TWO-SIDED SEMISIMPLE MAXIMAL QUOTIENT RINGS

BY

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Abstract. Let R be an associative ring with singular right ideal zero and finite right Goldie dimension; F. L. Sandomierski has shown that the (R. E. Johnson) maximal right quotient ring Q of R is then semisimple (artinian). In this paper necessary and sufficient conditions are sought that Q be also a left (necessarily the maximal) quotient ring of R. Flatness of Q as a right R-module is shown to be such a condition. The condition that R have singular left ideal zero and finite left Goldie dimension, though necessary, is shown to be not sufficient in general. Conditions of two-sidedness of Q are also obtained in terms of the homogeneous components (simple subrings) of Q and the subrings of R, they induce.

0. **Introduction and notation.** Let R be an associative ring and M a right R-module. The module M is of finite dimension over R [6, p. 202] if every direct sum of nonzero submodules of M contains only a finite number of summands. Finite dimension over R for a left R-module is defined similarly. The ring R is of finite right (left) dimension according as the module R_R (resp. $_RR$) is of finite dimension over R. We say R is finite dimensional if R is of finite right and of finite left dimension.

Suppose M is of finite dimension over R; if an integer n (≥ 0) exists such that (a) every direct sum of nonzero submodules of M has at most n summands, and (b) there is a direct sum of nonzero submodules with n summands, then n is the dimension of M, in the sense of Goldie [6, p. 202], denoted $d(M_R)$ ($d(_RM)$ in case M is a left R-module).

The rings R considered in this paper (nonsingular) in particular, if they are finite dimensional, then they have a right dimension $d(R_R)$ and a left dimension $d(_RR)$ [6, Theorem 1.1, p. 202].

We denote by $Z(R_R)$ ($Z(_RR)$) the singular right (resp. left) ideal of R (e.g., [2, Introduction]) and we say R is nonsingular if $Z(_RR) = Z(R_R) = (0)$.

In case R is a semiprime, finite dimensional, nonsingular ring, the maximal right quotient (MRQ) ring of R [7, p. 106] is also the maximal left quotient (MLQ) ring of R and in particular $d(R_R) = d(_RR)$ [10, Theorem 1.7], [6, Theorem 5.5], [5, Theorem 1.1].

In this paper we look at the question of two-sidedness of the MRQ ring Q of an arbitrary (not semiprime) finite dimensional nonsingular ring R. In this case Q is semisimple (with d.c.c.) [10, Theorem 6, p. 115].

Throughout this paper semisimple (ring) means semisimple with d.c.c.

In §2 (Theorem 2.3) it is shown that Q is also a left quotient ring and hence the MLQ ring of R if, and only if, Q_R is flat.

In §3 (Theorem 3.3) it is shown that if $d(R_R) = d(_RR) = 2$ then Q is also the MLQ ring of R; if, however, $d(R_R) = d(_RR) \geq 3$ this need not be true (Theorem 3.4).

In §4 conditions for the two-sidedness of Q are obtained in terms of the simple components of Q.

In §5 we look at the case R is a rational subdirect sum of a finite collection of rings (Theorem 5.2). The concept of rational subdirect sum is an extension of the concept of irredundant subdirect sum of a collection of rings [9, p. 65].

Although by a ring it is meant an associative ring, it is not assumed that a ring has identity 1, as in some cases, notably in §4 (the subrings $R_i = R \cap Q_i$), this may not be the case. If a ring does have an identity, the modules over it are, of course, considered unitary.

For each right (left) R-module M_R (resp. $_RM$), $L(M_R)$ (resp. $L(_RM)$) denotes the lattice of large submodules of M_R (resp. $_RM$). For further notation and definitions the reader is referred to [2, Introduction and notation].

This paper constitutes a portion of the author's doctoral dissertation at the University of Wisconsin. The author is deeply indebted to Professor F. L. Sandomierski, his advisor, for supplying many of the ideas included in what follows.

1. **Nonsingular rings without identity.** In this section we give a simple result by which most questions about the maximal right quotient ring of a nonsingular ring, which does not necessarily possess an identity, can be reduced to ones about a nonsingular ring which has an identity.

