

NICE HOMOLOGY COALGEBRAS

BY

A. K. BOUSFIELD⁽¹⁾

1. Introduction. Several different unstable versions of the Adams spectral sequence for homotopy groups have recently been constructed ([3], [9], [14]). While these spectral sequences differ for general spaces, it will be shown in [2] that they coincide (at least mod 2) for spaces having *nice* (2.2) homology coalgebras, e.g. for spheres and loop spaces but not for $S^2 \vee S^2$. The purpose of this note is to prove two purely algebraic results on nice homology coalgebras which will be needed in [2]. The first (4.1) provides a homological characterization of niceness, and the second (§6) gives the structure of the Hopf algebra $\text{Cotor}^C(k, k)$ when C is a nice k -coalgebra. Throughout this note we employ a theory of derived functors for nonadditive functors which is due to André [1], Quillen [13], and others. For convenience, we outline a simplified version of this theory in an appendix (§7).

The definition (2.2) of niceness for a homology coalgebra C comes essentially from Moore-Smith [12, §4] and involves the existence of a certain presentation for C . Our homological characterization of niceness is based on the notion of primitive dimension (3.1). In particular we study (§3) right derived functors $R^n P$ of the primitive element functor P , and we show (4.1) that a homology coalgebra C is nice if and only if $R^n P(C) = 0$ for $n > 1$, i.e. C has primitive dimension ≤ 1 .

For a homology coalgebra C over a field k , $R^n P(C)$ is closely related to the Hopf algebra $\text{Cotor}^C(k, k)$ defined by Eilenberg-Moore [7] (see (5.1)). We construct (5.3) a spectral sequence of Hopf algebras whose E^2 -term depends only on $R^n P(C)$ and which converges to $\text{Cotor}^C(k, k)$. Using this we determine (§6) the structure of $\text{Cotor}^C(k, k)$ when C is nice, and we thereby generalize certain results of [12]. It should be noted that when C is of finite type $\text{Cotor}^C(k, k)$ is isomorphic to $\text{Ext}_{C^*}(k, k)$ and thus §6 gives information on the cohomology of algebras.

Finally the author wishes to acknowledge that results similar to 4.1 and 6.1 for commutative rings have been proved by D. G. Quillen [16] using different methods.

2. Homology coalgebras and their derived functors. Let k be a fixed field. A *homology k -coalgebra* is a connected positively graded k -coalgebra with commutative comultiplication [11]. We let \mathcal{C}/k denote the category of homology k -coalgebras. Thus $H_*(X; k) \in \mathcal{C}/k$ for any connected space X .

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If V is a strictly positively graded k -module, let $U(V)$ be the homology k -coalgebra cogenerated by V (see [12]). This coalgebra is characterized by the existence of a map of graded k -modules

$$\pi(V): U(V) \rightarrow V$$

satisfying the universal mapping property: if $C \in \mathcal{C}/k$ and $f: C \rightarrow V$ is a map of graded k -modules, then there is a unique map $\bar{f}: C \rightarrow U(V)$ in \mathcal{C}/k such that $\pi(V) \circ \bar{f} = f$. A coalgebra $C \in \mathcal{C}/k$ is *cofree* if $C \approx U(V)$ for some V .

DEFINITION 2.1. A sequence

$$k \longrightarrow C' \xrightarrow{f'} C \xrightarrow{f''} C'' \longrightarrow k$$

in \mathcal{C}/k is an *injective extension sequence* [12] if:

- (i) f'' is an epimorphism (i.e. f'' is a surjection of the underlying graded k -modules).
- (ii) C is injective as a C'' comodule.
- (iii) f' is the kernel of f'' (i.e. f' is the natural map $C \square_{C''} k \rightarrow C$ of [11]).

DEFINITION 2.2. A coalgebra $C' \in \mathcal{C}/k$ is *nice* if there exists an injective extension sequence

$$k \longrightarrow C' \xrightarrow{f'} C \xrightarrow{f''} C'' \longrightarrow k$$

such that C and C'' are cofree.

2.3. EXAMPLES. A coalgebra $B \in \mathcal{C}/k$ is nice under any of the following conditions:

- (i) B is the coalgebra of some Hopf algebra (see [12, §4]).
- (ii) B is of finite type and its dual B^* is an algebra on one generator (or more generally B^* is the quotient of a free commutative algebra by a Borel ideal [15, p. 79]).
- (iii) $B \approx B' \otimes_k B''$ where $B', B'' \in \mathcal{C}/k$ are nice.

2.4. *Derived functors of coalgebras.* We now indicate how the general theory of derived functors in the Appendix (§7) applies to homology coalgebras. The class \mathcal{M} of all cofree coalgebras in \mathcal{C}/k is a class of injective models (7.2) for \mathcal{C}/k by 7.8. Thus if $T: \mathcal{C}/k \rightarrow \mathcal{A}$ is a (covariant) functor to an abelian category \mathcal{A} , then T has right derived functors (with respect to \mathcal{M})

$$R^n T: \mathcal{C}/k \rightarrow \mathcal{A}, \quad n \geq 0, \\ \varepsilon: T \rightarrow R^0 T.$$

To construct these explicitly, let \mathcal{V}/k be the category of strictly positively graded k -modules and let

$$J: \mathcal{C}/k \rightarrow \mathcal{V}/k, \quad U: \mathcal{V}/k \rightarrow \mathcal{C}/k$$

be the adjoint functors with U as above and with $J(C) \in \mathcal{V}/k$ equal to C in all positive degrees. Letting $K = UJ$, there is given by 7.7 an augmented cosimplicial (7.6) object KC for each $C \in \mathcal{C}/k$, with $KC^n = K^{n+1}C$ for $n \geq -1$. We call KC the

canonical resolution of C . Let $\tilde{K}C$ equal KC without the augmentation. By 7.8 and 7.9

$$R^nT(C) \approx H^n(T\tilde{K}C), \quad n \geq 0,$$

and $\varepsilon: T(C) \rightarrow R^0T(C)$ is induced by the augmentation of KC .

