TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 149, June 1970

QUASI-STATES ON C^*-ALGEBRAS

BY
JOHAN F. AARNES[1]

1. **Introduction.** Several attempts have been made in recent years to clarify and axiomatize the theoretical foundations of quantum mechanics. While the physical significance of these efforts remains unclear, they have yielded some highly interesting mathematical problems. In this paper we are going to discuss one of these problems—the question of additivity of the expectation-functional on the algebra of observables.

Basic in any modern treatment of quantum mechanics is the family of observables associated with the physical system under consideration, and the set of states in which the system can be found. The choice of an adequate mathematical model may be made in several ways. In his recent approach [16], Mackey identifies the family of observables $\mathcal{O}$ of a system with the set of all *question-valued measures* on $\boldsymbol{R}$. A *question* is an element of a *logic* $\mathcal{L}$ (an orthocomplemented, complete lattice). The states $\mathcal{S}$ of the system correspond to a convex set of probability measures on $\mathcal{L}$. A priori, no global assumption concerning the algebraic structure of $\mathcal{O}$ is made, although a certain local algebraic structure exists. Indeed, if $x \in \mathcal{O}$, and f is a real Borel-function on $\boldsymbol{R}$, then $f(x)$ is the observable given by $f(x)(E)=x(f^{-1}(E))$, where E is an arbitrary Borel subset of $\boldsymbol{R}$. It follows that if y and z are Borel-functions of a third observable x, then $y+z$ and yz exist as observables in a natural way.

If x is an observable and α is a state, then $\alpha_x(E)=\alpha(x(E))$, E a Borel subset of $\boldsymbol{R}$, defines a Borel-measure on $\boldsymbol{R}$. We refer to α_x as the *probability distribution* of x in the state α. The integral

$$\rho_\alpha(x) = \int_{-\infty}^{+\infty} \lambda \, d\alpha_x(\lambda),$$

if it exists, is called the *expectation* of the observable x in the state α. It is easily seen that the function ρ_α defined on $\mathcal{O}$ in this way is linear on each subset

$$\mathcal{O}_x = \{f(x) : f \text{ Borel}: \boldsymbol{R} \to \boldsymbol{R}\}.$$

Moreover, if $x_1, x_2 \in \mathcal{O}$ are given, there exists at most one $x_3 \in \mathcal{O}$ satisfying

$$\rho_\alpha(x_3) = \rho_\alpha(x_1)+\rho_\alpha(x_2)$$

for all $\alpha \in \mathcal{S}$ (assuming $\mathcal{S}$ to contain sufficiently many elements). If such an x_3

Received by the editors February 20, 1969.

(1) Supported in part by NSF GP-7683 (U.S.A.) and NAVF (Norway).

exists, we define it to be the sum of x_1 and x_2, and write $x_3 = x_1 + x_2$. It follows that the expectation-functional ρ_α is automatically additive whenever the sum of two observables exists (which, of course, need not exist in general).

The final and crucial assumption made by Mackey, is that $\mathscr{L}$ shall be taken to be (isomorphic to) the lattice of projections $\mathscr{P}$ on a separable Hilbert space H. The observables are then the projection-valued measures $x: E \to x(E)$; $x(E) \in \mathscr{P}$. By the spectral-theorem we obtain a one-to-one correspondence between the observables and the (not necessarily bounded) selfadjoint operators on H. The states correspond to probability measures on $\mathscr{P}$.

A remarkable theorem of Gleason [9] states that the probability measures on $\mathscr{P}$ are exactly the $\mathscr{P}$-restrictions of the set of positive, normal linear functionals of norm 1 on $\mathscr{L}(H)$ (the set of all bounded linear operators on H).

This result makes several things fall into place. The expectation-functional ρ_α on $\mathscr{O}$ carries over to the selfadjoint operators on H, and if A_x is the selfadjoint operator corresponding to the observable x, we have

$$\rho_\alpha(A_x) = \int_{-\infty}^{+\infty} \lambda \, d\alpha_x(\lambda).$$

By Gleason's theorem it now follows that ρ_α is linear on $\mathscr{L}(H)_h$ (the bounded selfadjoint linear operators on H), and that the sum of two observables always exists if the observables correspond to bounded operators.

Several reasonable modifications of Mackey's model suggest themselves:

(1) To take $\mathscr{L}$ to be the lattice of projections in any factor—or more generally—in any von Neumann algebra $\mathfrak{A}$.

(2) To assume only *finite* additivity of the states α on $\mathscr{L}$.

In both cases, however, the development is severely hampered by a missing analogue of Gleason's theorem. It remains true that the observables correspond to selfadjoint operators affiliated with $\mathfrak{A}$, but we are no longer able to say that this correspondence is additive (when the sum of two observables x and y exists), or that ρ_α is additive on the selfadjoint portion of $\mathfrak{A}$.

In the more general and abstract approach taken by Segal, Kadison and others, the bounded observables are identified with the selfadjoint elements of a C^*-algebra A, with the Jordan-algebraic structure accepted [11], [19]. Primarily, a physically meaningful state is considered as an assignment of a probability measure α_x to the spectrum $\sigma(x)$ of each observable $x \in A_h$. This gives rise to a function

$$\rho(x) = \int_{\sigma(x)} \lambda \, d\alpha_x(\lambda)$$

on A_h, with the obvious consistency requirement imposed, that if f is a continuous real-valued function on $\sigma(x)$, then

$$\rho(f(x)) = \int_{\sigma(x)} f(\lambda) \, d\alpha_x(\lambda).$$

Hence, mathematically speaking, a physical state is a real-valued function ρ on A_h, whose restriction to each singly generated C^*-subalgebra is linear. It is far from evident that ρ must be linear on A_h, so in the absence of a theorem to this effect, the linearity of ρ on all of A_h is taken as an extra assumption.

The present paper is devoted to the study of linearity of a physical state ρ (as described above) for an arbitrary C^*-algebra A. If A is abelian, there is a completely affirmative answer, a result which was proved in the author's paper [1]. If $A = \mathcal{L}\mathcal{C}(H) =$ the compact operators on a Hilbert-space H, physical states again turn out to be linear if dim $H \geqq 3$, and the same is true for any C^*-subalgebra of $\mathcal{L}\mathcal{C}(H)$. In the last part of the paper we show that, under the extra assumption that ρ is continuous on A, ρ is linear for quite large classes of C^*-algebras.

Needless to say, there is a close connection between this linearity problem and the missing analogue of Gleason's result for general von Neumann algebras. This connection will be made explicit in §3, and we shall see that Gleason's theorem provides an affirmative answer to the linearity problem mentioned above in an important special case, if A is a factor of type I on a Hilbert space of dimension $\geqq 3$. The situation where dim $H = 2$ is exceptional, and is discussed separately.

We are indebted to R. V. Kadison for calling our attention to these questions, for his helpfulness through several discussions on the subject, and for his steady encouragement. We also wish to express our gratitude to J. M. G. Fell, E. G. Effros and C. Akemann for valuable conversations.

We would like to point out that the problem of extending a measure on the projections of a von Neumann algebra $\mathfrak{A}$ to a normal state on $\mathfrak{A}$, was first recognized by Mackey [16]. Varadarajan [21] has a very interesting and illuminating discussion of this problem from the point of view of a noncommutative probability theory. In the case where the measure is the dimension-function on the projections of a type II_1 factor, the problem of extension is precisely the problem of the additivity of the trace [12], [17].

Throughout this paper concepts and results from the theory of C^*-algebras will be used quite freely. Our general reference is the book by Dixmier [6].

2. **Preliminaries.** Let A be a C^*-algebra. By a singly generated C^*-subalgebra of A we mean a norm-closed $*$-subalgebra $A(x)$ generated by a single selfadjoint element $x \in A$ (and the identity 1 if A has identity).

Definition. A positive *quasi-linear functional* is a function $\rho: A \to \mathbf{C}$ such that

(i) $\rho | A(x)$ is a positive linear functional for each $x \in A_h$.

(ii) $\rho(a) = \rho(a_1) + i\rho(a_2)$, when $a = a_1 + ia_2$ is the canonical decomposition of a in selfadjoint parts a_1, a_2.