Let R be a nonsingular ring not necessarily with 1. If Q and S are the maximal right and left, respectively, quotient rings of R, then they have identities 1_Q and 1_S, respectively, such that for any $a \in R$, $a1_Q = 1_Qa = a$ and $a1_S = 1_Sa = a$ [4, II, Prop. 6.2, p. 161]. If Z denotes the ring of integers, we let $R^* = \{r + m1_Q / r \in R, m \in Z\}$ and $\Lambda^* = \{r + m1_S / r \in R, m \in Z\}$, subrings of Q containing R and with identities 1_Q and 1_S respectively. We now have:

THEOREM. *Let R, Q, S, R^*, Λ^* be as in the paragraph above. The following statements are then true*:

(a) $_RR \in L(_RR^*)$, *i.e. R as a left R-module is large in the left R-module R^*.*

(b) $R^* \cong \Lambda^*$ *by a ring isomorphism ϕ, which extends the identity map on R.*

(c) *Q and S are the right and left, respectively, maximal quotient rings of R^*.*

Proof. (a) Let $0 \neq x = r + m1_Q \in R^*$. Now $Z(_RR) = (0)$ implies that $Z(_RQ) = (0)$ and since $0 \neq x \in Q$, it follows $Rx \neq 0$ so that there exists $a \in R$ such that $0 \neq ax = ar + a(m1_Q) = ar + ma \in R$. Clearly $_RR \in L(_RR^*)$.

(b) Define $\phi: \Lambda^* \to R^*$ by $\phi(r+m1_S)=r+m1_Q$. If $r+m1_S=0$ then for every $a \in R$, $0=a(r+m1_S)=ar+ma=ar+a(m1_Q)=a(r+m1_Q)$ so that $R(r+m1_Q)=(0)$ and hence $r+m1_Q=0$. Thus ϕ is well defined and easily seen to be a ring isomorphism, extending the identity map of R.

(c) It is clear that $R_{R^*}^* \in L(Q_{R^*})$ since $R_R^* \in L(Q_R)$ by [2, Proposition 1.1]. Now if A and B are right R^*-modules (unitary of course) then it is easy to check that $\mathrm{Hom}_R(A_R, B_R)=\mathrm{Hom}_{R^*}(A_{R^*}, B_{R^*})$, so that Q_{R^*} is R^*-injective since it is R-injective. It follows [7, p. 106] that Q is the maximal right quotient ring of R^*. From (b) it follows that S is (up to isomorphism) the maximal left quotient ring of R^*. The proof of the theorem is now complete.

We remark that statement (a) of the theorem implies that R has a two-sided maximal quotient ring if and only if R^* has.

2. The MRQ ring flat as a right R-module.

Necessary and sufficient conditions that Q, the MRQ ring of ring R with 1 and $Z(R_R)=(0)$, be flat as a *left* R-module were obtained in [2]. The nature of the question of flatness of Q as a *right* R-module (as far as symmetry is concerned) appears to be quite different. Some partial results in this direction, in particular Proposition 2.2, are obtained in this section.

LEMMA 2.1. *Let R be a ring with 1 and Q a right quotient ring of R, flat as a right R-module. Assume $\sum p_i \otimes q_i \to \sum p_i q_i$ is an isomorphism $Q \otimes_R Q \to Q$ (as Q-modules) and let $_R A$, $_R B$ be submodules of $_R Q$. Then $Q(A \cap B)=QA \cap QB$ (the left-right symmetric of this also holds).*

Proof. For any submodule $_R A$ of $_R Q$ we have $Q \otimes A \cong QA$ (by the given isomorphism) and now the proof proceeds as that of [2, Lemma 1.10].

PROPOSITION 2.2. *Let R be a ring with 1 and Q a right quotient ring of R, flat as a right R-module. If $Q \otimes_R Q \cong Q$ canonically (as in Lemma 2.1) then Q is also a left quotient ring of R.*

Proof. It is sufficient to show that for every $0 \neq q \in Q$, we have $R \cap Rq \neq (0)$. By Lemma 2.1 $Q(R \cap Rq)=QR \cap QRq=Q \cap Qq=Qq \neq (0)$. It follows that $R \cap Rq \neq (0)$.

It is appropriate to note here that the condition that $Q \otimes_R Q$ be canonically R- (or Q-) isomorphic to Q, in Proposition 2.2, need not be, in general, satisfied. In fact in case $Z(R_R)=(0)$ and Q is the MRQ ring of R, the condition imposes rather strong restrictions on R [3].

THEOREM 2.3. *Let R be a ring with $Z(R_R)=(0)$ and $d(R_R)<\infty$. Then Q, the MRQ ring of R, is also the MLQ ring of R if and only if Q_R is flat.*

Proof. Observe that the epimorphism $\sum p_i \otimes q_i \to \sum p_i q_i$ of $Q \otimes_R Q$ onto Q is a monomorphism if and only if $Q \otimes_R Q$ is a nonsingular right Q- or R-module.

Indeed suppose $\sum p_i q_i = 0$. The right ideal $I = \{r \in R/q_i r \in R, \text{ all } i\}$ is large in R_R [2, Proposition 1.1] and for every $t \in I$ we have

$$\left(\sum p_i \otimes q_i\right)t = \sum p_i \otimes (q_i t) = \sum p_i(q_i t) \otimes 1 = \left(\sum p_i q_i\right)t \otimes 1 = 0.$$

It follows that $\sum p_i \otimes q_i = 0$ since $Z(Q \otimes_R Q) = (0)$. Now by assumption Q is semisimple [10] and hence $Q \otimes_R Q \cong Q$ since every Q-module is projective [1, p. 11] and $Z(Q) = (0)$.

