda^{-ip} s^{ip-\lambda p|\alpha|} |f^{(i)}(s)|^p \\ &\leqq \text{const } \lambda^{-p} s^{p-\lambda p|\alpha|}.\end{aligned}$$

The final inequality follows because whenever $D^\alpha \theta(x) \neq 0$, $s < \delta^{1/\lambda} \leqq 1$, and also $|f^{(i)}(s)| \leqq \text{const}$ for $1 \leqq i \leqq |\alpha|$.

In the following lemma we consider several $(n-k)$-tubes in E_n simultaneously. Hence all the related quantities θ, r, σ, z', s, H carry subscripts ranging from 1 to N.

LEMMA 4. *Let S be a bounded open set in E_n. Let $H_1, \ldots, H_N$ be a finite collection of $(n-k)$-planes which intersect $\bar{S}$. Let m be a positive integer and let $\varepsilon > 0$. If either $p>1$ and $mp \leqq k$ or $p=1$ and $m<k$ then there exists a function $\psi \in C^\infty(E_n)$ with the properties*:

(i) $\psi(x)=0$ *for x near* $\bigcup_{i=1}^N H_i$,

(ii) $0 \leqq \psi(x) \leqq 1$ *for all x*,

(iii) $\psi(x)=1$ *for x in* $E_n - \bigcup_{i=1}^N T_\delta(H_i)$, $\delta>0$,

(iv) $\|D^\alpha \psi\|_{0,p,S} \leqq \varepsilon$ *for* $0<|\alpha| \leqq m$.

Proof. We first consider the case that no two of the planes H_i intersect in $\bar{S}$. It is then possible to choose $\delta \leqq 1$ small enough so that if $T_i = T_\delta(H_i)$ then $T_i \cap T_j \cap S$ is empty if $i \neq j$. By Lemma 3 there exist tube functions θ_i for T_i satisfying

$$|D^\alpha \theta_i(x)|^p \leqq \text{const } \lambda^{-p} s_i^{p-\lambda p|\alpha|}$$

where $s_i^\lambda = r_i = \text{dist}\,(x, H_i)$ and the constant is independent of λ and α for $0<|\alpha| \leqq m$. Let $\psi(x) = \theta_1(x)\theta_2(x)\cdots\theta_N(x)$. Clearly ψ satisfies (i)–(iii). Note that $D^\alpha\psi = 0$ outside $\bigcup_{i=1}^N T_i$ and that $D^\alpha\psi(x) = D^\alpha\theta_i(x)$ in T_i. We have

$$\begin{aligned}\|D^\alpha\psi\|_{0,p,S}^p &= \sum_{i=1}^N \int_{S\cap T_i} |D^\alpha\theta_i(x)|^p\,dx \\ &\leqq \text{const } \lambda^{-p} \sum_{i=1}^N \int_{S\cap T_i} s_i^{p-\lambda p|\alpha|} r_i^{k-1}\,dr_i\,d\sigma_i\,dz_i' \\ &\leqq \text{const } \lambda^{1-p} \int_0^1 s_i^{p-\lambda p|\alpha|+\lambda k-1}\,ds_i.\end{aligned}$$

The final constant depends on α, p, n, k, N and diam S but not on λ. If $|\alpha| \leqq m$ and $mp \leqq k$ then $p - \lambda p|\alpha| + \lambda k > 0$ and so

$$\|D^\alpha\psi\|_{0,p,S}^p \leqq \text{const } \lambda^{1-p}(p+\lambda k - \lambda p|\alpha|)^{-1}.$$

The expression on the right can be made arbitrarily small for sufficiently large λ provided either $p>1$ or $m<k$. This establishes (iv).

The case of intersecting H_i remains to be considered. Again pick $\delta<1$ small enough so that $T_i \cap T_j \cap S$ is empty whenever $H_i \cap H_j \cap \bar{S}$ is empty. Define θ_i and ψ as above. The general Leibniz formula states

$$D^\alpha\psi(x) = \sum_{\beta_1+\cdots+\beta_N=\alpha} \binom{\alpha}{\beta_1, \ldots, \beta_N} D^{\beta_1}\theta_1(x)\cdots D^{\beta_N}\theta_N(x). \tag{7}$$