For our applications we need a more general type of resolution than KC .

DEFINITION 2.5. A cosimplicial resolution of $C \in \mathcal{C}/k$ consists of an augmented cosimplicial object X over \mathcal{C}/k such that:

- (i) $X^{-1} = C$.
- (ii) For $n \geq 0$, X^n is cofree.
- (iii) For $n \geq -1$, $H^n(JX) = 0$.

Let \tilde{X} denote X without the augmentation.

REMARK 2.6. It is easily shown that KC is a cosimplicial resolution of C .

PROPOSITION 2.7. If X is a cosimplicial resolution of $C \in \mathcal{C}/k$ and $T: \mathcal{C}/k \rightarrow \mathcal{A}$ is as above, then there are natural isomorphisms

$$R^nT(C) \approx H^n(T\tilde{X}), \quad n \geq 0.$$

Proof. We must show that the chain complex $\text{ch}^+ X$ (7.6) is a right \mathcal{M} -resolution (7.1) of C . For the acyclicity condition 7.1(ii) it suffices, by the adjointness of U and J , to show that the chain complex of the augmented simplicial abelian group

$$Z \text{ Hom}_{\mathcal{V}/k}(JX, V), \quad V \in \mathcal{V}/k$$

is acyclic, where $Z(\)$ is the free abelian group functor. This follows by standard simplicial methods (see [10]) since the chain complex of the augmented simplicial k -module $\text{Hom}_{\mathcal{V}/k}(JX, V)$ is acyclic.

3. Primitive dimension of coalgebras. Let

$$R^nP: \mathcal{C}/k \rightarrow \mathcal{V}/k, \quad n \geq 0$$

be the right derived functors (2.4) of the primitive [11] element functor P for homology k -coalgebras.

DEFINITION 3.1. A coalgebra $C \in \mathcal{C}/k$ has primitive dimension $\leq d$ if $R^nP(C) = 0$ for all $n > d$.

In this section we develop the general properties of the functors R^nP , and leave unstated the obvious corollaries on primitive dimension.

PROPOSITION 3.2. For each $C \in \mathcal{C}/k$, $\varepsilon: P(C) \xrightarrow{\sim} R^0P(C)$ is an isomorphism.

Proof. By the definition of P there is an exact sequence

$$0 \rightarrow P \rightarrow J \rightarrow J \otimes J$$

and this induces an exact sequence

$$0 \rightarrow R^0P \rightarrow R^0J \rightarrow R^0(J \otimes J).$$

Clearly

$$\varepsilon: J \xrightarrow{\sim} R^0J, \quad \varepsilon: J \otimes J \xrightarrow{\sim} R^0(J \otimes J)$$

and so $\varepsilon: P \xrightarrow{\sim} R^0P$.

PROPOSITION 3.3. *If $B, C \in \mathcal{C}/k$ then*

$$R^nP(B \otimes C) \approx R^nP(B) \oplus R^nP(C), \quad n \geq 0.$$

Proof. If X, Y are cosimplicial resolutions (2.5) for B, C , then the degreewise tensor product $X \otimes Y$ is a cosimplicial resolution for $B \otimes C$ by the Eilenberg-Zilber theorem (see [5]). Since

$$P(\tilde{X} \otimes \tilde{Y}) \approx P\tilde{X} \oplus P\tilde{Y}$$

the result follows.

PROPOSITION 3.4. *A map $h: B \rightarrow C$ in \mathcal{C}/k is an isomorphism if and only if*

$$R^nP(h): R^nP(B) \rightarrow R^nP(C)$$

is an isomorphism for $n=0$ and a monomorphism for $n=1$.

An equivalent result is given in [12, 3.2]. For our proof we take X and $\tilde{K}B$ and $Y = \tilde{K}C$ in the following lemma.

LEMMA 3.5. *Let $f: X \rightarrow Y$ be a map of dimensionwise cofree cosimplicial objects over \mathcal{C}/k . If*

$$(Pf)^*: H^q(PX) \rightarrow H^q(PY)$$

is isomorphic for $q \leq n$ and monomorphic for $q = n + 1$, then (viewing X, Y as cosimplicial graded k -modules)

$$f^*: H^q(X) \rightarrow H^q(Y)$$

is isomorphic for $q \leq n$ and monomorphic for $q = n + 1$.

Proof. Filter X by cosimplicial graded k -modules $\{F_s X\}_{s \geq 0}$ where $F_s X$ is the kernel of the composite

$$X \xrightarrow{\Delta_{n+1}} X^{(n+1)} \longrightarrow (JX)^{(n+1)}$$

with Δ_{n+1} the $(n+1)$ -fold diagonal. The resulting spectral sequence $\{E_r X\}_{r \geq 0}$ converges to $H^*(X)$, and there is a natural isomorphism $E_0 X \approx U(PX)$ since each X^q is cofree. The hypothesis of 3.5 implies [5] that

$$(UPf)^*: H^q(UPX) \rightarrow H^q(UPY)$$

is isomorphic for $q \leq n$ and monomorphic for $q = n + 1$. A simple spectral sequence argument now completes the proof.

THEOREM 3.6. *Let*

$$k \longrightarrow C' \xrightarrow{i} C \xrightarrow{j} C'' \longrightarrow k$$

be an injective extension sequence in \mathcal{C}/k . Then there is a natural long exact sequence

$$\begin{aligned} 0 \longrightarrow R^0P(C') \xrightarrow{i_*} R^0P(C) \longrightarrow \dots \\ \longrightarrow R^{n-1}P(C'') \xrightarrow{\delta} R^nP(C') \xrightarrow{i_*} R^nP(C) \xrightarrow{j_*} R^nP(C'') \xrightarrow{\delta} \dots \end{aligned}$$

with i_, j_* induced by i, j .*

This extends the six term exact sequence of [12, 3.3]. The proof will occupy the rest of §3.