If Q_R is flat, Q is a left quotient ring by Proposition 2.2. It follows from a result of Utumi's [10, 1.16, p. 4] that Q is (up to isomorphism) the MLQ ring of R.

If Q is also the MLQ ring of R, it is flat as a right R-module, e.g. [2, Theorem 2.1, Remark, (iii), p. 247]. This completes the proof of the theorem.

REMARK (R AND Q AS IN THEOREM 2.3). If M_Q is a module of finite dimension, say n, and A_R is a submodule of M_R such that $A_R \in L(M_R)$ then $d(A_R) = d(M_Q)$. Indeed any submodule B_R of A_R has an R-injective hull E_R which may be chosen to be a right Q-module as well [10, Theorem 2.6, p. 118]. Since $Z(M_R) = Z(M_Q) = (0)$ it is easy to show that $\text{Hom}_R(E_R, M_R) = \text{Hom}_Q(E_Q, M_Q)$ so a copy of the injective hull of B_R can be found in M as Q-submodule of M. It follows that any direct sum of nonzero submodules of A_R has at most n summands; on the other hand there are simple Q-submodules $M_1, \ldots, M_n$ of M_Q such that $M = M_1 \oplus \cdots \oplus M_n$ and it is clear that $(M_1 \cap A) \oplus \cdots \oplus (M_n \cap A)$ is large in A_R and also $d((M_i \cap A)_R) = 1$, $i = 1, \ldots, n$. Thus we have $d(A_R) = d(M_R) = d(M_Q) = n$. In particular $d(Q_Q) = d(Q_R) = d(R_R)$ and since Q is semisimple (ring) we have $d(_Q Q) = d(Q_Q)$.

So if Q is the MLQ ring of R as well, then necessarily $Z(_R R) = (0)$ and $d(_R R) = d(R_R)$.

3. **Finite dimensional nonsingular rings.** If R is a finite dimensional nonsingular ring such that $d(R_R) = 1$, then, by the last remark, $d(Q_Q) = 1$, where Q is the MRQ ring of R. In particular any nonzero right ideal of Q is large in Q and since Q is semisimple [10, p. 115] it has no proper ($\neq (0), Q$) right ideals and hence it is a division ring. It follows that R is a prime ring which is right uniform (i.e. $d(R_R) = 1$) and since $Z(_R R) = (0)$ and $d(_R R) < \infty$, Q is also the MLQ ring of R [5, Theorem 2, p. 594]. In this case Q is also a classical quotient ring of R.

Our first main result of this section shows that if $d(_R R) = d(R_R) = 2$, R nonsingular, then the MRQ ring of R is also the MLQ ring of R.

We need the following:

LEMMA 3.1. *Let R be a nonsingular ring (with 1) with a semisimple MRQ ring Q. The following statements are then true:*

(a) *If $_R I \in L(_R R)$, then $QI = Q$.*

(b) *If $_R B$ is any R-module and $_R A \in L(_R B)$, then the left Q-homomorphism $1 \otimes i: Q \otimes A \to Q \otimes B$ is an epimorphism.*

Proof. (a) Since Q is semisimple, $QI = Qe$ for some $e^2 = e \in Q$. Now $Qe(1 - e) = (0)$ so that $I(1 - e) = (0)$ and since $Z(_R Q) = (0)$ it follows that $1 - e = 0$. We have $QI = Q$.

(b) The set $E = \{1 \otimes b : b \in B\}$ clearly generates $Q \otimes B$ over Q. Let $1 \otimes b \in E$; since $_RA \in L(_RB)$, $_RI = \{r \in R | rb \in A\} \in L(_RR)$ [2, Proposition 1.1] and by (a) $QI = Q$. It follows that $1 = \sum q_i \lambda_i$, $q_i \in Q$, $\lambda_i \in I$ and $\sum q_i \otimes \lambda_i b \in Q \otimes A$ since $Ib \subset A$. Now $\sum q_i \otimes \lambda_i b = \sum q_i \lambda_i \otimes b = 1 \otimes b$ so that $(1 \otimes i)(\sum q_i \otimes \lambda_i b) = 1 \otimes b$ and $1 \otimes i$ is an epimorphism.   Q.E.D.

A left R-module M is uniform if $d(_RM) = 1$, equivalently $M \neq (0)$ and every non-zero submodule of M is large in M; a left ideal U of R is uniform if the module $_RU$ is.