For estimates of $|D^\alpha\psi(x)|$ we may drop from terms in the Leibniz expression any factor $D^{\beta_i}\theta_i(x)$ for which $\beta_i = 0$ because $|\theta_i(x)| \leqq 1$. For simplicity consider a term $D^{\beta_1}\theta_1(x)\cdots D^{\beta_N}\theta_N(x)$ where no β_i is zero. Decompose S into the union of N subregions S_j such that in S_j we have $s_j \leqq s_i$ for $i \neq j$. We now obtain via Lemma 3 in the manner above, noting that whenever $D^{\beta_i}\theta_i(x) \neq 0$ then $s_i < \delta \leqq 1$

$$\begin{aligned}\int_S |D^{\beta_1}\theta_1(x)\cdots D^{\beta_N}\theta_N(x)|^p\,dx &= \sum_{j=1}^N \int_{S_j} |D^{\beta_i}\theta_1(x)\cdots D^{\beta_N}\theta_N(x)|^p\,dx \\ &\leqq \text{const}\,\lambda^{-Np}\sum_{j=1}^N \int_{S_j} s_1^{p-\lambda p|\beta_1|}\cdots s_N^{p-\lambda p|\beta_N|}\,dx \\ &\leqq \text{const}\,\lambda^{-Np}\sum_{j=1}^N \int_{S_j} s_j^{Np-\lambda p|\alpha|}\,dx \\ &\leqq \text{const}\,\lambda^{1-Np}\int_0^1 s^{Np-\lambda p|\alpha|+\lambda k-1}\,ds \\ &\leqq \text{const}\,\lambda^{1-Np}(Np+\lambda k-\lambda p|\alpha|)^{-1}.\end{aligned}$$

Similar estimates can be found for all terms in (7) and (iv) follows once more by taking λ sufficiently large.

DEFINITION. Let G be an open set in E_n. G is called a *regular domain* if bdry $G = \bigcup_{k=0}^{n-1} G_k$ where G_k is the union of a locally finite collection of smooth manifolds of dimension k in E_n. A regular domain G whose boundary manifolds are all segments of planes of various dimensions will be called a *regular flatly-bounded domain.* An unbounded regular domain G is called 0-*quasibounded* if it is quasibounded, i.e. if there exist at most finitely many disjoint congruent balls Q in G, having any specified positive radius, which do not intersect the boundary of G. G is called *k-quasibounded* $(1 \leqq k \leqq n-1)$ if there exist at most finitely many disjoint congruent balls Q in $\bar{G}$, having any specified positive radius, such that $Q \cap$ bdry $G \subset \bigcup_{i=0}^{k-1} G_i$; i.e. if dist $(x, \bigcup_{i=k}^{n-1} G_i) \to 0$ as $|x| \to \infty$, $x \in G$. For $1 \leqq k \leqq n-1$ the condition of k-quasiboundedness is stronger than that of $(k-1)$-quasiboundedness.

THEOREM 1. *Let G be a regular, quasibounded, flatly-bounded domain in E_n. Let k be the smallest integer $(1 \leqq k \leqq n)$ for which G is $(n-k)$-quasibounded. If either $mp \leqq k$ and $p > 1$ or $m < k$ and $p = 1$ then no imbedding of the form*

$$W_0^{m,p}(G) \to W_0^{j,r}(G)$$

can be compact.

Proof. For $2 \leqq k \leqq n$ since G is not $(n-k+1)$-quasibounded there is a sequence of congruent open balls $\{Q_i\}_{i=1}^{\infty}$ in $\bar{G}$ such that $Q_i \cap$ bdry G is contained in the union of finitely many $(n-k)$-planes. If $k=1$ balls Q_i with this property exist trivially since G is regular and flatly-bounded. Let Q denote any one of these balls and let $H_1, \dots, H_N$ be the corresponding $(n-k)$-planes. Let $\varphi \in C_0^\infty(Q)$ be a function for which

$$\|\varphi\|_{0,r,Q} = 2C > 0, \qquad \|\varphi\|_{m,p,Q} = K < \infty.$$

There exists a constant M such that for all $x \in E_n$ and for all α with $0 \leqq |\alpha| \leqq m$, $|D^\alpha\varphi(x)| \leqq M$. Choose $\delta_0 > 0$ small enough so that the sum of the volumes of the intersections with Q of the $(n-k)$-tubes $T_{\delta_0}(H_i)$, $i = 1, \dots, N$, does not exceed $(C/M)^r$. Let

$$\varepsilon = K\left[M \sum_{|\alpha|\leqq m}\sum_{\beta<\alpha}\binom{\alpha}{\beta}\right]^{-1}.$$

By Lemma 4 there exists for some $\delta \leqq \delta_0$ a function $\psi \in C_0^\infty(E_n - \bigcup_{i=1}^N H_i)$ satisfying $0 \leqq \psi(x) \leqq 1$, $\psi(x) = 1$ outside $\bigcup_{i=1}^N T_\delta(H_i)$ and $\|D^\alpha \psi\|_{0,p,Q} \leqq \varepsilon$ for $0 < |\alpha| \leqq m$. Let $\gamma = \varphi \cdot \psi = \varphi - \varphi(1-\psi)$. Clearly $\gamma \in C_0^\infty(Q \cap G)$. Putting $T_i = T_\delta(H_i)$ we have

$$\begin{aligned}\|\gamma\|_{0,r,G} &\geqq \|\varphi\|_{0,r,Q} - \|\varphi\|_{0,r,Q \cap (T_1 \cup \cdots \cup T_N)} \\ &\geqq 2C - M[\operatorname{vol} Q \cap (T_1 \cup \cdots \cup T_N)]^{1/r} \geqq C.\end{aligned}$$