3.7. *Mapping cones.* Let $f: X \rightarrow Y$ be a map of cosimplicial objects over \mathcal{C}/k . The *mapping cone* $M(f)$ is the cosimplicial object over \mathcal{C}/k given by:

- (i) $M(f)^0 = X^0, M(f)^n = X^n \otimes Y^{n-1} \otimes \dots \otimes Y^0, n \geq 1$.
- (ii) $d^0: M(f)^n \rightarrow M(f)^{n+1}$ equals $((d^0 \otimes f)\Delta) \otimes 1 \otimes \dots \otimes 1$ where $\Delta: X^n \rightarrow X^n \otimes X^n$ is the comultiplication.
- (iii) $d^i: M(f)^n \rightarrow M(f)^{n+1}, 0 < i < n+1$ equals $d^i \otimes \dots \otimes d^1 \otimes ((d^0 \otimes 1)\Delta) \otimes 1 \otimes \dots \otimes 1$ where $\Delta: Y^{n-i} \rightarrow Y^{n-i} \otimes Y^{n-i}$ is the comultiplication.
- (iv) $d^{n+1}: M(f)^n = M(f)^n \otimes k \rightarrow M(f)^{n+1}$ equals $d^{n+1} \otimes \dots \otimes d^1 \otimes \eta$ where $\eta: k \rightarrow Y^0$ is the coaugmentation.
- (v) $s^i: M(f)^{n+1} \rightarrow M(f)^n, 0 \leq i < n+1$ equals $s^i \otimes \dots \otimes s^0 \otimes \alpha \otimes 1 \otimes \dots \otimes 1$ where $\alpha: Y^{n-i} \rightarrow k$ is the counit.

For any $B \in \mathcal{C}/k$ let $c(B)$ denote the constant cosimplicial object with $c(B)^n = B$ for all $n \geq 0$.

PROPOSITION 3.8. *If $g: B' \rightarrow B \in \mathcal{C}/k$ and $cg: c(B') \rightarrow c(B)$ is the induced cosimplicial map, then*

$$H^n(M(cg)) \approx \text{Cotor}_{-n,*}^B(B', k), \quad n \geq 0.$$

Proof. For $1: B \rightarrow B$, the cosimplicial object $M(c1)$ augmented by $d^0: k \rightarrow M(c1)^0$ may be viewed as an augmented cosimplicial object over the category of left B comodules. The associated chain complex is an injective resolution of the left B comodule k . (It is essentially the cobar resolution.) Since $M(cg) \approx B' \square_B M(c1)$, where \square is the cotensor product [11], the result follows.

Let

$$\begin{array}{ccc} X & \xrightarrow{u} & R \\ \downarrow f & & \downarrow g \\ Y & \xrightarrow{v} & S \end{array}$$

be a commutative diagram of cosimplicial objects over \mathcal{C}/k .

PROPOSITION 3.9. *If*

$$u^*: H^*(X) \rightarrow H^*(R), \quad v^*: H^*(Y) \rightarrow H^*(S)$$

are isomorphic, then

$$(u, v)^*: H^*(M(f)) \rightarrow H^*(M(g))$$

is isomorphic.

Proof. First suppose X, Y, R, S are dimensionwise cofree and that

$$(Pu)^*: H^*(PX) \rightarrow H^*(PR), \quad (Pv)^*: H^*(PY) \rightarrow H^*(PS)$$

are isomorphisms. It is not hard to show

$$H^*(PMf/PX) \approx H^{*-1}(PY), \quad H^*(PMg/PR) \approx H^{*-1}(PS)$$

and hence by the five lemma u, v induce $H^*(PMf) \approx H^*(PMg)$. Thus

$$(u, v)^*: H^*(Mf) \rightarrow H^*(Mg)$$

is isomorphic by 3.5.

For the general case consider the diagram of cosimplicial objects

$$\begin{array}{ccc} K^n X & \xrightarrow{K^n u} & K^n R \\ \downarrow K^n f & & \downarrow K^n g \\ K^n Y & \xrightarrow{K^n v} & K^n S \end{array}$$

where $n \geq 1$ and K^n is as in 2.4. Since $PK^n X$ is of the form $T(JX)$ it follows by [5] that the diagram satisfies the hypotheses of our first case, and hence

$$(*) \quad H^*(MK^n f) \xrightarrow{\approx} H^*(MK^n g)$$

is an isomorphism. There is a double cosimplicial object $M\tilde{K}f$ (and also $M\tilde{K}g$) with

$$(M\tilde{K}f)^{i,j} = (MK^{i+1}f)^j$$

and we consider the two natural spectral sequences converging to the total homology $H^*(M\tilde{K}f)$, viewing $M\tilde{K}f$ as a double chain complex. One spectral sequence shows

$$H^*(M\tilde{K}f) \rightarrow H^*(M\tilde{K}g)$$

is an isomorphism because of (*), and the other spectral sequence collapses to show

$$H^*(Mf) \approx H^*(M\tilde{K}f), \quad H^*(Mg) \approx H^*(M\tilde{K}g)$$

which proves the proposition.

3.10. **Proof of 3.6.** Let

$$k \longrightarrow C' \xrightarrow{i} C \xrightarrow{j} C'' \longrightarrow k$$

be an injective extension sequence. There is a commutative diagram

$$\begin{array}{ccc}
 cC & \xrightarrow{\epsilon} & \tilde{K}C \\
 \downarrow cj & & \downarrow \tilde{K}j \\
 cC'' & \xrightarrow{\epsilon''} & \tilde{K}C''
 \end{array}$$

where ϵ, ϵ'' are induced by the augmentations of KC, KC'' . Since

$$\begin{aligned}
 \text{Cotor}_{n,*}^{C''}(C, k) &= 0, & n > 0, \\
 &= C', & n = 0,
 \end{aligned}$$