We need the following characterization of flatness of Q_R:

PROPOSITION 3.2. *Let R be a nonsingular ring with 1 and $d(_RR) = d(R_R) = n < \infty$; let Q be the MRQ ring of R. The following statements are equivalent*:

(a) $Q \otimes U$ *is uniform as a left Q-module, for every uniform left ideal U of R.*

(b) Q_R *is flat (Q is then the MLQ ring of R also).*

NOTE. In the interest of simplifying notation in the proof that follows as well as the proof of Theorem 3.3 we write $d(M)$ for $d(_QM)$, whenever M is a left Q-module.

Proof (of Proposition 3.2). (a) *implies* (b). In view of [7, p. 135, Ex. 1] it suffices to show that for any large left ideal I, $0 \rightarrow Q \otimes I \rightarrow Q \otimes R$ is exact. Let $_RI \in L(_RR)$. By [6, Theorem 1.1, p. 102] $d(_RI) = n$ and there exist uniform left ideals of R, $U_1, \ldots, U_n$ contained in $_RI$ such that the sum $U_1 + \cdots + U_n$ is direct and large in $_RI$. Since $Q \otimes (U_1 \oplus \cdots \oplus U_n) \cong (Q \otimes U_1) \oplus \cdots \oplus (Q \otimes U_n)$ and $d(Q \otimes U_i) = 1$, $i = 1, \ldots, n$, it follows that $d(Q \otimes (U_1 \oplus \cdots \oplus U_n)) = n$. Now by Lemma 3.1 the following maps are epimorphisms:

$$Q \otimes (U_1 \oplus \cdots \oplus U_n) \rightarrow Q \otimes I \rightarrow Q \otimes R = Q.$$

This gives

$$n = d(Q \otimes (U_1 \oplus \cdots \oplus U_n)) \geq d(Q \otimes I) \geq d(Q \otimes R) = d(Q) = n$$

so that $d(Q \otimes I) = n$. Now if $_QK = \ker(Q \otimes I \rightarrow Q \otimes R)$, the exact sequence $0 \rightarrow K \rightarrow Q \otimes I \rightarrow Q \otimes R \rightarrow 0$ splits over Q as Q is semisimple. It follows [6, p. 202] that $n = d(Q \otimes I) = d(K) + d(Q) = d(K) + n$ so that $d(K) = 0$ or $K = (0)$.

Thus $0 \rightarrow Q \otimes I \rightarrow Q \otimes R$ is exact for any $_RI \in L(_RR)$.

(b) *implies* (a). Let U be a uniform left ideal of R. By [6, Theorem 1.1] there exist uniform left ideals $U_2, \ldots, U_n$ such that the sum $U_1 + \cdots + U_n$ is direct and a large left ideal of R, where $U_1 = U$. Since $U_i \neq (0)$ and $1 \in Q$ we have $QU_i \neq (0)$ and thus from the canonical epimorphism $Q \otimes U_i \rightarrow QU_i$ we obtain $d(Q \otimes U_i) \geq 1$, $i = 1, \ldots, n$. On the other hand we have

$$Q = Q(U_1 \oplus \cdots \oplus U_n) \cong Q \otimes (U_1 \oplus \cdots \oplus U_n)$$
$$\cong (Q \otimes U_1) \oplus \cdots \oplus (Q \otimes U_n)$$

(the first equality by Lemma 3.1, the first isomorphism by flatness of Q_R). It follows

that $d(Q \otimes U_1) + \cdots + d(Q \otimes U_n) = n$, which implies $d(Q \otimes U_i) = 1$ for each i, and in particular $d(Q \otimes U) = 1$ for any uniform left ideal of R.   Q.E.D.

We can now state and prove the following:

THEOREM 3.3. *Let R be a nonsingular ring (with 1) and $d(R_R) = d(_R R) = 2$. If Q is the MRQ ring of R then it is also the MLQ ring of R.*

Proof. We shall show that Q_R is flat. The conclusion will then follow from Theorem 2.3. It, hence, suffices to show that $Q \otimes U$ is uniform as a left Q-module for every uniform left ideal U of R (Proposition 3.2). Now if $0 \neq u \in U$, then $Ru \in L(_R U)$ since $d(_R U) = 1$, and we have epimorphisms

(*) $Q \otimes Ru \to Q \otimes U \to QU$

the first one (left) by Lemma 3.1, the second (right) canonical. Since $d(Q \otimes Ru) \geq d(Q \otimes U) \geq d(QU) \geq 1$, it follows that $d(Q \otimes U) = 1$ if $d(Q \otimes Ru) = 1$. In particular (*) says $Q \otimes Ru \neq (0)$. Observe that $_R(0 : u) = \{x \in R / xu = 0\} \neq (0)$ since if $_R(0 : u) = (0)$ then $_S(0 : u) = \{t \in S / tu = 0\} = (0)$ where S is the MLQ ring of R, and semisimple [10]; it follows that u is a nonzero divisor of R [8, Lemma 2.8, p. 139] and $Ru \in L(_R S)$ or $1 = d(Ru) = 2$. Now the map $q \to q \otimes u$ is an epimorphism of Q onto $Q \otimes Ru$ and by semisimplicity of Q we have $Q = _Q A \otimes [Q \otimes_Q Ru]$. The module $_Q A$ cannot be zero, since if it is we have: $Q = Q \otimes Ru \cong Q \otimes R/_R(0 : a) \cong Q/_Q(0 : u) \cong Qe$ for $e^2 = e \in Q$, $e \neq 1, 0$. This implies $2 = d(Q) = d(Qe) = 1$ so that $_Q A \neq (0)$. Since $Q \otimes Ru \neq (0)$ we have $d(Q \otimes Ru) = 1$ and hence $d(Q \otimes U) = 1$, which was to be shown.