If in addition

(iii) $\sup\{\rho(a): a \in A, \|a\| \leqq 1, a \geqq 0\} = 1$, then we say that ρ is a *quasi-state* on A.

Consequently, the restriction of a quasi-state ρ to A_h is a physical state. The reason for condition (ii) above is to avoid trivial complications. Indeed, as shown in simple examples in [1], there are functions $\rho: A \to C$ which satisfy (i) and (iii), and whose restriction to A_h is linear, but still fails to satisfy (ii). As our interest centers around the linearity of physical states, condition (ii) imposes no restriction on the problem.

Observe that if A has an identity, then (iii) is equivalent to the condition $\rho(1)=1$. We may also note that if two positive quasi-linear functionals ρ and γ coincide on each singly generated C^*-subalgebra of A, then $\rho=\gamma$ by (ii). Clearly (i) implies that ρ is real on selfadjoint elements; so by (ii) it follows that $\rho(a^*)=\bar{\rho}(a)$ for all $a \in A$. Let us use the notation

$$\|\rho\| = \sup\{\rho(a) : a \in A,\ \|a\| \leqq 1,\ a \geqq 0\}.$$

It is easily seen (by [6, 2.1.5vi]) that if A does not have an identity, then a positive quasi-linear functional ρ may be extended to a positive quasi-linear functional $\tilde{\rho}$ on $\tilde{A}$ (=the C^*-algebra obtained by adjoining an identity to A), by defining $\tilde{\rho}(1)=K$, if $K \geqq \|\rho\|$. In particular a quasi-state on A extends to a quasi-state on $\tilde{A}$.

In [1] the following theorem was proved.

THEOREM 1. *Any positive quasi-linear functional ρ on an abelian C^*-algebra A is linear.*

The following results are all preparatory for §§5 and 6. Let A be a C^*-algebra. Q=the set of all positive quasi-linear functionals ρ such that $\|\rho\| \leqq 1$. We give Q the topology of pointwise convergence on A.

PROPOSITION 1. *Q is a compact convex set.*

Proof. Convexity is obvious. For each $x \in A$, let $I_x=\{r \in R : |r| \leqq 2\|x\|\}$, and let $I=X\{I_x : x \in A\}$. By the Tychonov theorem, I is compact. We consider the map $\phi: Q \to I$ given by $(\phi\rho)_x=\rho(x)$: $\rho \in Q$, $x \in A$. We must verify that $\phi\rho \in I$. Note that if $x \in A_h$, then $|\rho(x)| \leqq \|x\|$, since $\rho|A(x)$ is a positive linear functional of norm $\leqq 1$. In general $x=x_1+ix_2$; with $x_1, x_2 \in A_h$ and $\|x_1\|, \|x_2\| \leqq \|x\|$. Hence $|\rho(x)| \leqq |\rho(x_1)|+|\rho(x_2)| \leqq \|x_1\|+\|x_2\| \leqq 2\|x\|$, so $(\phi\rho)_x \in I_x$ and $\phi\rho \in I$. The map ϕ is clearly one-to-one, and by the definition of the topology on Q, a homeomorphism of Q into I. It remains to verify that $\phi(Q)$ is a closed subset of I. Let $\alpha \in I$ be in the closure of $\phi(Q)$, and let $\{\rho_v\} \subseteq Q$ be a net such that $\phi\rho_v \to \alpha$. Take an arbitrary element $a \in A_h$, and let $x, y \in A(a)$. Then

$$\begin{aligned}\alpha_{x+y} &= \lim_v (\phi\rho_v)_{x+y} = \lim_v \rho_v(x+y) = \lim_v \rho_v(x)+\lim_v \rho_v(y) \\ &= \lim_v (\phi\rho_v)_x+\lim_v (\phi\rho_v)_y = \alpha_x+\alpha_y.\end{aligned}$$

The other properties α must have, to belong to $\phi(Q)$, are equally simple to verify, and we conclude that $\phi(Q)$ is closed in I, so Q is compact. The proof is complete.

Note. The set S of positive linear functionals on A of norm less than or equal to one is contained in Q as a closed convex subset. The relative topology for S coincides with the w^*-topology for S as a subset of the norm-dual A^* of A.

Let us say that a quasi-state ρ is *pure* if each element $\gamma \in Q$ such that $\gamma \leqq \rho$ (i.e., $\rho-\gamma \in Q$) is of the form $\gamma=\lambda\rho$, with $0 \leqq \lambda \leqq 1$; $\lambda \in R$.

PROPOSITION 2. *A quasi-state ρ is pure if and only if ρ is an extreme point of Q different from* 0. 0 *is an extreme point of Q.*

Proof. Let ρ be a pure quasi-state, and suppose $\rho=\lambda\rho_1+(1-\lambda)\rho_2$ with $0<\lambda<1$, $\rho_1, \rho_2 \in Q$. We claim that $\|\rho_1\|=\|\rho_2\|=1$. Suppose not, so for example $\|\rho_1\|=\alpha<1$. For any $x \in A^+$; $\|x\| \leqq 1$ we get

$$\rho(x) = \lambda\rho_1(x)+(1-\lambda)\rho_2(x) \leqq \lambda\alpha+(1-\lambda) = 1-\lambda(1-\alpha).$$

Since $\|\rho\|=1$, this is a contradiction, and the claim is proved. Now $\rho \geqq \lambda\rho_1$; so by assumption, $\lambda\rho_1=\mu\rho$; $0 \leqq \mu \leqq 1$. Now $\lambda=\|\lambda\rho_1\|=\|\mu\rho\|=\mu$, so $\rho_1=\rho$, and hence $\rho_2=\rho$. This shows that ρ is an extreme point of Q.

Conversely, let $\rho \neq 0$ be an extreme point of Q. Clearly $\|\rho\|=1$. Suppose $\rho>\rho_1>0$; $\rho_1 \in Q$, and put $\rho_2=\rho-\rho_1$; $\rho_2 \in Q$. Let $\lambda=\|\rho_1\|$. By an argument similar to that given above, $\|\rho_2\|=1-\lambda$. Let $\gamma_1=\rho_1/\lambda$ and $\gamma_2=\rho_2/(1-\lambda)$, so $\rho=\lambda\gamma_1+(1-\lambda)\gamma_2$ with $\gamma_1, \gamma_2 \in Q$. Since ρ is extreme, we obtain $\rho=\gamma_1=\gamma_2$, and hence $\rho_1=\lambda\rho$, which shows that ρ is a pure quasi-state.

Finally, suppose that 0 is not an extreme point of Q. Then there is $\rho \neq 0$ in Q, such that $-\rho$ also belongs to Q. Hence $\rho(x)=0$ for all $x \geqq 0$ in A. If $x \in A_h$, then $x=x^+-x^-$, with $x^+, x^- \in A(x)^+$, so by linearity of ρ on $A(x)$ we get $\rho(x)=0$. But then $\rho=0$ by (ii) in the definition of quasi-states. This is a contradiction, so 0 is an extreme point of Q. The proof is complete.

Note. The proofs of Propositions 1 and 2 are, with small modifications, the standard arguments used for states. For a model of Proposition 2, in the case of states, see, for instance, [6, 2.5.5].

The next result is supposedly well known. The proof is included for the sake of completeness.

If ρ is a positive linear functional on a C^*-algebra A, the function

$$n_\rho\colon x \to \rho(x^*x)^{1/2}; \qquad x \in A,$$

is a seminorm on A. We have

(1) $|\rho(x)|^2 \leqq \|\rho\| \cdot n_\rho(x)^2$; $x \in A$

[6, 2.1.5], so ρ is always continuous with respect to n_ρ. Also

(2) $n_\rho(xy) \leqq \|x\| \cdot n_\rho(y)$; $x, y \in A$.

In fact,

$$n_\rho(xy) = \rho((xy)^*(xy))^{1/2} = \rho(y^*x^*xy)^{1/2} \leqq \|x^*x\|^{1/2} \cdot \rho(y^*y)^{1/2} = \|x\| \cdot n_\rho(y)$$

[6, 2.1.5].