3.8 and 3.9 imply that $M(\tilde{K}j)$ is a cosimplicial resolution (2.5) of C' with augmentation omitted. Thus

$$R^n P(C') \approx H^n(PM(\tilde{K}j)), \quad n \geq 0.$$

The commutative diagram

$$\begin{array}{ccccc}
 ck & \longrightarrow & \tilde{K}C & \xrightarrow{1} & \tilde{K}C \\
 \downarrow \eta & & \downarrow \tilde{K}j & & \downarrow \alpha \\
 \tilde{K}C'' & \xrightarrow{1} & \tilde{K}C'' & \longrightarrow & ck
 \end{array}$$

gives a sequence of mapping cones

$$M(\eta) \rightarrow M(\tilde{K}j) \rightarrow M(\alpha) = \tilde{K}C.$$

Applying P we obtain a short exact sequence

$$(*) \quad 0 \rightarrow PM(\eta) \rightarrow PM(\tilde{K}j) \rightarrow P\tilde{K}C \rightarrow 0$$

with

$$H^n(PM(\eta)) \approx R^{n-1}P(C''), \quad H^n(PM(\tilde{K}j)) \approx R^n P(C'), \quad H^n(P\tilde{K}C) \approx R^n P(C).$$

The first isomorphism is given by the coboundary map for the sequence of type (*) obtained using the injective extension sequence

$$k \longrightarrow k \longrightarrow C'' \xrightarrow{1} C'' \longrightarrow k.$$

The cohomology sequence of (*) is the exact sequence required for 3.6.

4. Niceness and primitive dimension.

THEOREM 4.1. *A coalgebra $C \in \mathcal{C}/k$ is cofree if and only if C is of primitive dimension ≤ 0 , and C is nice if and only if C is of primitive dimension ≤ 1 .*

Proof. If C is cofree then $R^n P(C) = 0, n \geq 1$, by 7.4. If C is nice there is an injective extension sequence

$$k \rightarrow C \rightarrow B \rightarrow B'' \rightarrow k$$

with B and B'' cofree. The exact sequence 3.6 shows that $R^n P(C) = 0, n \geq 2$. The theorem will now follow from 4.2 and 4.4.

PROPOSITION 4.2. *If $R^1 P(C) = 0$ for $C \in \mathcal{C}/k$ then C is cofree.*

Proof. Choose a map $j: C \rightarrow PC$ of graded vector spaces with $ji = 1$ where $i: PC \subset C$. Let $\bar{j}: C \rightarrow U(PC)$ be the induced (§2) map of coalgebras. Then 3.4 shows that \bar{j} is isomorphic, so C is cofree.

LEMMA 4.3. *Let $j: B \rightarrow B''$ in \mathcal{C}/k be such that*

$$\text{Cotor}_{-1,*}^{B''}(B, k) = 0.$$

Then

$$k \longrightarrow B \square_{B''} k \xrightarrow{i} B \xrightarrow{j} B'' \longrightarrow k$$

is an injective extension sequence where i is the natural inclusion.

Proof. Let C denote $B \square_{B''} k$, and choose a map $r: B \rightarrow C$ of graded vector spaces such that $ri = 1$. Let $h: B \rightarrow C \otimes B''$ be the extension [11, p. 221] of r to a map of right B'' comodules. Then h is monomorphic by [11, 2.5] since h induces an isomorphism

$$B \square_{B''} k \xrightarrow{\cong} (C \otimes B'') \square_{B''} k.$$

Also h is epimorphic since its cokernel D satisfies $D \square_{B''} k = 0$ as shown by the long exact sequence for $\text{Cotor}^{B''}(\cdot, k)$. Thus h is an isomorphism of right B'' -comodules and the lemma follows.

PROPOSITION 4.4. *If $R^2 P(C) = 0$ for $C \in \mathcal{C}/k$, then C is nice.*

Proof. Let X be any cosimplicial resolution (2.5) of C , and consider the normalized chain complex NPX of PX :

$$\begin{aligned} NPX^0 = PX^0, \quad NPX^n = PX^n / (\text{Im } d^1 + \dots + \text{Im } d^n), \quad n \geq 1, \\ \delta: NPX^{n-1} \rightarrow NPX^n, \end{aligned}$$

with δ induced by the operator d^0 .

By the normalization theorem (see [5]) $H^2(NPX) \approx H^2(PX) \approx R^2 P(C) = 0$.

Choose a map $h: NPX^1 \rightarrow M \in \mathcal{V}/k$ such that the chain map

$$\begin{array}{ccccccc} 0 & \longrightarrow & NPX^0 & \xrightarrow{\delta} & NPX^1 & \xrightarrow{\delta} & NPX^2 \xrightarrow{\delta} \dots \\ & & \downarrow 1 & & \downarrow h & & \downarrow \\ 0 & \longrightarrow & PX^0 & \xrightarrow{h\delta} & M & \longrightarrow & 0 \xrightarrow{0} \dots \end{array}$$

induces cohomology isomorphisms in degrees ≤ 2 .

Now construct a map $j: X^1 \rightarrow M \in \mathcal{V}/k$ such that:

- (i) The restriction of j to PX^1 induces $h: NPX^1 \rightarrow M$.

(ii) The composite $X^0 \xrightarrow{d^1} X^1 \xrightarrow{j} M$ is zero.

Let $j: X^1 \rightarrow U(M)$ be the map of coalgebras induced by j , and define $g: X^0 \rightarrow U(M)$ by $g = jd^0$.

We claim that

$$(*) \quad k \longrightarrow X^{-1} \xrightarrow{d^0} X^0 \xrightarrow{g} U(M) \longrightarrow k$$

is an injective extension sequence, and this will show $C = X^{-1}$ is nice. The map

$$cg: cX^0 \rightarrow cU(M)$$

of constant cosimplicial objects has mapping cone (3.7) $M(CG)$. Letting \tilde{X} equal X without augmentation, there is a unique cosimplicial map

$$F: \tilde{X} \rightarrow M(CG)$$

over \mathcal{C}/k such that:

(i) $F^0 = 1: X^0 \rightarrow X^0$.

(ii) The composite

$$X^1 \xrightarrow{F^1} X^0 \otimes U(M) \xrightarrow{\alpha \otimes 1} k \otimes U(M)$$

equals \bar{j} where $\alpha: X^0 \rightarrow k$ is the counit.