Q_R is flat and the MLQ ring also.   Q.E.D.

If $2 < d(_R R) = d(R_R) < \infty$ in the preceding theorem, then Q is not necessarily the MLQ ring also. We proceed with an example followed by a more general result in that direction:

Let D be a division ring and D_3 the complete ring of 3×3 matrices over D. Consider all matrices in D_3 of the form

$$\begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & z \end{pmatrix}, \qquad x, y, z, a \in D.$$

They form a ring R with identity (of D_3). If e_{ij} denotes the matrix of D_3 with 1 in the (i, j) position and zeros elsewhere, we have as usual

$$e_{ij} e_{kl} = 0 \quad \text{if } j \neq k,$$
$$= e_{il} \quad \text{if } j = k,$$

and if α is any element of R then

(1) $\alpha = (e_{11} + e_{22})a + e_{33}z + e_{13}x + e_{23}y, \qquad x, y, z, a \in D.$

If $0 \neq \delta = (d_{ij}) \in D_3$ with say the jth column nonzero then δe_{j3} is an element of R

with the 3rd column $\neq 0$, so we have $R_R \in L((D_3)_R)$ and D_3 is the MRQ ring of R, as D_3 is its own [11, 1.16, p. 4]. In particular this shows that $Z(R_R) = (0)$ and $d(R_R) = d(Q_R) = d(Q_Q) = 3$.

Now the left ideals $R(e_{11} + e_{22})$, Re_{13}, Re_{23} are easily shown to be simple; furthermore $I = Re \oplus Re_{13} \oplus Re_{23}$, $e = e_{11} + e_{22}$, is large in $_RR$. If $z = 0$ in (1) then $\alpha \in I$ so suppose $z \neq 0$; then $e_{23}\alpha = e_{23}z \neq 0$ and $e_{23}z \in Re_{23} \subseteq I$. Now $d(_RI) = 3$ and since $_RI \in L(_RR)$ it follows that $d(_RR) = 3$. Since a large ideal of R must contain all its simple left ideals, the (left) annihilator of any element α of $Z(_RR)$ contains $e = e_{11} + e_{22}$, e_{13} and e_{23}. We thus have $0 = e\alpha = ea + e_{13}x + e_{23}y$ so that $a = x = y = 0$, and $\alpha = e_{33}z$. But we have $0 = e_{13}\alpha = e_{13}e_{33}z = e_{13}z$ so that $z = 0$ and $\alpha = 0$. Thus $Z(_RR) = (0)$. Finally we show that D_3 is not an essential extension of R as a left R-module, hence D_3 is not the MLQ ring R. For example $0 \neq e_{21} \in D_3$ and $\alpha e_{21} = e_{21}a$ for all $\alpha \in R$, but $e_{21}a \notin R$.

We thus have a ring R with the properties: $Z(_RR) = Z(R_R) = (0)$, $d(_RR) = d(R_R) = 3$ and Q, the MRQ of R is not the MLQ ring R.

This example can be generalized to the theorem below, whose proof proceeds in the lines of the argument used above:

THEOREM 3.4. *For each ordered pair (k, n) of integers k and n such that $3 \leq n < \infty$ and $n \leq k \leq 2(n - 1)$, there exists a ring R with 1 and the following properties*:

  (a) $Z(_RR) = Z(R_R) = (0)$.

  (b) $d(_RR) = k$, $d(R_R) = n$.

  (c) *The MRQ ring Q, of R, is not a left quotient ring of R (in case $k \neq n$, (c) is, of course, a consequence of (b))*.

Proof. Let D be a division ring and D_n the complete ring of $n \times n$ matrices over D. The ring R is then the set of matrices (a_{ij}) of D_n defined by:

$a_{11} = a_{22} = \cdots = a_{mm}$, $m = 2n - k - 1$,

a_{jj}, arbitrary for $m < j \leq n$,

$a_{1n}, a_{2n}, a_{3n}, \ldots, a_{n-1,n}$ arbitrary,

all other a_{ij} are zero.