LEMMA 1. *Let A be a C^*-algebra, J a closed two-sided ideal in A. Suppose that $\rho \in A^*$; $\rho \geqq 0$ and $\|\rho | J\|=\|\rho\|$. Then J is n_ρ-dense in A.*

Proof. Let $\{u_i\}_{i\in I}$ be an approximate identity for J. If A does not have an identity, adjoin one, and define $\rho(1)=\|\rho\|$. It then follows that $n_\rho(u_i-1)\to 0$; $i\in I$ [6, 2.1.5] since $\|\rho|J\|=\|\rho\|$. Let x be an arbitrary element in A. Then $xu_i\in J$ for all $i\in I$, and

$$n_\rho(x-xu_i) = n_\rho(x(1-u_i)) \leqq \|x\|\cdot n_\rho(1-u_i)$$

by (2). Hence $xu_i\to x$ with respect to the seminorm n_ρ, and the proof is complete.

Proposition 3. *Let A be a C^*-algebra, J a closed two-sided ideal in A. If ρ is a positive linear functional on J, then ρ has a unique extension ρ' to A such that ρ' is a positive linear functional on A satisfying $\|\rho'\|=\|\rho\|$.*

Proof. Let $\{u_i\}_{i\in I}$ be an increasing approximate identity for J. We claim that for any $x\in A$, the net $\{\rho(u_ixu_i)\}_{i\in I}$ converges. Let $i,j\in I$:

$$\begin{aligned}|\rho(u_ixu_i)-\rho(u_jxu_j)| &\leqq |\rho(u_ixu_i-u_ixu_j)|+|\rho(u_ixu_j-u_jxu_j)|\\ &= |\rho(u_ix(u_i-u_j))|+|\rho((u_i-u_j)xu_j)|\\ &\leqq n_\rho(x^*u_i)n_\rho(u_i-u_j)+n_\rho(xu_j)n_\rho(u_i-u_j)\\ &\leqq (\|\rho\|^{1/2}\cdot\|x^*u_i\|+\|\rho\|^{1/2}\cdot\|xu_j\|)n_\rho(u_i-u_j)\\ &\leqq 2\cdot\|\rho\|^{1/2}\cdot\|x\|\cdot n_\rho(u_i-u_j).\end{aligned}$$

By [6, 2.1.5], $n_\rho(u_i-u_j)\to 0$ as i,j become large in I, and the claim follows. Now define $\rho'(x)=\lim_{i\in I}\rho(u_ixu_i)$, $x\in A$. We observe that $\rho'|J=\rho$. Indeed, ρ is continuous on J [6, 2.1.8] and

$$\begin{aligned}\|x-u_ixu_i\| &\leqq \|x-u_ix\|+\|u_ix-u_ixu_i\|\\ &\leqq \|x-u_ix\|+\|x-xu_i\|\to 0 \quad \text{if } x\in J.\end{aligned}$$

ρ' is clearly linear on A, and if $x\in A$; $x\geqq 0$, then $u_ixu_i\geqq 0$ for all $i\in I$, so ρ' is positive. For any $x\in A$:

$$|\rho(u_ixu_i)| \leqq \|x\|\rho(u_i^2) \leqq \|\rho\|\cdot\|x\|.$$

Hence $|\rho'(x)|\leqq\|\rho\|\cdot\|x\|$, so $\|\rho'\|=\|\rho\|$. The uniqueness of ρ' follows from Lemma 1 and formula (1). The proof is complete.

3. Quasi-states and probability measures on projections.

An immediate consequence of Theorem 1 is that if ρ is a quasi-state on a C^*-algebra A, then ρ is linear on each maximal abelian C^*-subalgebra of A.

Let A be a von Neumann algebra and let $\mathscr{P}$ denote the lattice of projections in A. A function $\mu:\mathscr{P}\to \boldsymbol{R}^+$ such that $\mu(0)=0$ is called a *finitely additive measure* on $\mathscr{P}$ if

$$\mu\left(\sum_{i=1}^n e_i\right) = \sum_{i=1}^n \mu(e_i)$$

for any finite family $\{e_i\}_{i=1,\dots,n}$ of mutually orthogonal projections $e_i\in\mathscr{P}$. μ is a *completely additive measure* if

$$\mu\left(\sum_{i\in I} e_i\right) = \sum_{i\in I} \mu(e_i)$$

for any family $\{e_i\}_{i\in I}$ of mutually orthogonal projections. μ is a *probability measure* if $\mu(1)=1$.

PROPOSITION 1. *Let A be a von Neumann algebra. There is a natural one-to-one correspondence between the quasi-states ρ on A and the finitely additive probability measures μ on $\mathscr{P}$. This correspondence is given by*

$$\mu(a) = \int_{\sigma(a)} \lambda \, d\mu(e_\lambda); \qquad \mu = \rho|\mathscr{P},$$

where $\{e_\lambda\}$ is the spectral resolution of the selfadjoint element $a \in A$.

Proof. Let ρ be a quasi-state on A, and let $\mu=\rho|\mathscr{P}$. If $\{e_i\}_{i=1,\dots,n}$ is a finite family of mutually orthogonal projections, then they commute with each other, and are consequently contained in an abelian C^*-subalgebra of A. Hence μ is additive. Let $a \in A_h$, and let $\{e_\lambda\}$ be the spectral resolution of a. Let B be an abelian C^*-subalgebra of A containing a and $\{e_\lambda\}$ (and hence also 1). $\rho|B$ is a state, and is therefore norm-continuous. The Riemann-Stieltjes integral $a=\int_{\sigma(a)} \lambda \, de_\lambda$ exists in the norm topology of B, so

$$\rho(a) = \int_{\sigma(a)} \lambda \, d\rho(e_\lambda) = \int_{\sigma(a)} \lambda \, d\mu(e_\lambda),$$

where the integrals now are ordinary Riemann-Stieltjes integrals. This shows that the map $\rho \to \rho|\mathscr{P}$ is injective.

Now let μ be a finitely additive probability measure on $\mathscr{P}$. Let $a \in A_h$ have spectral resolution $\{e_\lambda\}$. The function $\lambda \to \mu(e_\lambda)$ is monotone increasing, and $0 \leqq \mu(e_\lambda) \leqq \mu(1)=1$. It follows that the Riemann-Stieltjes integral $\int_{\sigma(a)} \lambda \, d\mu(e_\lambda)$ exists, and we put $\rho(a)$ equal to its value. We extend ρ to all of A putting $\rho(a)=\rho(a_1)+i\rho(a_2)$, if $a=a_1+ia_2$ is the canonical decomposition of a in selfadjoint parts. To show that ρ is a quasi-state on A, let $b \in A(a)_h$. There is a real continuous function f on $\sigma(a)$ such that $b=f(a)$. Let $\{e'_\gamma\}$ be the spectral resolution of b. Then we have $e'_\gamma=e_{f^{-1}(]-\infty,\gamma])}$ so

$$\rho(b) = \int_{\sigma(b)} \gamma \, d\mu(e'_\gamma) = \int_{\sigma(b)} \gamma \, d\mu(e_{f^{-1}(]-\infty,\gamma])}) = \int_{\sigma(a)} f(\lambda) \, d\mu(e_\lambda).$$

This shows that $\rho|A(a)_h$ is linear, and consequently $\rho|A(a)$ is linear. $\rho|A(a)$ is clearly positive, and $\rho(1)=1$, so ρ is a quasi-state on A. Evidently $\rho|\mathscr{P}=\mu$, so $\rho \to \rho|\mathscr{P}$ is surjective. The proof is complete.

Note. Proposition 1 remains true for any C^*-algebra A which contains the spectral-projections of each of its selfadjoint elements.

As mentioned before, the main problem under consideration is to determine when a quasi-state on A is linear on A. The result above tells us that for von Neumann algebras this problem is equivalent to the following problem: If μ is a finitely additive measure on the projections $\mathscr{P}$, does there exist a state ρ on A such that $\rho|\mathscr{P}=\mu$? Moreover, if such a state exists, then it must satisfy $\rho(a)=\int_{\sigma(a)} \lambda \, d\mu(e_\lambda)$,

$a \in A_h$, and is therefore uniquely determined on A_h—and hence on A since ρ is a linear.