It is easily shown that

$$NPF^*: H^n(NP\tilde{X}) \rightarrow H^n(NPM(CG))$$

is an isomorphism for $n \leq 2$. Hence by 3.5

$$F^*: H^n(\tilde{X}) \rightarrow H^n(M(CG))$$

is an isomorphism for $n \leq 1$. Since $H^0(\tilde{X}) \approx X^{-1}$ and $H^1(\tilde{X}) = 0$, it follows by 3.8 that

$$X^0 \square_{U(M)} k \approx X^{-1}, \quad \text{Cotor}_{-1,*}^{U(M)}(X^0, k) = 0,$$

where $g: X^0 \rightarrow U(M)$ gives the comodule structure. Hence 4.3 shows that (*) is an injective extension sequence.

REMARK 4.5. Suppose $C \in \mathcal{C}/k$ is nice and $i: C \rightarrow B$ is any monomorphism in \mathcal{C}/k with B cofree. Then there is an injective extension sequence

$$k \longrightarrow C \xrightarrow{i} B \xrightarrow{j} B'' \longrightarrow k$$

with B'' cofree. This may be shown by constructing a cosimplicial resolution X of C with

$$\left(X^{-1} \xrightarrow{d^0} X^0 \right) = \left(C \xrightarrow{i} B \right)$$

and then using X in the proof of 4.4.

5. A spectral sequence for Cotor. For a homology coalgebra $C \in \mathcal{C}/k$ we shall construct a spectral sequence of Hopf algebras $\{E^r(C)\}$ converging to $\text{Cotor}^C(k, k)$

and such that $E^2(C)$ depends only on $R^*P(C)$. This technical machinery will be applied in §6.

5.1. *Recollections on Cotor.* For $C \in \mathcal{C}/k$ there is defined (see [7], [12]) a second quadrant bigraded Hopf algebra

$$\text{Cotor}^C(k, k) = \{\text{Cotor}_{n,q}^C(k, k)\}_{n \leq 0 \leq q}$$

where n is the homological degree, and $n+q$ is the total degree. By [12]

$$\text{Cotor}^C(k, k) = H(\Omega C)$$

where $\Omega(C)$ is the differential Hopf algebra given by the reduced cobar construction. Thus $\Omega(C)$ is a bigraded tensor algebra with

$$\Omega_{-n,*}(C) = \bigotimes^n (JC)$$

where the comultiplication is determined by the condition $\Omega_{-1,*}(C) \subset P\Omega(C)$, and where the differential is of degree $(-1, 0)$.

If C is cofree, it is well known (see [12]) that

$$\text{Cotor}^C(k, k) \approx G(PC)$$

where G is the following functor from \mathcal{V}/k to bigraded Hopf algebras. If k is of characteristic 2, then $G(M)$ is the exterior algebra generated by $G_{-1,*}(M) = M$; and if k is of characteristic not 2, then $G(M)$ is the free commutative algebra generated by $G_{-1,*}(M) = M$. The comultiplication of $G(M)$ is determined by the condition that $G_{-1,*}(M) \subset PG(M)$.

5.2. *The Hopf algebra $E^2(C)$.* For $C \in \mathcal{C}/k$ we now construct the E^2 -term of our spectral sequence (5.3). Consider the cosimplicial object $GP\check{K}C$ over the category of bigraded Hopf algebras, with $\check{K}C$ as in 2.4. Define

$$E_{s,n,q}^2(C) = H^{-s}(G_{n,q}P\check{K}C), \quad s, n \leq 0 \leq q,$$

with $s+n+q$ called the total degree, and give $E^2(C)$ the following structure as a trigraded Hopf algebra. Let φ' denote the composition

$$\begin{aligned} H^{-s}(G_{n,q}P\check{K}C) \otimes H^{-h}(G_{i,j}P\check{K}C) &\xrightarrow{f} H^{-s-h}(G_{n,q}P\check{K}C \otimes G_{i,j}P\check{K}C) \\ &\xrightarrow{\mu^*} H^{-s-h}(G_{n+i,q+j}P\check{K}C) \end{aligned}$$

where f is induced by the dual to the Alexander-Whitney map [8, p. 241] and μ is the multiplication in $G(\cdot)$. The multiplication φ of $E^2(C)$ is defined by

$$\varphi = (-1)^{s(i+j)}\varphi': E_{s,n,q}^2(C) \otimes E_{h,i,j}^2(C) \rightarrow E_{s+h,n+i,q+j}^2(C).$$

Let Δ' denote the composition

$$\begin{aligned} H^{-s-h}(G_{n+i,q+j}P\check{K}C) &\xrightarrow{\psi^*} H^{-s-h}(G_{n,q}P\check{K}C \otimes G_{i,j}P\check{K}C) \\ &\xrightarrow{g} H^{-s}(G_{n,q}P\check{K}C) \otimes H^{-h}(G_{i,j}P\check{K}C) \end{aligned}$$

where ψ is a component of the diagonal map of $G(\cdot)$, and g is induced by the dual to the shuffle map [8, p. 243]. The diagonal map Δ of $E^2(C)$ has components defined by

$$\Delta = (-1)^{s(i+j)}\Delta': E_{s+h, n+i, q+j}^2(C) \rightarrow E_{s, n, q}^2(C) \otimes E_{h, i, j}^2(C).$$

One shows that $E^2(C)$ is a Hopf algebra with commutative multiplication and commutative diagonal. (The twisting map is defined using total degrees.) Clearly

$$\begin{aligned} E_{s, 0, q}^2(C) &= k \quad \text{if } s, q = 0, \\ &= 0 \quad \text{otherwise;} \\ E_{-s, -1, *}(C) &= R^sP(C). \end{aligned}$$

In general $E^2(C)$ depends only on $R^*P(C)$ since by [5] $H^*(GP\tilde{K}C)$ depends only on $H^*(P\tilde{K}C)$.