It should be remarked here that to show $d(_RR) = k$ suffices to show that the left ideals:

$$R(e_{11} + \cdots + e_{mm}), Re_{m+1,m+1}, \ldots, Re_{n-1,n-1}, Re_{1n}, Re_{2n}, \ldots, Re_{n-1,n}$$

are all simple (k of them) and their (direct) sum I is large in $_RR$. The proof, otherwise, proceeds as in the example.

4. The simple components of the semisimple MRQ ring. If the MRQ ring Q of R is semisimple we write $Q = Q_1 \oplus \cdots \oplus Q_n$ where each Q_i is a simple ring and in fact $Q_i = e_iQ$, e_i a central idempotent of Q. Since

$$d(Q_Q) = \sum_{i=1}^{n} d(Q_{i_{Q_i}})$$

we have $n \leq d(Q_Q)$. We set $R_i = R \cap Q_i$, a two-sided ideal of R, hence a subring of R not necessarily with an identity. We then have:

THEOREM 4.1. *If the MRQ ring Q of a ring R is semisimple then Q is also the MLQ ring of R if and only if Q_i is the MLQ ring of R_i for each i.*

Before proceeding with the proof of this theorem we note the following: If Q is a right quotient ring of a ring R with $Z(R_R) = (0)$ and $e^2 = e$ is a central idempotent of Q, then eQ is a right quotient ring of $R \cap eQ$. Indeed let $0 \neq eq \in eQ$. For any $x \in Q$, the right ideal $(R:x) = \{r \in R / xr \in R\}$ is large in R_R and so in particular $(R:e)$, $(R:eq)$ and $(R:e) \cap (R:eq)$ [2, Proposition 1.1]. Since $Z(Q_R) = (0)$ there exists $r \in (R:e) \cap (R:eq)$ such that $0 \neq eqr$. Now $0 \neq eqr = (eq)(er) \in R \cap eQ$ and $er \in R \cap eQ$ so that eQ is a right quotient ring of $R \cap eQ$.

Now to return to the proof of Theorem 4.1:

Only if. Follows from the argument (on the left) given above and the fact that a self-injective quotient ring of a ring is the maximal one [11, 1.16, p. 4].

If. If $0 \neq q \in Q$ we write $q = e_1 q_1 + \cdots + e_n q_n$. Since $0 \neq q$, $e_i q \neq 0$ for some i and since $_{R_i} R_i \in L(_{R_i} Q_i)$, there exists $r_i \in R_i$, hence $r_i \in R$, such that $0 \neq r_i e_i q \in R_i \subseteq R$. Observe that $r_i e_j q = 0$ for $j \neq i$ so that $0 \neq r_i q = r_i e_i q \in R$. It follows that $_R R \in L(_R Q)$ and Q is the MLQ ring of R, also.   Q.E.D.

If in the preceding theorem $d(Q_Q) = n$ in $Q = Q_1 \oplus \cdots \oplus Q_n$, then $d(Q_{i_{Q_i}}) = 1$ for each i, so that each Q_i is a division ring, by an earlier observation. In this case Q is a classical right quotient ring and if $d(_R R) < \infty$ then it is also a classical left. This is contained in the following:

THEOREM 4.2. *Suppose R is a finite dimensional ring with the MRQ ring Q, a finite direct sum of division rings. Then Q is a classical two-sided quotient ring of R.*

Proof. Let $Q = D_1 \oplus \cdots \oplus D_n$ where each D_i is a division ring. It follows [5, Theorem 11, p. 604] that D_i is a two-sided maximal (classical) quotient ring of $R_i = R \cap D_i$, for each i. Q is the MLQ ring of R follows from Theorem 4.1.

If $a \in R$ we write $a = (d_1, \ldots, d_n)$, $d_i \in D_i$, and observe that a is a nonzero divisor of R if and only if $d_i \neq 0$ for each i; it follows that if a is a nonzero divisor of R, it is, then, invertible in Q. Let, now, $q \in Q$; to show that $q = ad^{-1}$, $a \in R$, d a nonzero divisor of R, suffices to show that $(R:q) = \{r \in R / qr \in R\}$ contains a nonzero divisor of R. Write $q = (q_1, \ldots, q_n)$; since $R_{i_{R_i}} \in L(D_{i_{R_i}})$ there exists, for each i, $0 \neq a_i \in R_i$ such that $0 \neq q_i a_i \in R_i \subseteq R$. Clearly $d = (a_1, \ldots, a_n)$ is a nonzero divisor of R and $qd = a \in R$. Hence $q = ad^{-1}$. Thus Q is a classical right quotient ring of R and also a left one [6, Theorem 5.5, p. 217].   Q.E.D.

The following is a special result on finite dimensional rings.

PROPOSITION 4.3. *Let R be finite dimensional, nonsingular ring with $d(R_R) = d(_R R) = 3$. If the MRQ ring Q of R is not simple, then Q is also the MLQ ring of R.*

Proof. If $Q = Q_1 \oplus Q_2 \oplus Q_3$, then $d(Q_{i_{Q_i}}) = 1$ and the conclusion follows from Theorem 4.3 above.