Let us say that a positive quasi-linear functional ρ on a von Neumann algebra A is *normal*, if ρ is ultraweakly continuous when restricted to any maximal abelian $*$-subalgebra.

PROPOSITION 2. *Let ρ be a quasi-state on a von Neumann algebra A. Then ρ is normal if and only if its associated measure on $\mathscr{P}$ is completely additive.*

Proof. Suppose $\mu=\rho|\mathscr{P}$ is completely additive. ρ is linear on any maximal abelian $*$-subalgebra, so it follows by a well-known result [20] that ρ is normal. Conversely, let ρ be normal. Since each family $\{e_i\}_{i\in I}$ of mutually orthogonal projections is contained in a maximal abelian $*$-subalgebra, the result follows.

Note. If A is countably decomposable, a quasi-state ρ is normal if (and only if) it is ultraweakly continuous on any singly generated sub-von Neumann algebra. Indeed, let this condition be satisfied, and let $\{e_i\}_{i\in N}$ be any family of mutually orthogonal projections. Let $a=\sum_{i=1}^{\infty} 2^{-i}e_i$. This series converges in norm, and represents the spectral resolution for a. Hence $\{e_i\}_{i\in N}$ is contained in the singly generated sub-von Neumann algebra of A containing a. The claim now follows by the proposition above.

4. **Quasi-states on $\mathscr{L}(H)$ and $\mathscr{L}\mathscr{C}(H)$.** $\mathscr{L}(H)$ denotes the C^*-algebra of all bounded linear operators on a Hilbert-space H, $\mathscr{L}\mathscr{C}(H)$ is the C^*-subalgebra of all compact operators on H.

The main result of this section is that if ρ is a quasi-state on $\mathscr{L}(H)$, then a Lebesgue-decomposition is possible: $\rho=\rho_1+\rho_2$ where ρ_1 and ρ_2 are positive quasi-linear functionals on $\mathscr{L}(H)$, with ρ_1 normal and linear, and $\rho_2(\mathscr{L}\mathscr{C}(H))=0$. In particular, it follows from this that a quasi-state on $\mathscr{L}\mathscr{C}(H)$ is always linear. We obtain this by a slight extension of Gleason's result [9], which we now restate in terms of quasi-linear functionals.

THEOREM 2. *A normal positive quasi-linear functional ρ on $\mathscr{L}(H)$ is linear if* $\dim H \geqq 3$.

Proof. We may suppose $\rho(1)=1$. If ρ is normal then its associated probability measure $\mu=\rho|\mathscr{P}$ is completely additive (Proposition 2, §3). Then, by [9, Theorem 4.1], there is a normal state ϕ on $\mathscr{L}(H)$ such that $\phi|\mathscr{P}=\mu$. By Proposition 1, §3 we obtain $\rho=\phi$ and the proof is finished.

A central role in Gleason's paper is played by the so-called frame-functions:

DEFINITION. A real-valued function f on the unit sphere $S=\{\xi \in H : \|\xi\|=1\}$ of H, is called a *frame-function of weight* w if $\sum_{i\in I} f(\xi_i)=w$ for any orthonormal basis $\{\xi_i\}_{i\in I}$ in H.

Gleason goes on to prove that each nonnegative frame-function f on a separable real or complex Hilbert-space of dimension $\geqq 3$ is *regular* in the sense that there exists a positive operator a of trace-class on H, such that

$$(*) \qquad f(\xi) = (a\xi, \xi); \qquad \xi \in S.$$

A scrutiny of his arguments makes it clear that the assumption about separability is unnecessary, and is really never used in the proof. The requirement that $\dim H \geqq 3$ is more essential. For $\dim H = 1$, every frame-function is trivially regular; the irregularity occurs for $\dim H = 2$. It is not hard to see why. We can obtain a frame-function for $\boldsymbol{R}^2$ by defining it arbitrarily on the first quadrant of the unit circle. Such a function need not be continuous, a condition which evidently must be fulfilled if the relation $(*)$ is to hold. From this observation it follows that if ρ is a quasi-state on the algebra of 2 by 2 matrices (real or complex), then ρ need not be linear on the set of hermitian matrices. In what follows, we will therefore assume that $\dim H \geqq 3$.

PROPOSITION 1. *Let f be a bounded, nonnegative function on the unit sphere of H, and suppose that f is a frame-function when restricted to any finite-dimensional subspace of H. Then there is a bounded, positive operator a on H, such that $f(\xi) = (a\xi, \xi)$ for each unit vector $\xi \in H$.*

Proof. f preserves its properties by restriction to completely real subspaces of H (a real-linear subspace K of H is completely real if the inner product is real on $K \times K$). Every completely real two-dimensional subspace can be imbedded in a completely real 3-dimensional subspace, since $\dim H \geqq 3$. f is a nonnegative frame-function on each such subspace ($\approx \boldsymbol{R}^3$) and is therefore regular there—and then also, by restriction, on any two-dimensional completely real subspace of H. Lemmas 3.2 and 3.3 of [9] go through without modification, so f is a regular frame-function on every two-dimensional subspace of H. The proof of Lemma 3.4 of [9] goes through without modification, and the proof is complete.

PROPOSITION 2. *Let μ be a finitely additive measure on the orthogonal projections $\mathscr{P}$ in H. There exists a unique completely additive measure ν on $\mathscr{P}$ such that $\nu \leqq \mu$ and $\mu - \nu$ vanishes on finite-dimensional projections.*

Proof. For $\xi \in H$, let e_ξ denote the orthogonal projection on the one-dimensional subspace containing ξ. Let $g(\xi) = \mu(e_\xi)$ for each unit vector $\xi \in H$. If e is a finite-dimensional projection on H, let $\{\xi_i\}_{i=1,\dots,n}$ be an orthogonal basis for $e(H)$. Then

$$\sum_{i=1}^{n} g(\xi_i) = \sum_{i=1}^{n} \mu(e_{\xi_i}) = \mu\left(\sum_{i=1}^{n} e_{\xi_i}\right) = \mu(e),$$

so g is a frame-function on each finite-dimensional subspace of H. g is nonnegative and bounded, since $g(\xi) = \mu(e_\xi) \leqq \mu(1)$ for each unit vector $\xi \in H$. So g satisfies the conditions of the preceding proposition. Hence there is a positive operator a on H such that $g(\xi) = (a\xi, \xi)$, $\xi \in H$, $\|\xi\| = 1$. We prove that a is of trace class. Let

$\{\xi_i\}_{i\in I}$ be an orthonormal basis for H, and let Γ be the collection of finite subsets of the index set I. For $\gamma \in \Gamma$ put $a_\gamma = \sum_{i\in\gamma} e_{\xi_i}$. Then $\mu(e_\gamma)$ is a monotone, ascending net of real numbers satisfying $\mu(e_\gamma) \leqq \mu(1)$ for all $\gamma \in \Gamma$, so that $\lim_{\gamma\in\Gamma} \mu(e_\gamma) = \alpha \leqq \mu(1)$ exists. Consequently

$$\sum_{i\in I} (a\xi_i, \xi_i) = \sum_{i\in I} g(\xi_i) = \sum_{i\in I} \mu(e_{\xi_i}) = \lim_{\gamma\in\Gamma} \mu(e_\gamma) = \alpha < \infty,$$

where we have used finite additivity of μ on $\mathscr{P}$. So a is of trace class on H, and the sum $\sum_{i\in I} g(\xi_i)$ is independent of the basis $\{\xi_i\}_{i\in I}$. Hence g is a regular frame-function of weight $\alpha \leqq \mu(1)$ on H. Let e be an arbitrary element of $\mathscr{P}$, and let $\{\eta_i\}_{i\in I}$ be an orthogonal basis for $e(H)$. Define $\nu(e) = \sum_{i\in I} g(\eta_i) \leqq \mu(e)$. Then ν is easily seen to be a completely additive measure on $\mathscr{P}$, and $\mu-\nu$ vanishes on finite-dimensional projections. The uniqueness of ν is obvious and the proof is complete.