Moreover, for $0 < |\alpha| \leqq m$

$$D^\alpha \gamma = \psi D^\alpha \varphi + \sum_{\beta < \alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha - \beta} \psi,$$

$$\|D^\alpha \gamma\|_{0,p,Q} \leqq \|D^\alpha \varphi\|_{0,p,Q} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} M\varepsilon \leqq 2K.$$

Thus if $K_1^p = \sum_{|\alpha| \leqq m} 1$ we have $\|\gamma\|_{m,p,G} \leqq 2KK_1$.

Now let φ_i, $i = 1, 2, \ldots$ be a translate of φ with support in Q_i and let γ_i be constructed from φ_i as γ from φ above, so that

$$\|\gamma_i\|_{0,r,G} \geqq C, \qquad \|\gamma_i\|_{m,p,G} \leqq 2KK_1.$$

The sequence $\{\gamma_i\}$, though bounded in $W_0^{m,p}(G)$ has no subsequence converging in $L^r(G)$. In fact if $i \neq j$ then $\|\gamma_i - \gamma_j\|_{0,r,G} \geqq 2^{1/r}c$. Thus the imbedding

$$W_0^{m,p}(G) \to W_0^{0,r}(G) = L^r(G)$$

if it exists cannot be compact. Neither can the imbedding $W_0^{m,p}(G) \to W_0^{j,r}(G)$ for if this latter imbedding were compact then so would be the composition

$$W_0^{m,p}(G) \to W_0^{j,r}(G) \to L^r(G).$$

3. **Flatly-bounded domains—compact imbeddings.** Let H be a k-plane in E_n and let $a \in H$. We denote by $T_{\delta,\rho}(H, a)$ the *tube segment* of radius δ and length 2ρ having axis H and centre a. Thus, if P is the orthogonal projection operator on H then

$$T_{\delta,\rho}(H, a) = \{x \in E_n : \operatorname{dist}(x, H) < \delta, \operatorname{dist}(Px, a) < \rho\}.$$

LEMMA 5. *Let H be an $(n-k)$-plane in E_n $(1 \leqq k \leqq n)$ and let $a \in H$. If either $p > k$ or $p = k = 1$ then there exists a constant depending only on p and k such that for each δ, $\rho > 0$*

$$\|\gamma\|_{0,p,T} \leqq \text{const } \delta |\gamma|_{1,p,T}$$

for all $\gamma \in C_0^\infty(E_n - H \cap T)$ where $T = T_{\delta,\rho}(H, a)$.

Proof. First consider the case $p > k$. Let (r, σ, z') denote cylindrical polar coordinates in E_n with origin at a and cylindrical axis H. By Hölder's inequality for $(r, \sigma, z') \in T$

$$
\begin{aligned}
|\gamma(r, \sigma, z')|^p r^{k-1} &\leqq \delta^{k-1}\left|\int_0^r \frac{d}{dt}\gamma(t, \sigma, z')\, dt\right|^p \\
&\leqq \delta^{k-1}\int_0^\delta |D_t\gamma(t, \sigma, z')|^p t^{k-1}\, dt\left\{\int_0^\delta t^{-(k-1)/(p-1)}\, dt\right\}^{p-1} \\
&\leqq \left(\frac{p-1}{p-k}\right)^{p-1}\delta^{p-1}\int_0^\delta |D_t\gamma(t, \sigma, z')|^p t^{k-1}\, dt.
\end{aligned}
$$

Integrating over Σ and Z the domains of the variables σ and z' respectively in T we obtain

$$
\begin{aligned}
\|\gamma\|_{0,p,T}^p &= \int_\Sigma d\sigma \int_Z dz' \int_0^\delta |\gamma(r, \sigma, z')|^p r^{k-1}\, dr \\
&\leqq \left(\frac{p-1}{p-k}\right)^{p-1}\delta^{p-1}\int_0^\delta dr\int_\Sigma d\sigma\int_Z dz'\int_0^\delta |D_t\gamma(t, \sigma, z')|^p t^{k-1}\, dt \\
&\leqq \text{const } \delta^p|\gamma|_{1,p,T}^p.
\end{aligned}
$$

For the special case $p=k=1$ we have

$$
|\gamma(x)| = |\gamma(r, z')| \leqq \int_0^\delta |D_t\gamma(t, z')|\, dt,
$$

$$
\|\gamma\|_{0,1,T} \leqq \int_Z dz' \int_0^\delta dr \int_0^\delta |D_t\gamma(t, z')|\, dt \leqq \delta|\gamma|_{1,1,T}.
$$

COROLLARY. *Under the conditions of the lemma, if* $1 \leqq q \leqq p$ *then there exists a constant depending only on* p, q, k, n *such that for all* $\delta > 0$

$$
\|\gamma\|_{0,q,T} \leqq \text{const } \delta^{1+n/q-n/p}|\gamma|_{1,p,T}
$$

for all $\gamma \in C_0^\infty(E_n - H \cap T)$ *where* $T = T_{\delta,\delta}(H, a)$.