Assume $Q = Q_1 \oplus Q_2$ with, say, $d(Q_{1_{Q_1}}) = 1$ and $d(Q_{2_{Q_2}}) = 2$. Now $R_1 \oplus R_2$ is large in R_R where $R_i = R \cap Q_i$, so that

$$d((R_1 \oplus R_2)_R) = 3 = d(R_{1_{R_1}}) \oplus d(R_{2_{R_2}}).$$

Since $d(Q_{1_{Q_1}}) = 1$ we have $d(_{R_1}R_1) = d(R_{1_{R_1}}) = 1$. We also have $d(_R(R_1 \oplus R_2)) \leqq d(_R R) = 3$ so that $d(_{R_2}R_2) \leqq 2$. That Q_2 is also the MLQ ring of R_2 now follows from Theorem 3.3 (as it is true if $d(_R R) \leqq 2$ and $d(R_R) = 2$), and hence Q is also the MLQ ring of R.   Q.E.D.

5. Rational subdirect sums.

Let R be any ring and let $R^\#$ be the ring obtained from R by formally adjoining the integers. If M_R is a right R-module, a submodule B_R of M_R is rational in M_R (M_R is a rational extension of B_R) if for every pair of elements x, y of M_R, with $x \neq 0$, there exists $r \in R^\#$ such that $xr \neq 0$ and $yr \in B_R$. If $Z(B_R) = (0)$, then B_R is rational in M_R if and only if B_R is large in M_R.

As in [9, p. 65] we say that a ring R is a subdirect sum of a set of rings $\{R_\alpha : \alpha \in A\}$ if there exists an isomorphism h of R into the (complete) direct product $\prod_\alpha R_\alpha$ such that $\pi_\alpha h(R) = R_\alpha$ for each $\alpha \in A$, where π_α is the usual projection of $\prod_\alpha R_\alpha$ onto R_α. The subdirect sum is irredundant if for every $\beta \in A$, the kernel of the map $h_{(\beta)}: r \to \{\pi_\alpha h(r) : \alpha \neq \beta\}$ of R into $\prod_{\alpha \neq \beta} R_\alpha$ is nonzero or equivalently $h(R) \cap R_\beta \neq (0)$ for each $\beta \in A$. It is easy to see that $h(R) \cap R_\beta$ is a two-sided R_β- and $h(R)$-ideal. We say R is a right rational subdirect sum (right RSS) of the rings $\{R_\alpha\}$, if $h(R) \cap R_\alpha$ is rational in R_α as a right R_α-ideal, for each $\alpha \in A$. If R is a right RSS of the rings $\{R_\alpha\}$ then R is an irredundant subdirect sum (ISS) of them, clearly, since then $h(R) \cap R_\alpha \neq (0)$ for each $\alpha \in A$. The converse is not true as the following example shows:

Let D be a division ring and consider the following rings:

$$R = \left\{ \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & z \end{pmatrix} : a, x, y, z \in D \right\},$$

$$\Lambda = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in D \right\}.$$

It was shown in the preceding section that D_3 is the MRQ ring of R and it is easy to check that D_2 is the MRQ ring of Λ, also the MLQ ring. In particular $Z(\Lambda_\Lambda) = Z(_\Lambda\Lambda) = (0)$. The mapping:

$$h: \begin{pmatrix} a & 0 & x \\ 0 & a & y \\ 0 & 0 & z \end{pmatrix} \to \left(\begin{pmatrix} a & x \\ 0 & z \end{pmatrix}, \begin{pmatrix} a & y \\ 0 & z \end{pmatrix} \right)$$

is an isomorphism of R into $\Lambda^{(1)} \oplus \Lambda^{(2)}$, $\Lambda^{(i)} = \Lambda$, $i = 1, 2$, and determines an ISS representation of R in terms of $\Lambda^{(1)}$ and $\Lambda^{(2)}$. In fact

$$A_i = h(R) \cap \Lambda^{(i)} = \left\{ \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} : d \in D \right\} \neq (0).$$

If A_i were rational in $\Lambda^{(i)}$ it would then be large in $\Lambda^{(i)}$. Since $Z(\Lambda^{(i)}) = (0)$ and $A_i A_i = (0)$, it follows that A_i is not rational in $\Lambda^{(i)}$. This shows that the isomorphism h does not determine a right RSS representation of R in terms of the rings $\Lambda^{(1)}$, $\Lambda^{(2)}$. Proposition 5.1, below shows that R is not a right RSS of $\Lambda^{(i)}$, $i = 1, 2$.