Let us say that a positive quasi-linear functional ρ on $\mathscr{L}(H)$ is *singular* if ρ vanishes on $\mathscr{L}\mathscr{C}(H)$.

LEMMA. *A quasi-state ρ is singular if and only if its associated measure vanishes on finite-dimensional projections.*

Proof. If ρ is singular, then clearly $\mu = \rho|\mathscr{P}$ vanishes on finite-dimensional projections, since these are compact. Conversely, suppose that μ vanishes on finite-dimensional projections. If $a \in \mathscr{L}\mathscr{C}(H)$ is selfadjoint, then the eigen-manifolds of a are all finite-dimensional. From the construction of the spectral resolution and the formula of Proposition 1, §3 it follows that $\rho(a)=0$, and the proof is complete.

THEOREM 3. *A quasi-state ρ on $\mathscr{L}(H)$ may be written uniquely as a sum $\rho=\rho_1+\rho_2$ with $\rho_1, \rho_2 \in Q$, ρ_1 normal and linear and ρ_2 singular.*

Proof. Let μ be the finitely additive measure on $\mathscr{P}$ associated with ρ. Apply Proposition 2 to obtain the completely additive measure ν on $\mathscr{P}$. By Gleason's theorem there is a positive, normal linear functional ρ_1 on $\mathscr{L}(H)$ such that $\rho_1|\mathscr{P}=\nu$. Since $\nu \leqq \mu$ we clearly get $\rho_1 \leqq \rho$. $\rho_2 = \rho - \rho_1$ is the quasi-linear functional associated with the measure $\mu-\nu$, so ρ_2 is singular by the lemma above. The uniqueness of ρ_1 is a consequence of the uniqueness of ν, and the one-to-one correspondence between quasi-states and measures established in Proposition 1, §3. The proof is complete.

COROLLARY 1. *If ρ is a quasi-state on $\mathscr{L}(H)$ which is an extreme point of the convex set of quasi-states Q, then either ρ is a normal state or a singular quasi-state.*

Proof. By Theorem 3 there is a positive normal linear functional ϕ on $\mathscr{L}(H)$ satisfying $0 \leqq \phi \leqq \rho$. By Proposition 2, §2, $\phi = \lambda\rho$; $0 \leqq \lambda \leqq 1$. If $\lambda=0$ then ρ is singular, if $0<\lambda\leqq 1$ then ρ is a normal state.

REMARK. Combined with Proposition 1, §2, and with the aid of the Krein-Milman theorem [15], this last result reduces the question of whether a quasi-state on $\mathscr{L}(H)$ is linear, to the same question for the pure, singular quasi-states.

COROLLARY 2. *If ρ is a quasi-state on $\mathscr{L}\mathscr{C}(H)$, then ρ is linear.*

Proof. We consider the C^*-algebra A generated by $\mathscr{L}\mathscr{C}(H)$ and $\mathfrak{1}$, and put $\rho(\mathfrak{1})=1$ so ρ becomes a quasi-state on A. For each unit vector $\xi \in H$ we put $g(\xi)=\rho(e_\xi)$, which is possible since $e_\xi \in A$. By the same argument as in the proof of Proposition 2, we see that g is a nonnegative, regular frame-function on H, and therefore determines a unique positive normal linear functional ϕ on $\mathscr{L}(H)$. Clearly $\phi=\rho$ on the finite-dimensional projections, so $\rho=\phi$.

Later (Corollary 4, §6) we shall see that linearity is also automatic for quasi-states on C^*-subalgebras of $\mathscr{L}\mathscr{C}(H)$.

The next results are immediate consequences of Theorem 2, and are included here for later reference. A C^*-algebra A is called *uniformly hyperfinite* (UHF) if there is an increasing sequence of factors $M_1 \subseteq M_2 \subseteq \cdots \subseteq A$ of types $\mathrm{I}_{n_1}, \mathrm{I}_{n_2}, \ldots$ such that A is the norm-closure of $\bigcup_{i=1}^\infty M_i$. It is assumed that $n_i \to \infty$ as $i \to \infty$. A von Neumann algebra R is said to be *hyperfinite* if R is the weak closure of a uniformly hyperfinite C^*-algebra A acting on a Hilbert-space H [10].

PROPOSITION 3. *If ρ is a norm-continuous quasi-state on a* UHF-*algebra A, then ρ is linear.*

Proof. Immediate from the definition of UHF-algebras, and Theorem 2.

PROPOSITION 4. *If ρ is a weakly continuous quasi-state on a hyperfinite von Neumann algebra R, then ρ is linear.*

Proof. Evident.

We also record the following generalization of Theorem 2.

PROPOSITION 5. *Let A be a type* I *von Neumann algebra with discrete center and no type* I_2 *factor. Any normal quasi-state ρ on A is linear.*

Proof. A is the direct sum of factors $\{A_i\}_{i\in I}$, with A_i of type I_n, $n\neq 2$. Each element $x \in A_h$ may be written $x=\sum_{i\in I} x_i$; $x_i \in (A_i)_h$, in the weak operator topology. We may regard x and all the x_i as contained in an abelian sub-von Neumann algebra of A. Since ρ is normal we have $\rho(x)=\sum_{i\in I}\rho(x_i)$. Since $\rho|A_i$ is linear for all $i \in I$ by Theorem 2, it now follows that ρ is linear on A. The proof is complete.

5. **Quasi-states on the central extension of a C^*-algebra.** The rest of this paper is devoted to the question of linearity of quasi-states on more general C^*-algebras than those previously considered. On the basis of what we already know, it is natural to ask whether quasi-states on tensor products of the type $B \otimes \mathscr{L}\mathscr{C}(H)$; where B is an abelian C^*-algebra, are linear. It turns out that much is gained by looking at this problem more generally—that is to say, in terms of fields of C^*-algebras. While norm-continuity for ordinary states on a C^*-algebra is automatic [6, 2.1.8], this is far from clear for quasi-states. However, if continuity is assumed, quasi-states prove to be linear on quite large classes of C^*-algebras.

The present section is mainly preparatory for §6.

Let A be a C^*-algebra; $T = \mathrm{Prim}\, A$, its structure space with the hull-kernel topology. T is locally compact, and compact if A has an identity [6, 3.3.8]. Let $\mathscr{C}^b(T)$ be the abelian C^*-algebra of all bounded, continuous, complex-valued functions on T, and let $\mathscr{C}^0(T)$ be the closed $*$-ideal of $\mathscr{C}^b(T)$ consisting of the functions vanishing at infinity. If A has an identity, $\mathscr{C}^b(T)$ is canonically isomorphic to the center of A [3]. If A has no identity, the situation is more complicated. In fact, if $\tilde{A}$ is the C^*-algebra obtained from A by adjoining an identity, and $\tilde{T} = \mathrm{Prim}\, \tilde{A}$, then $\mathscr{C}^b(\tilde{T})$ may be reduced to scalar multiples of the function equal to one on $\tilde{T}$, while $\mathscr{C}^b(T)$ is quite large. This is due to the fact that while we may identify T with the complement of a closed point ω in $\tilde{T}$, it may happen that ω is in the closure of every point of T.

However, as shown in [7], there is an extension A' of A, in general larger than $\tilde{A}$, such that A' is a C^*-algebra with identity, and with center Z' isomorphic to $\mathscr{C}^b(T)$. Moreover, $A' = A + Z'$; $A \cap Z'$ is the center of A, and A is a closed two-sided ideal in A'. If A has an identity, we have $A' = A$. We shall call A' the *central extension* of A, and Z' the *ideal center* of A.

The main tool in Dixmier's proof of this theorem is the following recent result [3], which will be used extensively in the following: If A is a C^*-algebra, $x \in A$ and $h \in \mathscr{C}^b(T)$, then there is an element $y \in A$ such that $y \bmod t = h(t)\ (x \bmod t)$ for all $t \in T$.