Proof. By Hölder's inequality and since vol $T=$const δ^n

$$
\begin{aligned}
\|\gamma\|_{0,q,T} &\leqq \|\gamma\|_{0,p,T}[\text{vol } T]^{1/q-1/p} \\
&\leqq \text{const } \delta^{1+n/q-n/p}|\gamma|_{1,p,T}.
\end{aligned}
$$

DEFINITION. Let G be an unbounded, regular, flatly-bounded domain in E_n. We shall say that G has the *k-tube property* if for every sufficiently large positive number R there exists a positive number $\delta = \delta(R)$ with the properties:

(i) $\delta(R) \to 0$ as $R \to \infty$,

(ii) for each $x \in G_R = \{y \in G : |y| > R\}$ there exists a k-plane H and a point $a \in H$ such that $x \in T_{\delta,\delta}(H, a)$ and $H \cap T_{2\delta,2\delta}(H, a) \subset \text{bdry } G$.

It is clear that if G has the k-tube property then G is k-quasibounded. Of course the converse is not true as the planar segments comprising the boundary of G may have too many gaps to satisfy condition (ii). For domains G whose boundaries consist only of whole planes the k-tube property is equivalent to k-quasiboundedness. Other examples of domains with the k-tube property are not difficult to construct—for example Clark's "spiny urchin" [7] has the 1-tube property in E_2.

LEMMA 6 (A VARIANT ON POINCARÉ'S INEQUALITY). *Let G be an unbounded, regular, flatly-bounded domain in E_n having the $(n-k)$-tube property for some k $(1 \leqq k \leqq n)$. If $1 \leqq r \leqq p$ where either $p > k$ or $p = k = 1$ then there exists a constant depending only on n, k, p and r such that for all $u \in W_0^{1,p}(G)$ and all sufficiently large R*

$$\|u\|_{0,r,G_R} \leqq \text{const } [\delta(R)]^{1+n/r-n/p} |u|_{1,p,G}.$$

Proof. Fix R large enough so that $\delta = \delta(R)$ exists. If α is an n-tuple of integers (not necessarily nonnegative) let $Q_\alpha = \{x \in E_n : \alpha_i n^{-1/2}\, \delta \leqq x_i \leqq (\alpha_i + 1) n^{-1/2}\delta\}$. Then $E_n = \bigcup_\alpha Q_\alpha$. If $x \in G_R$ then $x \in Q_\alpha$ for some α and there exists an $(n-k)$-plane H and a point $a \in H$ such that $x \in T_{\delta,\delta}(H, a) = T$ and $H \cap T' \subset \text{bdry}\, G$ where $T' = T_{2\delta,2\delta}(H, a)$. Clearly $Q_\alpha \subset T'$. For any $\gamma \in C_0^\infty(G)$ since γ vanishes near $H \cap T'$ we have by the corollary of Lemma 5

$$\begin{aligned} \|\gamma\|_{0,r,Q_\alpha \cap G_R} &\leqq \|\gamma\|_{0,r,T'} \\ &\leqq \text{const } (2\delta)^{1+n/r-n/p} |\gamma|_{1,p,T'} \\ &\leqq \text{const } \delta^{1+n/r-n/p} |\gamma|_{1,p,Q'_\alpha} \end{aligned}$$

where Q'_α is the union of all the cubes Q_β which intersect T'. There is a number N depending only on n such that any $N+1$ of the sets Q'_α have empty intersection. Summing the above inequality over all α for which Q_α meets G_R we obtain

$$\|\gamma\|_{0,r,G_R} \leqq \text{const } N \cdot \delta^{1+n/r-n/p} |\gamma|_{1,p,G}$$

and this inequality extends by completion from $C_0^\infty(G)$ to $W_0^{1,p}(G)$.