THEOREM 5.3. For $C \in \mathcal{C}/k$ there is a natural spectral sequence $\{E^r(C), d^r\}_{r \geq 2}$ of trigraded differential Hopf algebras such that:

- (i) $E^2(C)$ is given by 5.2.
- (ii) The differential d^r is of degree $(-r, r-1, 0)$.
- (iii) The spectral sequence $\{E^r(C)\}$ converges to $\text{Cotor}^C(k, k)$ in the naive sense, i.e. there is a decreasing filtration on $\text{Cotor}^C(k, k)$ compatible with the Hopf algebra structure and with

$$E^0 \text{Cotor}^C(k, k) \approx E^\infty(C)$$

as Hopf algebras. In particular $E_{s, n, q}^\infty(C)$ is a subquotient of $\text{Cotor}_{s+n, q}^C(k, k)$.

- (iv) The image of the edge morphism

$$R^sP(C) = E_{-s, -1, *}(C) \rightarrow \text{Cotor}_{-s-1, *}^C(k, k)$$

lies in $P \text{Cotor}^C(k, k)$.

Proof. Let Y denote the cosimplicial object $\tilde{K}C$. Let $T(Y)$ be the trigraded k -module

$$T_{s, n, q}(Y) = \Omega_{n, q}(Y^{-s}), \quad s, n \leq 0 \leq q$$

with $\Omega(\cdot)$ as in 5.1, and with $s+n+q$ called the total degree. The total differential ∂_T on $T_{s, n, q}(Y)$ is given by

$$\partial_T = \partial + (-1)^{n+q}\delta$$

where $\partial: T_{s, n, q}(Y) \rightarrow T_{s, n-1, q}(Y)$ is the differential of $\Omega(Y^{-s})$ and

$$\delta: T_{s, n, q}(Y) \rightarrow T_{s-1, n, q}(Y)$$

is the coboundary for the cosimplicial object $\Omega_{n, q}(Y)$. Let φ' denote the composition

$$\Omega_{n, q}(Y^{-s}) \otimes \Omega_{i, j}(Y^{-h}) \xrightarrow{f} \Omega_{n, q}(Y^{-s-h}) \otimes \Omega_{i, j}(Y^{-s-h}) \xrightarrow{\mu} \Omega_{n+i, q+j}(Y^{-s-h})$$

where f is dual to the Alexander-Whitney map [8, p. 241] and μ is the multiplication in $\Omega(\cdot)$. The multiplication map

$$\varphi = (-1)^{s(i+j)}\varphi': T_{s, n, q}(Y) \otimes T_{h, i, j}(Y) \rightarrow T_{s+h, n+i, q+j}(Y)$$

makes $(T(Y), \partial_T)$ a differential graded algebra. Similarly $(T(Y), \partial_T)$ is a differential graded coalgebra with diagonal map defined using the dual to the shuffle map [8, p. 243] and the diagonal in $\Omega(\cdot)$. We do not claim that $T(Y)$ is a Hopf algebra.

Filter $T(Y)$ by $\{F_sTY\}_{s \geq 0}$ where F_sTY is generated as a graded k -module by all $T_{m,n,q}(Y)$ with $m \leq s$. The filtration is compatible with the algebra and coalgebra structures of $T(Y)$ and we claim that the associated spectral sequence satisfies 5.3. By construction, each $E^r(C)$ is a differential algebra and coalgebra; and one easily shows $E^2(C)$ is the Hopf algebra of 5.2. Hence $E^r(C)$, $r \geq 2$, is a Hopf algebra. The convergence condition 5.3 (iii) follows from the fact that $T(Y)$ has total homology satisfying

$$H(TY) \approx \text{Cotor}^C(k, k)$$

which may be proved by a spectral sequence argument. Part 5.3 (iv) follows since elements of $E_{s,-1,q}^\infty(C)$ are represented by cycles in $T_{s,-1,q}(Y)$ which must be primitive in $T(Y)$ for reasons of degree.

6. **Cotor for nice coalgebras.** For a homology k -coalgebra C , it is well known (see [12]) that the Hopf algebra $\text{Cotor}^C(k, k)$ is primitively generated. Let

$$P_{*,*}C = P \text{Cotor}^C(k, k)$$

be the bigraded Lie algebra of primitive elements, and note that $P_{*,*}C$ is restricted [11] if the characteristic of k is $p \neq 0$. By [11] the Hopf algebra $\text{Cotor}^C(k, k)$ is completely determined by the (restricted) Lie algebra $P_{*,*}C$.

THEOREM 6.1. *If $C \in \mathcal{C}/k$ with k of characteristic 0, there are natural isomorphisms $R^sP(C) \approx P_{-s-1,*}C$, $s \geq 0$.*

REMARK. If C is nice in 6.1, then the computation of $\text{Cotor}^C(k, k)$ is reduced to determining $P_{-1,*}C = PC$, $P_{-2,*}C$, and the Lie product

$$[\ , \]: P_{-1,*}C \otimes P_{-1,*}C \rightarrow P_{-2,*}C.$$

Proof. In the spectral sequence 5.3,

$$E_{s,-1,*}^2(C) \approx R^sP(C), \quad s \geq 0,$$

and it is straightforward to show using [5] that $E^2(C)$ is the free commutative algebra generated by $E_{*,-1,*}^2(C)$. Thus $E^2(C)$ is the tensor product of a polynomial algebra generated by elements of even total degree in $E_{*,-1,*}^2(C)$ with an exterior algebra generated by elements of odd total degree. It follows that $E^2(C) = E^\infty(C)$ and so the edge morphism

$$\sigma: R^sP(C) \rightarrow P_{-s-1,*}C, \quad s \geq 0,$$

is monomorphic. Using the structure of $E^\infty(C)$, together with the Poincaré-Birkhoff-Witt theorem [11] applied to $\text{Cotor}^C(k, k)$, one easily shows that σ is an isomorphism.