THEOREM 4. *Let A be a A^*-algebra with $T\,(=\mathrm{Prim}\, A)$ Hausdorff. If ρ is a positive quasi-linear functional on A, then ρ extends to a positive quasi-linear functional ρ' on A' such that*

(i) $\|\rho'\| = \|\rho\|$;

(ii) *ρ' is continuous if and only if ρ is continuous*;

(iii) *ρ' is linear if and only if ρ is linear*;

(iv) *ρ' is pure if ρ is pure, and in this case ρ' is unique and $\rho'|Z'$ is pure.*

The proof of this theorem will be broken up in several steps. Note that if A has an identity, then the only thing to prove is that the restriction of ρ to the center of A is pure if ρ is pure.

For each $t \in T$, let

$$\begin{aligned} A(t) &= A/t && \text{if } A/t \text{ has an identity,} \\ &= (A/t)^{\sim} && \text{if } A/t \text{ does not have an identity.} \end{aligned}$$

Hence $A(t)$ becomes a C^*-algebra with identity $1(t)$ for each $t \in T$. It will be convenient for our purpose to realize A as a C^*-algebra of vector-fields on T with values in the $A(t)$; i.e. as a C^*-subalgebra of $\Gamma = \prod_{t \in T} A(t) =$ the C^*-direct product of the C^*-algebras $A(t)$ [6, 1.3.3]. For each $t \in T$, let $\phi_t \colon A \to A(t)$ be the canonical map $A \to A/t$ followed by the injection of A/t in $A(t)$. For each $x \in A$, let $x(t) = \phi_t(x)$; $t \in T$. Since $\|\phi_t(x)\| \leqq \|x\|$ for all $t \in T$, the vector-field $t \to x(t)$ belongs to Γ. In this way each element $x \in A$ defines an element of Γ, and it is easily verified that the map $\Phi: A \to \Gamma$ is a $*$-homomorphism, since each ϕ_t is a $*$-homomorphism. Φ is also injective, since $\bigcap \{t : t \in T\} = (0)$ [6, 2.7.3], so Φ is an isometry [6, 1.8.3]. We may therefore identify A with a C^*-subalgebra of Γ, and will denote by x the vector-field $t \to x(t)$; $t \in T$, for each $x \in A$. We observe that

$$\|x\| = \sup_{t \in T} \|x(t)\|; \qquad x \in A.$$

Now let $h \in \mathscr{C}^b(T)$, and consider the vector-field $t \to h(t) \cdot 1(t)$. Since $h(t)1(t) \in A(t)$ for all $t \in T$, and

$$\|h(t)1(t)\| = |h(t)| \leqq \|h\|_\infty; \qquad t \in T$$

we see that h defines an element of Γ which we also will denote by h. This is permissible, since the correspondence function $\to$ vector-field is an injective $*$-homomorphism of $\mathscr{C}^b(T)$ into Γ. We identify $\mathscr{C}^b(T)$ with its image in Γ and observe that $\|h\| = \sup_{t \in T} |h(t)|$. At this point, it is rather easy to get hold of the central extension of A. Indeed, let $A' = A + \mathscr{C}^b(T)$ in Γ. The Dauns-Hofmann result, quoted above, now tells us that if $x \in A$ and $h \in \mathscr{C}^b(T)$, then $y = hx$ belongs to A. Hence A' is a $*$-subalgebra of Γ. Let A'' be the closure of A' in Γ, so A'' is a C^*-algebra. A is a closed two-sided ideal of A''. In fact, Z is evidently a closed $*$-subalgebra of A'' since A is complete, moreover, if $x \in A''$ then there is a sequence $x_n + h_n \to x$, with $x_n \in A$; $h_n \in \mathscr{C}^b(T)$. Hence, if $y \in A$, then $xy = \lim_n (x_n y + h_n y) \in A$, and similarly $yx \in A$. $\mathscr{C}^b(T)$ is a C^*-subalgebra of A'', so $A' = A + \mathscr{C}^b(T)$ is closed in A'' [6, 1.8.4]. Hence $A'' = A'$ and A' is a C^*-algebra.

$\mathscr{C}^b(T) \cap A$ is clearly contained in the center Z of A. On the other hand, let $z \in Z$. For each $t \in T$, A/t is primitive, so the center of A/t is either (0) if A/t does not have an identity, or equal to scalar multiples of the identity of A/t. Now $\phi_t(z)$ is central in A/t for each $t \in T$, so there is a complex valued function h_z on T such that

$$z(t) = h_z(t)1(t); \qquad t \in T.$$

Dixmier's argument in the proof of Theorem 5 in [7] show that $h_z \in \mathscr{C}^b(T)$, and that the map $z \to h_z$ is a *-isomorphism. Consequently $Z = \mathscr{C}^b(T) \cap A$. It follows that A' is the central extension of A, and that $\mathscr{C}^b(T)$ is the center Z' of A'.

A certain mapping of A into the center of A' turns out to be useful. For each $x \in A$, let k_x be the function on T defined by $k_x(t) = \|x(t)\|$; $t \in T$. The function k_x vanishes at infinity [6, 3.3.7], and since T is assumed to be Hausdorff, it is continuous on T [14]. Hence $k_x \in \mathscr{C}^0(T)$ for each $x \in A$. For later reference we record some properties of the map $x \to k_x$:

(1) If $x \in A_h$, then $-k_x \leqq x \leqq k_x$.

(2) $k_{x+y} \leqq k_x + k_y$; $k_{\lambda x} = |\lambda| \cdot k_x$; $x, y \in A$, $\lambda \in \boldsymbol{C}$.

(3) $\|x\| = \|k_x\|$ so $k_x = 0 \Leftrightarrow x = 0$.

(4) If $x, y \in A$ then $\|k_x - k_y\| \leqq \|x - y\|$.

Proof of (1). For each $t \in T$ we have

$$-\mathbb{1}(t)\cdot\|x(t)\| \leqq x(t) \leqq \|x(t)\|\cdot\mathbb{1}(t); \quad \text{i.e. } -k_x(t) \leqq x(t) \leqq k_x(t).$$

Hence (1).

Proof of (4). $|k_x(t)-k_y(t)| = |\,\|x(t)\|-\|y(t)\|\,| \leqq \|x(t)-y(t)\| \leqq \|x-y\|$ for all $t \in T$. Hence (4) is true. (2) and (3) are almost evident and are left to the reader.

LEMMA 1. *Let B be an abelian C^*-algebra of A, closed under multiplication by $\mathscr{C}^b(T)$. Then $B' = B + \mathscr{C}^b(T)$ is an abelian C^*-subalgebra of A', and B is a closed ideal in B'.*

Proof. Argue as we did to show that A' $(=A+\mathscr{C}^b(T))$ is a C^*-algebra.

In what follows we assume that ρ is an arbitrarily given pure quasi-state on A.

Let us temporarily fix an element $x \in A$; $x > 0$ such that $\rho(x) > 0$, and let B be a fixed abelian C^*-subalgebra of A which is closed under multiplication by $\mathscr{C}^b(T)$, and contains the element x. Let $B' = B + \mathscr{C}^b(T)$. By Theorem 1, §2, $\rho|B$ is a nonzero positive linear functional. Then, by Lemma 1 and Proposition 3, §2, $\rho|B$ has a unique extension ρ' to B' such that $\rho'(\mathbb{1}) = \|\rho|B\|$.

Define, for $h \in \mathscr{C}^b(T)$:

$$\bar{\rho}(h) = \|\rho|B\|^{-1}\rho'(h)$$

so $\bar{\rho}$ becomes a state on $\mathscr{C}^b(T)$. We are going to show that $\bar{\rho}$ is pure, and independent of the choice of x and B.