THEOREM 2. *Let G be an unbounded, regular, flatly-bounded domain in E_n having the $(n-k)$-tube property $(1 \leqq k \leqq n)$. If either $p > k$ or $p = k = 1$ then the imbedding $W_0^{m+1,p}(G) \to W_0^{m,r}(G)$ (exists and) is compact for $m = 0, 1, 2, \ldots$ and $1 \leqq r < \infty$ if $p \geqq n$ or for $1 \leqq r < np(n-p)^{-1}$ if $p < n$.*

Proof. First consider the case $1 \leqq r \leqq p$, $m = 0$. To prove that the imbedding $W_0^{1,p}(G) \to L^r(G)$ is compact we use the following compactness criterion for sets in $L^r(G)$: a sequence $\{u_i\}_{i=1}^\infty$ which is bounded in $L^r(G)$ is precompact in $L^r(G)$ provided

(a) for every bounded $G' \subset G$ the sequence $\{u_i | G'\}$ is precompact in $L^r(G')$, and

(b) for each $\varepsilon > 0$ there exists $R > 0$ such that for all i, $\|u_i\|_{0,r,G_R} < \varepsilon$.

Lemma 6 and condition (i) of the $(n-k)$-tube property assures us that (b) is satisfied for any sequence $\{u_i\}$ bounded in $W_0^{1,p}(G)$. To establish (a) let G' be a bounded subset of G. Then for some R, $G' \subset K_R = \{x \in E_n : |x| \leqq R\}$. Let $W^{1,p}(G, R)$ denote the completion with respect to the norm $\|\cdot\|_{1,p,G \cap K_R}$ of the space $C_0^\infty(G)$. The imbedding $W^{1,p}(K_R) \to L^r(K_R)$ is known to be compact (Kondrašov's Theorem) and since an element of $W^{1,p}(G, R)$ can be extended to be zero outside its support so as to belong to $W^{1,p}(K_R)$ it follows that $W^{1,p}(G, R)$ is compactly imbedded in

$L^r(G \cap K_R)$. But $\{u_i|K_R\}$ is bounded in $W^{1,p}(G, R)$ and hence precompact in $L^r(G \cap K_R)$ whence $\{u_i|G'\}$ is precompact in $L^r(G')$ as required.

By Sobolev's Imbedding Theorem $W_0^{1,p}(G)$ is continuously imbedded in $L^q(G)$ for any q satisfying $p \leqq q < \infty$ if $p \geqq n$ or $p \leqq q \leqq np(n-p)^{-1}$ if $p<n$. Select such a q and a sequence $\{u_i\}$ bounded in $W_0^{1,p}(G)$ so that, say, $\|u_i\|_{0,q,G} \leqq C$. We may assume, passing to a subsequence if necessary, that $\{u_i\}$ converges in $L^p(G)$. By Hölder's Inequality if $p \leqq r < q$

$$\begin{aligned}\|u_i-u_j\|_{0,r,G} &\leqq \|u_i-u_j\|_{0,p,G}^{\lambda}\|u_i-u_j\|_{0,q,G}^{1-\lambda}\\ &\leqq (2C)^{1-\lambda}\|u_i-u_j\|_{0,p,G}^{\lambda}\end{aligned}$$

where $\lambda = p(q-r)r^{-1}(q-p)^{-1} > 0$. Hence $\{u_i\}$ converges in $L^r(G)$ and so the imbedding $W_0^{1,p}(G) \to L^r(G)$ is compact for $1 \leqq r < \infty$ if $p \geqq n$ and for $1 \leqq r < np(n-p)^{-1}$ if $p<n$.

Finally, if $\{u_i\}$ is bounded in $W_0^{m+1,p}(G)$ then for any α with $0 \leqq |\alpha| \leqq m$, $\{D^\alpha u_i\}$ is bounded in $W_0^{1,p}(G)$ and so has a subsequence convergent to an element v_α of $L^r(G)$. In particular (for a suitable subsequence) $u_i \to v_0$ in $L^r(G)$ and so in the sense of distributions. Since $D^\alpha u_i \to v_\alpha$ in $L^r(G)$ and $D^\alpha u_i \to D^\alpha v_0$ in the sense of distributions it follows that $v_\alpha = D^\alpha v_0$ and $u_i \to v_0$ in $W_0^{m,r}(G)$. This completes the proof.

This theorem affords for imbeddings of the sort $W_0^{1,p}(G) \to L^r(G)$ on domains G for which $(n-k)$-quasiboundedness is equivalent to the $(n-k)$-tube property, a complete converse to Theorem 1. For imbeddings $W_0^{m,p}(G) \to L^r(G)$, $m \geqq 2$, we do not fare quite so well.

THEOREM 3. *Let G be an unbounded, regular, flatly-bounded domain in E_n having the $(n-k)$-tube property* $(1 \leqq k \leqq n)$. *Then the imbedding*

$$W_0^{m,p}(G) \to W_0^{j,r}(G), \qquad 0 \leqq j < m,$$

is compact in any of the following cases:

(i) $m=p=k=1$,

(ii) $mp > n+p-np/k$, $p \leqq r < p^*$,

(iii) $mp > n+(j+1)p-np/k$, $1 \leqq r < p^*$,

where $p^* = np(n-mp+jp)^{-1}$ *if* $n > mp-jp$ *and* $p^* = \infty$ *if* $n \leqq mp-jp$.