THEOREM 6.2. *Let $C \in \mathcal{C}/k$ be nice with k of characteristic 2. Then $P_{-1,*}C \approx R^0P(C)$, $P_{-2,*}C \approx R^1P(C)$, and $P_{-n,*}C = 0$ unless $n = 2^t$ with $t \geq 0$. If $\{x_i\}_{i \in I}$ is any homogeneous k -basis for $P_{-2,*}C$, then $\{x_i^{[2^t]}\}_{i \in I}$ is a k -basis for $P_{-2^{t+1},*}C$. Furthermore all Lie products in $P_{*,*}C$ are trivial except (possibly) those of the form*

$$[\ , \]: P_{-1,*}C \otimes P_{-1,*}C \rightarrow P_{-2,*}C.$$

Proof. In the spectral sequence 5.3,

$$E_{-s,-1,*}^2(C) \approx R^sP(C), \quad s \geq 0,$$

and $R^sP(C) = 0$ for $s \geq 2$. It is straightforward to show using [5] that $E^2(C)$ is the tensor product of the exterior algebra generated by $E_{0,-1,*}^2(C)$ with the polynomial algebra generated by $E_{-1,-1,*}^2(C)$. Thus $E^2(C) = E^\infty(C)$ and 6.2 follows easily.

A coalgebra $C \in \mathcal{C}/k$ will be called *regular* [12] if C is nice and $R^1P(C)$ is trivial in odd degrees.

For $C \in \mathcal{C}/k$ let $P_{*,*}^+C$ be the subobject of $P_{*,*}C$ consisting of all elements of even total degree.

THEOREM 6.3. *Let $C \in \mathcal{C}/k$ be regular with k of characteristic p odd. Then $P_{-1,*}C \approx R^0P(C)$, $P_{-2,*}C \approx R^1P(C)$, and $P_{-n,*}C = 0$ unless either $n = p^t$ or $n = 2p^t$ with $t \geq 0$. If $\{x_i\}_{i \in I}$ is any homogeneous k -basis for $P_{-1,*}^+C$ (resp. $P_{-2,*}C$), then for $t \geq 1$, $\{x_i^{[p^t]}\}_{i \in I}$ is a k -basis for $P_{-p^t,*}C$ (resp. $P_{-2p^t,*}C$). Furthermore, all Lie products in $P_{*,*}C$ are trivial except (possibly) those of the form*

$$[\ , \]: P_{-1,*}C \otimes P_{-1,*}C \rightarrow P_{-2,*}C.$$

Proof. In the spectral sequence 5.3, it is straightforward to show by [5] that $E^2(C)$ is the free commutative algebra generated by $E_{0,-1,*}^2(C) = R^0P(C)$ and by $E_{-1,-1,*}^2(C) \approx R^1P(C)$. Thus $E^2(C) = E^\infty(C)$ and 6.3 follows easily.

REMARK 6.4. Under the hypotheses of 6.2 and 6.3, the problem of computing $\text{Cotor}^C(k, k)$ is reduced to that of determining $P_{-1,*}C = PC$, $P_{-2,*}C$,

$$[\ , \]: P_{-1,*}C \otimes P_{-1,*}C \rightarrow P_{-2,*}C$$

and in 6.2 ()^[2]: $P_{-1,*}C \rightarrow P_{-2,*}C$.

APPENDIX

7. Derived functors of nonadditive functors. Let

$$T: \mathcal{C} \rightarrow \mathcal{A}$$

be a (covariant) functor from an arbitrary category \mathcal{C} to an abelian category \mathcal{A} . If \mathcal{C} is equipped with a class \mathcal{M} of injective models (7.2) we will define right derived functors of T with respect to \mathcal{M}

$$R^nT: \mathcal{C} \rightarrow \mathcal{A}, \quad n \geq 0.$$

Let \mathcal{C}^+ denote the category whose objects are the same as those of \mathcal{C} and whose maps are formal sums of maps in \mathcal{C} , i.e. $\text{Hom}_{\mathcal{C}^+}(X, Y)$ is the free abelian

group on $\text{Hom}_{\mathcal{C}}(X, Y)$. Thus \mathcal{C} is a subcategory of \mathcal{C}^+ , and T extends to a unique additive functor

$$T^+ : \mathcal{C}^+ \rightarrow \mathcal{A}.$$

DEFINITION 7.1. If \mathcal{M} is a class of objects in \mathcal{C} , a *right \mathcal{M} -resolution* for $N \in \mathcal{C}$ is a chain complex

$$(*) \quad N \xrightarrow{\varepsilon} M^0 \xrightarrow{\delta} M^1 \xrightarrow{\delta} \dots$$

in \mathcal{C}^+ such that:

- (i) For $n \geq 0, M^n \in \mathcal{M}$.
- (ii) For each $M \in \mathcal{M}$ the functor $\text{Hom}_{\mathcal{C}^+}(\cdot, M)$ transforms (*) into an acyclic complex of abelian groups

$$0 \leftarrow \text{Hom}_{\mathcal{C}^+}(N, M) \leftarrow \text{Hom}_{\mathcal{C}^+}(M^0, M) \leftarrow \text{Hom}_{\mathcal{C}^+}(M^1, M) \leftarrow \dots$$

DEFINITION 7.2. A *class of injective models* for \mathcal{C} consists of a class \mathcal{M} of objects in \mathcal{C} such that each $N \in \mathcal{C}$ has a right \mathcal{M} -resolution.

7.3. *Right derived functors.* Let

$$T : \mathcal{C} \rightarrow \mathcal{A}$$

be as above, and let \mathcal{M} be a class of injective models for \mathcal{C} . The *right derived functors* of T with respect to \mathcal{M} are defined by the usual Cartan-Eilenberg [4] procedure. In particular for $N \in \mathcal{C}$, $R^nT(N)$ is isomorphic to the n th cohomology group of

$$0 \longrightarrow T^+(M^0) \xrightarrow{T^+(\delta)} T^+(M^1) \xrightarrow{T^+(\delta)} \dots$$

where

$$N \xrightarrow{\varepsilon} M^0 \xrightarrow{\delta} M^1 \xrightarrow{\delta} \dots$$

is a right \mathcal{M} -resolution of N . The obvious comparison theorem holds for right \mathcal{M} -resolutions and we obtain functors

$$R^nT : \mathcal{C} \rightarrow \mathcal{A}, \quad n \geq 0,$$

unique up to natural equivalence. Furthermore $\varepsilon : N \rightarrow M^0$ induces a natural map $\varepsilon : T(N) \rightarrow R^0T(N)$.