It is shown in [4, II; Proposition 6.4, p. 162] that if R has zero left annihilator (i.e. $aR = (0)$ if and only if $a = 0$) and A is a two-sided ideal of R, then A is rational in R as a right ideal if and only if A has zero left annihilator (in R).

We use this to prove the following, a generalization of a result of L. Levy's [9, Proposition, p. 72].

PROPOSITION 5.1. *Let $\{R_\alpha : \alpha \in A\}$ be a family of rings R_α with zero left annihilator and let Q_α be the MRQ ring of R_α for each α. If R is a right RSS of the rings $\{R_\alpha\}$, then $\prod_\alpha Q_\alpha$ is the MRQ ring of R. (R is identified with its image in $\prod R_\alpha$.)*

Proof. Since $\prod_\alpha Q_\alpha$ is the MRQ ring of $\prod_\alpha R_\alpha$ [7, p. 100] it is sufficient to show that R is rational in $\prod_\alpha R_\alpha$ as a right R-module. The conclusion will then follow by Utumi's Proposition 1.5 [11, p. 2]. Let $x, y \in \prod_\alpha R_\alpha$ with $x \neq 0$. For some $\beta \in A$, $\pi_\beta(x) \neq 0$, $\pi_\beta(x) \in R_\beta$. Since $A_\beta = R \cap R_\beta (\equiv h(R) \cap R_\beta)$ is rational in R_β, it follows that $0 \neq \pi_\beta(x) A$, so there exists $r \in A_\beta \subseteq R$ such that $\pi_\beta(x) r \neq 0$. Since A_β is a two-sided R_β ideal we have $yr = \pi_\beta(y)r \in A_\beta \subseteq R$. Thus: there exists $r \in R$ such that $0 \neq xr = \pi_\beta(x)r$ and $yr \in R$, so that R_R is rational in $(\prod_\alpha R_\alpha)_R$, and the proposition is established. Q.E.D.

We see now that the ring R is the example is not a right RSS of $\Lambda^{(i)}$, $i = 1, 2$ as we would have $D_3 = D_2 \oplus D_2$.

In [9, Theorem 6.1, p. 74] L. Levy showed that a semiprime ring R is an ISS of a finite number of prime rings $R_1, \ldots, R_n$ with a (classical) simple right quotient ring $Q_1, \ldots, Q_n$ (correspondingly) if and only if R has a (classical) semisimple right quotient ring Q, and then $Q \cong Q_1 \oplus \cdots \oplus Q_n$. In the proof of this it is shown [9, Lemma 6.2] that R is a right RSS of the rings $R_1, \ldots, R_n$. Replacing ISS by RSS we can generalize this result to arbitrary rings and their maximal right quotient ring. The generalization is otherwise false as the remark following Proposition 5.1 above shows. Hence we give:

THEOREM 5.2. *For any ring R the following statements are equivalent:*

(a) *R is a right RSS of finite number of rings $R_1, \ldots, R_n$ with simple MRQ rings $Q_1, \ldots, Q_n$ correspondingly.*

(b) *R has a semisimple MRQ ring $Q = Q_1 \oplus \cdots \oplus Q_n$.*

Proof. (a) *implies* (b). Follows from Proposition 5.1.

(b) *implies* (a). For each $i = 1, \ldots, n$, let e_i be the identity of Q_i, a central idempotent of Q such that $Q_i = e_i Q$. Set $R_i = e_i R$, $i = 1, 2, \ldots, n$. It is clear that $R \cap Q_i \subseteq e_i R = R_i$ so that Q_i is the MRQ ring of R_i for each i as it is the MRQ of $R \cap Q_i$ (Theorem 4.1). Since $1 = e_1 + \cdots + e_n$, the mapping $h: r \to e_1 r + \cdots + e_n r$ of R into $R_1 \oplus \cdots \oplus R_n$ is an isomorphism and determines an irredundant subdirect sum representation of R in $R_1 \oplus \cdots \oplus R_n$. It remains to show that $R \cap R_i$ is rational in R_i as right R_i-modules. Since $Z(R_{i_{R_i}}) = (0)$, rational coincides with large so we show that $R \cap R_i$ is large in $R_{i_{R_i}}$. Let $0 \neq x = e_i r \in R_i$. The right ideal $I = \{t \in R / e_i t \in R\}$ is large in R_R so that $xI \neq (0)$ and there exists $t \in I$ such that $xt \neq 0$. Clearly $xt \in R \cap R_i$ and since $xt = e_i r t = (e_i r)(e_i t)$ we have $y = e_i t \in R_i$ such that $0 \neq xy \in R \cap R_i$ so that $R \cap R_i$ is large in R_i.

It follows that R is a right RSS of $R_1, \ldots, R_n$.   Q.E.D.

It is easy to see that in the theorem above, Q is also the MLQ ring of R if and only if each Q_i is, and this gives another criterion of two-sidedness of the MRQ ring in terms of its simple components in case it is semisimple.

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