Proof. The case $m=1$ has already been proved. If $m \geqq 2$ the imbedding $W_0^{m,p}(G) \to W_0^{1,q}(G)$ is continuous for $p \leqq q \leqq np(n-mp+p)^{-1}$ if $n > mp-p$ and $p \leqq r < \infty$ if $n \leqq mp-p$. By Theorem 2 the imbedding $W_0^{1,q}(G) \to L^p(G)$ is compact provided $q > k$. Since $mp > n+p-np/k$ is equivalent to $np(n-mp+p)^{-1} > k$ such $q > k$ can always be chosen and so the composed imbedding $W_0^{m,p}(G) \to L^p(G)$ is compact.

By a standard interpolation theorem for Sobolev spaces [5, Lemma 6] there exists a constant K such that for $0 \leqq j < m$, $p \leqq q < p^*$ we have

$$\|u\|_{j,r,G} \leqq K\|u\|_{m,p,G}^{\lambda}\|u\|_{0,p,G}^{1-\lambda}$$

for all $u \in W_0^{m,p}(G)$ where $\lambda = (nr + jrp - np)(mrp)^{-1}$. Note that $0 \leqq \lambda < 1$ for all relevant values of j, m, n, r and p. If $\{u_i\}_{i=1}^{\infty}$ is a bounded sequence in $W_0^{m,p}(G)$ then it has a subsequence again denoted $\{u_i\}$ which is convergent in $L^p(G)$. Since

$$\begin{aligned}\|u_i - u_k\|_{j,r,G} &\leqq K\|u_i - u_k\|_{m,p,G}^{\lambda}\|u_i - u_k\|_{0,p,G}^{1-\lambda} \\ &\leqq K[2 \sup \|u_i\|_{m,p,G}]^{\lambda}\|u_i - u_k\|_{0,p,G}^{1-\lambda},\end{aligned}$$

it follows that $\{u_i\}$ is a Cauchy sequence and hence convergent in $W_0^{j,r}(G)$ proving case (ii).

If $mp > n + (j+1)p - np/k$ (and in particular if $j = 0$ in case (ii)) we have by Sobolev's theorem and Theorem 2

$$W_0^{m,p}(G) \to W_0^{j+1,np(n-mp+jp+p)^{-1}}(G) \to W_0^{j,r}(G)$$

the second imbedding being compact since $np(n - mp + jp + p)^{-1} > k$.

Remark. The condition $mp > n + p - np/k$ implies $mp > k$. The converse is, however, true for $m \geqq 2$ only if $k = n$ or $k \leqq p$. Thus even for domains G for which $(n-k)$-quasiboundedness is equivalent to the $(n-k)$-tube property imbeddings of $W_0^{m,p}(G)$ corresponding to the cases $1 < p < k < mp \leqq n + p - np/k$ and $1 = p < k \leqq m \leqq n + 1 - n/k$ fail to be covered either by Theorem 1 or Theorem 3.

4. Extensions to nonflatly-bounded domains.

Let G, G' be open sets in E_n. A one-to-one transformation M from G onto G' is called an *m-diffeomorphism of modulus C* if all the components of M and M^{-1} have continuous partial derivatives of all orders up to and including m, and these partials do not exceed C in modulus.

Lemma 7. *Let G, G' be open in E_n and let M be an m-diffeomorphism of modulus C from G onto G'. Let*

$$Au(y) = u(M^{-1}y), \qquad y \in G'.$$

Then A is a homeomorphism from $W^{m,p}(G)$ [respectively $W_0^{m,p}(G)$] onto $W^{m,p}(G')$ [resp. $W_0^{m,p}(G')$] and there exist constants C_1 and C_2 depending only on n, p and C and not on G or G' such that for all $u \in W^{m,p}(G)$

$$C_1\|u\|_{m,p,G} \leqq \|Au\|_{m,p,G'} \leqq C_2\|u\|_{m,p,G}.$$

Proof. A is a homeomorphism from $L^p(G)$ onto $L^p(G')$ for if $\partial M/\partial x$ represents the Jacobian determinant of M then

$$\begin{aligned}\left\{\sup_{y \in G'}\left|\frac{\partial M^{-1}(y)}{\partial y}\right|^{1/p}\right\}^{-1}\|u\|_{0,p,G} &\leqq \|Au\|_{0,p,G'} \\ &\leqq \sup_{x \in G}\left|\frac{\partial M(x)}{\partial x}\right|^{1/p}\|u\|_{0,p,G}.\end{aligned}$$

By induction and formal applications of the chain rule it is easily verified that in the sense of distributions on G'

$$D^{\alpha}(Au) = \sum_{|\beta| \leqq |\alpha|} M_{\alpha\beta}A(D^{\beta}u)$$

where $M_{\alpha\beta}$ is a polynomial of degree $|\beta|$ in the derivatives of the components of M^{-1} involving derivatives of orders not exceeding $|\alpha|$. It follows that

$$\begin{aligned}\|Au\|_{m,p,G'}^p &= \sum_{|\alpha|\leqq m}\left\|\sum_{|\beta|\leqq|\alpha|} M_{\alpha\beta}A(D^\beta u)\right\|_{0,p,G'}^p \\ &\leqq \text{const} \sum_{|\beta|\leqq m}\|A(D^\beta u)\|_{0,p,G'}^p \\ &\leqq \text{const}\ \|u\|_{m,p,G}^p.\end{aligned}$$

The reverse inequality follows in a similar manner.