LEMMA 2. *If T is Hausdorff, then there is a point $t_0 \in T$ such that*

$$\bar{\rho}(h) = h(t_0); \qquad h \in \mathscr{C}^b(T).$$

Proof. We first observe that $\|\bar{\rho}|\mathscr{C}^0(T)\| = 1$. In fact: let $\{x_i\}_{i\in N}$ be a sequence of positive elements in B; $\|x_i\| \leqq 1$ for all $i \in N$, such that $\rho(x_i) \to \|\rho|B\|$. Now $0 \leqq x_i \leqq k_{x_i}$; $k_{x_i} \in \mathscr{C}^0(T)$ and $\|k_{x_i}\| \leqq 1$ for all $i \in N$. Hence $0 \leqq \rho(x_i) \leqq \rho'(k_{x_i}) \leqq \rho'(\mathbb{1})$, so $\rho'(k_{x_i}) \to \|\rho|B\|$. Hence

$$\|\rho|B\| \leqq \|\rho'|\mathscr{C}^0(T)\| \leqq \rho'(\mathbb{1}) = \|\rho|B\|$$

so $\|\bar{\rho}|\mathscr{C}_0(T)\| = 1$. Let

$$J = \{h \in \mathscr{C}^0(T) : \bar{\rho}(h^*h) = 0\}.$$

J is a closed ideal in $\mathscr{C}^0(T)$, so there is a closed set $F \subseteq T$ such that $J = \{h \in \mathscr{C}^0(T) : h \equiv 0$ on $F\}$. Since $\bar{\rho}|\mathscr{C}^0(T) \neq 0$; $F \neq \varnothing$. Suppose $t_1, t_2 \in F$; $t_1 \neq t_2$. Let U_1, U_2 be disjoint neighbourhoods of t_1, t_2 respectively. Choose functions h_1, h_2 in $\mathscr{C}^0(T)$ such that $0 \leqq h_i(t) \leqq h_i(t_i) = 1$, $h_i \equiv 0$ outside U_i; $i = 1, 2$. We observe that $\bar{\rho}(h_i) > 0$; $i = 1, 2$. Indeed, if $\bar{\rho}(h_i) = 0$, then

$$0 \leqq \bar{\rho}(h_i^2) \leqq \bar{\rho}(h_i)\cdot\|h_i\| = 0$$

so $h_i \in J$—contradicting $h_i(t_i) = 1$. Now define, for $i = 1, 2$;

$$\phi_i(y) = \rho(h_i y); \qquad y \in A.$$

Since A is closed under multiplication by $\mathscr{C}^b(T)$, and $h_i y \geqq 0$ if $y \geqq 0$, ϕ_i is well defined, and becomes a positive quasi-linear functional on A. If $y \in A$ and $y \geqq 0$; then $h_i y \leqq y$ since $h_i \leqq 1$, $h_i y$ and y commute and belong to an abelian C^*-subalgebra of A; so

$$\phi_i(y) = \rho(h_i y) \leqq \rho(y),$$

i.e., $0 \leqq \phi_i \leqq \rho$; $i=1, 2$. Since ρ is pure, it follows that there are constants $0 \leqq \lambda_i \leqq 1$ such that $\phi_i = \lambda_i \rho$; $i=1, 2$ (Proposition 2, §2). Let $\{u_j\}_{j \in I}$ be an increasing approximate identity for B. Then, by Lemma 1, and Lemma 1, §2 we see that $\phi_i(u_j) = \rho(h_i u_j) \to \rho'(h_i) = \|\rho|B\| \bar{\rho}(h_i) > 0$. It follows that $\phi_i \neq 0$, i.e., $\lambda_i > 0$; $i=1, 2$. Thus, for some $j \in I$: $0 < \phi_2(u_j) = \rho(h_2 u_j) = (1/\lambda_1)\phi_1(h_2 u_j) = (1/\lambda_1)\rho(h_1 h_2 u_j) = 0$; since clearly $h_1 h_2 u_j = 0$. This is a contradiction, so $F = \{t_0\}$, and it follows that $\bar{\rho}(h) = h(t_0)$ for all $h \in \mathscr{C}^0(T)$. Since $\|\bar{\rho}|\mathscr{C}^0(T)\| = \bar{\rho}(\mathbb{1}) = 1$, $\bar{\rho}$ is uniquely determined on $\mathscr{C}^b(T)$ by its values on $\mathscr{C}^0(T)$ (Proposition 3, §2). Thus $\bar{\rho}(h) = h(t_0)$ for all $h \in \mathscr{C}^b(T)$. The proof is complete.

Before we can show that $\bar{\rho}$ is independent of B, we need another lemma.

Lemma 3. *Let B be as above.*

(i) *If $y \in B$ and $y(t_0) = 0$, then $\rho(y) = 0$.*

(ii) *If $y \in B$, $h \in \mathscr{C}^b(T)$ and $h(t_0) = 1$ then $\rho(y) = \rho(hy)$.*

Proof. (i) $\rho|B$ is a positive linear functional, so $0 \leqq |\rho(y)|^2 \leqq \|\rho|B\| \cdot \rho(y^*y)$ by 2.1.5(i) [6]. Let $z = y^*y \geqq 0$. Then $z(t_0) = 0$ and

$$0 \leqq z \leqq k_z \Rightarrow 0 \leqq \rho(z) = \rho'(z) \leqq \rho'(k_z) = \|\rho|B\| k_z(t_0) = 0.$$

Hence $\rho(y) = 0$.

(ii) If $y \in B$ and $h \in \mathscr{C}^b(T)$; then $hy \in B$, so $\rho(y) - \rho(hy) = \rho((\mathbb{1} - h)y) = 0$, since $(\mathbb{1} - h)y(t_0) = 0$.

The proof is complete.

Now let $x_1 \in A$ be any positive element such that $\rho(x_1) > 0$, and let B_1 be an abelian C^*-subalgebra of A such that $x_1 \in B$ and $hB_1 \subseteq B_1$; $\forall\, h \in \mathscr{C}^b(T)$. Let $B_1' = B_1 + \mathscr{C}^b(T)$, and extend $\rho|B_1$ uniquely to a positive linear form ρ_1' on B_1', such that $\rho_1'(\mathbb{1}) = \|\rho|B_1\|$. Let $\bar{\rho}_1 = \|\rho|B_1\|^{-1} \cdot \rho_1'|\mathscr{C}^b(T)$.

Lemma 4. $\bar{\rho}_1 = \bar{\rho}$.

Proof. By Lemma 2, there is a point $t_1 \in T$ such that $\bar{\rho}_1(h) = h(t_1)$ for all $h \in \mathscr{C}^b(T)$. Suppose $t_0 \neq t_1$. Choose functions h_0, h_1 in $\mathscr{C}^b(T)$ with support in disjoint neighbourhoods of t_0, t_1 respectively, such that $0 \leqq h_i(t) \leqq h_i(t_i) = 1$; $i=0, 1$, $t \in T$. Then $h_0 x$ and $h_1 x_1$ are positive with product 0, so they are contained in a common abelian C^*-subalgebra B_2 of A, closed under multiplication of $\mathscr{C}^b(T)$. Extending ρ from B_2, we obtain ρ_2' on $B_2' = B_2 + \mathscr{C}_b(T)$ and another point $t_2 \in T$.

Case 1. t_2 is different from t_1 and t_0. Choose disjoint neighbourhoods U_i, $i=0, 1, 2$ around t_0, t_1 and t_2, and functions $k_i \in \mathscr{C}^b(T)$, $0 \leqq k_i(t) \leqq k_i(t_i) = 1$;

$i=0$, 1, 2, with the support of k_i contained in U_i. Put $z=k_0h_0x+k_1h_1x_1 \in B_2$. Lemma 3 applies to B_2, so

$$\rho(z) = \rho(k_2z) = \rho(k_2k_0h_0x + k_2k_1h_1x_1) = 0.$$

On the other hand, since $\rho|B_2$ is linear, and Lemma 3 applies to B and B_1,

$$\rho(z) = \rho(k_0h_0x) + \rho(k_1h_1x_1) = \rho(x) + \rho(x_1) > 0,$$

a contradiction.

Case 2. t_2 is equal to t_0 or t_1, say $t_2=t_0$. Let $z=h_0x+h_1x_1$. Then $\rho(z)=\rho(h_0z) =\rho(h_0^2x+h_0h_1x_1)=\rho(h_0^2x)=\rho(x)$. On the other hand

$$\rho(z) = \rho(h_0x)+\rho(h_1x) = \rho(x)+\rho(x_1) > \rho(x),$$

a contradiction. The proof is complete.