It is easy to show:

PROPOSITION 7.4. *For $M \in \mathcal{M}$, $\varepsilon : T(M) \rightarrow R^0T(M)$ is isomorphic and $R^nT(M) = 0$ for $n > 0$.*

REMARK 7.5. If \mathcal{C} is an abelian category with enough injectives, then [1] the injectives form a class of injective models for \mathcal{C} , and the above right derived functors coincide with the classical [4] ones when T is additive.

7.6. *Cosimplicial objects.* In studying derived functors it is convenient to use cosimplicial methods. A *cosimplicial object* X (over a category \mathcal{C}) consists of

- (i) for every integer $n \geq 0$ an object $X^n \in \mathcal{C}$.
- (ii) for every pair of integers (i, n) with $0 \leq i \leq n$ *coface* and *codegeneracy* maps

$$d^i: X^{n-1} \rightarrow X^n \in \mathcal{C}, \quad s^i: X^{n+1} \rightarrow X^n \in \mathcal{C},$$

satisfying the identities

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, & i < j, \\ s^j d^i &= d^i s^{j-1}, & i < j, \\ &= id, & i = j, j + 1, \\ &= d^{i-1} s^j, & i > j + 1, \\ s^j s^i &= s^{i-1} s^j, & i > j. \end{aligned}$$

Thus a cosimplicial object over \mathcal{C} corresponds to a simplicial (see [10]) object over the dual category \mathcal{C}^* .

An *augmentation* for X consists of a map $d^0: X^{-1} \rightarrow X^0 \in \mathcal{C}$ such that

$$d^1 d^0 = d^0 d^0: X^{-1} \rightarrow X^1.$$

If X is an (augmented) cosimplicial object over \mathcal{C} , there is a chain complex $ch^+ X$ over \mathcal{C}^+ defined by

$$ch^+ X^n = X^n, \quad \delta = \sum_{i=0}^{n+1} (-1)^i d^i: ch^+ X^n \rightarrow ch^+ X^{n+1}$$

for $n \geq 0$ (for $n \geq -1$ if X augmented).

If \mathcal{C} is an abelian category there is a chain complex chX over \mathcal{C} defined by omitting the “+” in the definition of $ch^+ X$; and we then let $H^n(X)$ denote $H^n(chX)$.

7.7. *Triples and cosimplicial objects.* For arbitrary categories \mathcal{C} and \mathcal{D} let

$$F: \mathcal{C} \rightarrow \mathcal{D}, \quad G: \mathcal{D} \rightarrow \mathcal{C},$$

be functors with F left adjoint to G , and denote the adjunction maps by

$$\varphi: 1_{\mathcal{C}} \rightarrow GF, \quad \psi: FG \rightarrow 1_{\mathcal{D}}.$$

There is associated a triple [6]

$$K: \mathcal{C} \rightarrow \mathcal{C}, \quad \varphi: 1_{\mathcal{C}} \rightarrow K, \quad \mu: KK \rightarrow K,$$

where

$$K = GF, \quad \mu = 1\psi 1: G(FG)F \rightarrow G1F.$$

For $N \in \mathcal{C}$ let $K^0 N = N$ and $K^n N = KK^{n-1} N$. Define an augmented cosimplicial object KN by:

$$\begin{aligned} KN^n &= K^{n+1} N, \quad n \geq -1, \\ \left(KN^{n-1} \xrightarrow{d^i} KN^n \right) &= \left(K^n N \xrightarrow{K^i \varphi K^{n-i}} K^{n+1} N \right), \quad 0 \leq i \leq n, \\ \left(KN^{n+1} \xrightarrow{s^i} KN^n \right) &= \left(K^{n+2} N \xrightarrow{K^i \mu K^{n-i}} K^{n+1} N \right), \quad 0 \leq i \leq n. \end{aligned}$$

The following proposition provides many examples in which the above theory of derived functors may be applied. We assume the conditions of 7.7.

PROPOSITION 7.8. *Let \mathcal{M} be the class of all $M \in \mathcal{C}$ with $M \approx GD$ for some $D \in \mathcal{D}$. Then \mathcal{M} is a class of injective models for \mathcal{C} , and $ch^+(\mathbf{KN})$ is a right \mathcal{M} -resolution for $N \in \mathcal{C}$.*

Proof. Clearly $\mathbf{KN}^n \in \mathcal{M}$ for all $n \geq 0$, so it suffices to show that

$$\mathrm{Hom}_{\mathcal{C}^+}(ch^+(\mathbf{KN}), GD)$$

is an acyclic chain complex for each $D \in \mathcal{D}$. A contracting homotopy

$$\bar{s}: \mathrm{Hom}_{\mathcal{C}^+}(ch^+(\mathbf{KN})^{n-1}, GD) \rightarrow \mathrm{Hom}_{\mathcal{C}^+}(ch^+(\mathbf{KN})^n, GD)$$

for $n \geq 0$ is induced by the following map

$$s: \mathrm{Hom}_{\mathcal{C}}(K^n N, GD) \rightarrow \mathrm{Hom}_{\mathcal{C}}(K^{n+1} N, GD).$$

For $f: K^n N \rightarrow GD \in \mathcal{C}$ let $s(f) = G(f')$: $GFK^n N \rightarrow GD$ where f' is adjoint to f .

REMARK 7.9. Under the hypotheses of 7.8, right derived functors with respect to \mathcal{M}

$$R^n T: \mathcal{C} \rightarrow \mathcal{A}, \quad n \geq 0,$$

may be constructed as follows. For $N \in \mathcal{C}$

$$R^n T(N) \approx H^n(T\tilde{\mathbf{K}}N)$$

where $T\tilde{\mathbf{K}}N$ is the cosimplicial object over \mathcal{A} obtained from \mathbf{KN} by omitting the augmentation and applying T dimensionwise.

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BRANDEIS UNIVERSITY,
WALTHAM, MASSACHUSETTS 02154