LEMMA 8. *Under the hypotheses of Lemma 7 $W^{m,p}(G)$ is compactly imbedded in $W^{j,r}(G)$ if and only if $W^{m,p}(G')$ is compactly imbedded in $W^{j,r}(G')$. A similar statement holds for the spaces $W_0^{m,p}(G)$.*

Proof. Suppose the imbedding $W^{m,p}(G)\to W^{j,r}(G)$ is compact. Let $\{u_i\}_{i=1}^\infty$ be a bounded sequence in $W^{m,p}(G')$. Then $\{A^{-1}u_i\}$ is bounded in $W^{m,p}(G)$ and so has a subsequence converging in $W^{j,r}(G)$. The corresponding subsequence of $\{u_i\}$ is convergent in $W^{j,r}(G')$ whence the imbedding $W^{m,p}(G')\to W^{j,r}(G')$ is compact. The other cases are proved similarly.

Of course Lemma 8 can be used to obtain immediately the conclusions of Theorems 1–3 for any domain G which is m-diffeomorphic to an unbounded, regular, flatly-bounded domain G' satisfying the conditions of the particular theorem. As most quasibounded domains do not have this property we obtain generalizations of these theorems with localized hypotheses.

THEOREM 4. *Let G be a regular, unbounded domain in E_n. Let k be the largest integer $(1\leqq k\leqq n)$ for which for some constant C there exist infinitely many mutually disjoint open sets U in $\bar G$ each of which is m-diffeomorphic with modulus not greater than C to the unit ball B in E_n in such a way that $U\cap$ bdry G is mapped into a subset of the union of finitely many $(n-k)$-planes. (In particular G is not $(n-k+1)$-quasi-bounded.) If either $mp\leqq k$, $p>1$ or $m<k$, $p=1$ then no imbedding of the form $W_0^{m,p}(G)\to W_0^{j,r}(G)$ can be compact.*

Proof. Let $\{U_i\}_{i=1}^\infty$ be a sequence of mutually disjoint open sets in $\bar G$ for which there correspond m-diffeomorphisms M_i: $U_i\to B$ having modulus$\leqq C$ and such that $M_i(U_i\cap \text{bdry } G)\subset B\cap P_i$ where P_i is the union of finitely many $(n-k)$-planes. Let $\varphi\in C_0^\infty(B)$ be such that

$$\|\varphi\|_{0,r,B}=C_1>0,\qquad \|\varphi\|_{m,p,B}=K_1<\infty.$$

By the method used in the proof of Theorem 1 we can construct functions $\gamma_i\in C_0^\infty(B-P_i)$ such that

$$\|\gamma_i\|_{0,r,B}\geqq C_2>0,\qquad \|\gamma_i\|_{m,p,B}\leqq K_2<\infty,$$

the constants C_2 and K_2 being independent of i. Denoting by A_i the operator for which $A_i u(y) = u(M_i^{-1} y)$, $y \in B$ we have by Lemma 7 that there exist constants C_3 and K_3 again independent of i such that

$$\|A_i^{-1}\gamma_i\|_{0,r,G} \geqq C_3 > 0, \qquad \|A_i^{-1}\gamma_i\|_{m,p,G} \leqq K_3 < \infty$$

and

$$A_i^{-1}\gamma_i \in C_0^\infty(U_i \cap G).$$

The noncompactness of the imbedding $W_0^{m,p}(G) \to W_0^{j,r}(G)$ now follows as in Theorem 1.

As an analog of the compactness Theorems 2 and 3 we have

THEOREM 5. *Let G be an unbounded open set in E_n with the property that there exist constants C, R_0 and K such that for each $R \geqq R_0$ there exist positive numbers $d(R)$ and $\delta(R)$ with the following properties*:

(i) $d(R)+\delta(R) \to 0$ *as* $R \to \infty$,

(ii) $d(R)/\delta(R) \leqq K$, $R \geqq R_0$,

(iii) *for each $x \in G_R = \{x \in G : |x| > R\}$ the ball $B_\delta(x)$ of radius $\delta(R)$ and center x can be mapped by a 1-diffeomorphism M of modulus $\leqq C$ onto a set S in E_n such that for some $(n-k)$-plane H $(1 \leqq k \leqq n)$ and some point $a \in H$ we have $S \subset T_{d(R),d(R)}(H, a)$ and $H \cap T_{d(R),d(R)}(H, a) \subset M(\text{bdry } G \cap B_\delta(x))$.*

Then the imbedding $W_0^{m,p}(G) \to W_0^{j,r}(G)$, $0 \leqq j < m$ is compact in any of the following cases:

(a) $m=p=k=1$,

(b) $mp > n+p-np/k$, $p \leqq r < p^*$,

(c) $mp > n+(j+1)p-np/k$, $1 \leqq r < p^*$,

where $p^* = np(n-mp+jp)^{-1}$ if $n > mp-jp$ and $p^* = \infty$ if $n \leqq mp-jp$.

Proof. The conclusion is the same as that of Theorem 3 and the proof is identical if we reprove Lemma 6 (Poincaré's inequality) under the conditions of this theorem. Thus, let $p > k$ or $p=k=1$ and let $1 \leqq r \leqq p$. Fix $R \geqq R_0$ and let $d=d(R)$ and $\delta=\delta(R)$. Define the cubes Q_α as in the proof of Lemma 6. If $x \in G_R$ then for some α, $x \in Q_\alpha \subset B_\delta(x)$. There exists a 1-diffeomorphism M of $B=B_\delta(x)$ onto $S \subset E_n$ having modulus $\leqq C$ and there exists an $(n-k)$-plane H and a point $a \in H$ such that $S \subset T = T_{d,d}(H, a)$ and $H \cap T \subset M(B \cap \text{bdry } G)$. For any $\gamma \in C_0^\infty(G)$ we have that $A\gamma$ (defined by $A\gamma(y) = \gamma(M^{-1}y)$, $y \in S$) vanishes near $H \cap T$. Thus by the corollary of Lemma 5, Lemma 7 and the fact that $d \leqq K\delta$

$$\begin{aligned} \|\gamma\|_{0,r,Q_\alpha \cap G_R} &\leqq \|\gamma\|_{0,r,B} \leqq \text{const } \|A\gamma\|_{0,r,S} \\ &\leqq \text{const } d^{1+n/r-n/p}\|A\gamma\|_{1,p,T} \\ &\leqq \text{const } \delta^{1+n/r-n/p}\|\gamma\|_{1,p,M^{-1}(T)} \\ &\leqq \text{const } \delta^{1+n/r-n/p}\|\gamma\|_{1,p,Q'_\alpha} \end{aligned}$$

where Q'_α is the union of all the cubes Q_β which intersect $M^{-1}(T)$. Since the modulus of M is bounded, M^{-1} is Lipschitzian and there exists a constant λ such that $M^{-1}(T) \subset B_{\lambda d}(x) \subset B_{\lambda K\delta}(x)$. Thus there is a constant N independent of R and x

such that any $N+1$ of the sets Q'_α have empty intersection. Summing the above inequality over those α for which Q_α meets G_R we obtain, as in Lemma 6, the required form of Poincaré's inequality.

5. **An application to differential operators.** Let L be a linear partial differential operator of order $2m$ in G given by

$$Lu(x) = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha u(x)$$

with coefficients a_α infinitely differentiable, bounded, complex functions on G. Suppose L is such that it satisfies the boundedness condition

$$\left| \int_G L\varphi(x)\overline{\psi(x)}\,dx \right| \leqq c_0 \|\varphi\|_{m,2,G} \|\psi\|_{m,2,G}$$

for all φ, $\psi \in C_0^\infty(G)$, and also Garding's inequality

$$\operatorname{Re} \int_G L\varphi(x)\overline{\varphi(x)}\,dx \geqq c_1 \|\varphi\|^2_{m,2,G} - c_2 \|\varphi\|^2_{0,2,G}$$

for all $\varphi \in C_0^\infty(G)$, where c_0, $c_1 > 0$ and c_2 are constants. The realization of L in $L^2(G)$ corresponding to null Dirichlet boundary data is an operator T in $L^2(G)$ defined by

$$\begin{aligned} \operatorname{Dom}(T) &= W_0^{m,2}(G) \cap \{f \in L^2(G) : Lf \in L^2(G)\} \\ Tf &= Lf, \qquad f \in \operatorname{Dom}(T). \end{aligned}$$

THEOREM 6. *If G is open in E_n and satisfies the conditions of either Theorem 3 or Theorem 5 with $2m > n+2-2n/k$ then T as defined above is a closed linear operator in $L^2(G)$; the spectrum $\sigma(T)$ is discrete and has no finite limit points; for $\lambda \notin \sigma(T)$ the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$ is completely continuous.*

The proof is identical to that of the standard theorem of this type. A sketch can be found in [6].

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