So, the point $t_0 \in T$ and the pure state $\bar{\rho}$ on $\mathscr{C}^b(T)$ are uniquely determined by ρ. To proceed, we first need:

LEMMA 5. $\rho|Z=\bar{\rho}|Z$.

Proof. Z, the center of A, is a closed ideal in $\mathscr{C}^0(T)$, so there is a closed set $E\subseteq T$ such that

$$Z = \{h \in \mathscr{C}^0(T) : h \equiv 0 \text{ on } E\}.$$

There are two possibilities:

(1) $t_0 \in E$. In this case, $\bar{\rho}(h)=h(t_0)=0$ if $h \in Z$. Let B be any abelian C^*-subalgebra of A, $Z\subseteq B$, and B closed under multiplication by $\mathscr{C}^b(T)$, such that $\|\rho|B\| \neq 0$. Let ρ' be the extension of ρ to $B'=B+\mathscr{C}^b(T)$. If $y=h \in Z$, then $y(t_0)=0$. Hence $\rho(y)=0$ by Lemma 3. Hence, in this case $\rho|Z=\bar{\rho}|Z=0$.

(2) $t_0 \notin E$. There is a function $h \in \mathscr{C}^0(T)$ such that $0\leqq h(t)\leqq h(t_0)=1$; $h\equiv 0$ on E. Hence $h \in Z$. We claim that $\rho(h)=1$. Suppose that $\rho(h)=\alpha<1$. Since ρ is a quasi-state of A, we can choose $x \in A$; $x>0$ and $\|x\|\leqq 1$, such that $\rho(x)>\alpha$.

Let B be an abelian C^*-subalgebra of A containing x and Z, closed under multiplication by $\mathscr{C}^b(T)$. By Lemma 3,

$$\rho(x) = \rho(xh), \quad \text{so } \rho(x)^2 = \rho(xh)^2 \leqq \rho(x^2)\cdot\rho(h^2) \leqq \rho(x)\rho(h).$$

Hence $\rho(x)\leqq \rho(h)=\alpha$—a contradiction. Since $\rho(h)=1$, $h \in Z$, it follows that $\|\rho|B\| =1$; $\rho'|\mathscr{C}^b(T)=\bar{\rho}$. Hence $\rho=\bar{\rho}$ on Z; and the proof is complete.

We are now in a position to extend ρ to all of A'. Suppose first that $x' \in A'_h$, and that $x'=x+h=y+h$, $x, y \in A_h$, $h, k \in \mathscr{C}^b(T)$.

LEMMA 6. $\rho(x)+\bar{\rho}(h)=\rho(y)+\bar{\rho}(k)$.

Proof. Since $x+h=y+k$, we have $x-y=k-h \in Z$. Thus x and y commute and belong to an abelian C^*-subalgebra of A on which ρ is linear. So, by Lemma 5

$$\rho(x)-\rho(y) = \rho(x-y) = \bar{\rho}(k-h) = \bar{\rho}(k)-\bar{\rho}(k),$$

and the lemma follows.

Definition. If $x' = x + h \in A'_h$, let $\rho'(x') = \rho(x) + \bar{\rho}(h)$. For arbitrary x' in A', let $\rho'(x') = \rho'(x'_1) + i\rho'(x'_2)$, where $x' = x'_1 + ix'_2$ is the decomposition of x' into selfadjoint parts.

Lemma 7. *ρ' is a quasi-state on A' and $\rho'|A = \rho$.*

Proof. That $\rho'|A = \rho$ is clear. Let $x' = x + h$ be a selfadjoint element of A'; $x \in A_h$; $h \in \mathscr{C}^b(T)$; and let B_1 be the C^*-subalgebra of A' generated by x' and $\mathfrak{1}$. Let B be the C^*-subalgebra of A generated by $\{xk; k \in \mathscr{C}^b(T)\}$, and put $B' = B + \mathscr{C}^b(T)$. B' is abelian, and clearly $B_1 \subseteq B'$. Consequently, to see that ρ' is a quasi-state on A', it is sufficient to show that $\rho'|B'$ is a state. $\rho'|B'$ is linear. Indeed, let $y', z' \in B'$; $\lambda \in \mathbf{C}$. We have $y' = y + h$, $z' = z + k$; $y, z \in B$; $h, k \in \mathscr{C}^b(T)$. Note that $\rho|B$ is linear by Theorem 1, §2. So

$$\begin{aligned} \rho'(\lambda y' + z') &= \rho'(\lambda y + \lambda h + z + k) = \rho'(\lambda y + z + \lambda h + k) = \rho(\lambda y + z) + \bar{\rho}(\lambda h + k) \\ &= \lambda\rho(y) + \rho(z) + \lambda\bar{\rho}(h) + \bar{\rho}(k) = \lambda\rho'(y') + \rho'(z'). \end{aligned}$$

We have $\rho'(\mathfrak{1}) = \bar{\rho}(\mathfrak{1}) = 1$, so it remains to verify that $\rho'|B'$ is positive. Let $y' = y + h \geqq 0$ in B', $y \in B$, $h \in \mathscr{C}^b(T)$.

Case 1. $h(t_0) \geqq 0$. If $\rho(y) = 0$, then $\rho'(y') = h(t_0) \geqq 0$, and we are finished. If $\rho(y) \neq 0$, then $\|\rho|B\| \neq 0$ and $\rho|B$ extends uniquely to a positive linear functional ρ_1 on B' such that $\rho_1(\mathfrak{1}) = \|\rho|B\|$. Hence

$$\begin{aligned} \rho'(y') &= \rho(y) + \bar{\rho}(h) = \rho(y) + h(t_0) \geqq \rho(y) + \|\rho|B\| h(t_0) \\ &= \rho(y) + \rho_1(h) = \rho_1(y') \geqq 0, \end{aligned}$$

which shows that $\rho'(y')$ is positive in this case.

Case 2. $h(t_0) < 0$. Then there is a neighbourhood U of t_0 such that $h(t) < -\alpha$ for all $t \in U$, for some $\alpha > 0$. Hence $y(t) \geqq -h(t) > \alpha \cdot \mathfrak{1}(t)$ for all $t \in U$. Let $y_1 = \max(\alpha \cdot \mathfrak{1}, y)$ in B' which is abelian. We claim that $y_1(t) = y(t)$ if $t \in U$. Fix an arbitrary $t \in U$, and let ϕ be the homomorphism $x \to x(t)$ of B' into $A(t)$; $x \in B'$. We may identify B' with $\mathscr{C}(X)$; where X is a compact Hausdorff space. Let C be the image of B' under ϕ in $A(t)$, so C is an abelian C^*-algebra with identity $\mathfrak{1}(t)$. If γ is a pure state on C, then $\gamma \cdot \phi$ is a positive, multiplicative linear functional on B', and $(\gamma \cdot \rho)(\mathfrak{1}) = 1$, so $\gamma \cdot \phi$ is a pure state on $B' = \mathscr{C}(X)$. Hence there is a unique point $s \in X$ such that

$$x(s) = (\gamma \cdot \phi)(x); \qquad x \in B'.$$

Now $t \in U$, so $y(t) > \alpha \cdot \mathfrak{1}(t)$. Thus, $(\gamma \cdot \phi)(y) \geqq (\gamma \cdot \phi)(\alpha \cdot \mathfrak{1})$, so $y(s) \geqq \alpha \cdot \mathfrak{1}(s) = \alpha$. It follows that $y_1(s) = y(s)$ for all points $s \in X$ obtained from pure states γ on C. But then $\gamma(y_1(t)) = \gamma(y(t))$ for all pure states γ on C, so $y_1(t) = y(t)$. The claim is proved.

Now y_1 has an inverse $u \in B'$, and we define $x = uy$. Since B is an ideal in B' and $y \in B$, x belongs to B. Moreover, if $t \in U$:

$$x(t) = u(t)y(t) = u(t)y_1(t) = \mathfrak{1}(t),$$

since y and y_1 coincide on U. Let $k \in \mathscr{C}^b(T)$ be chosen such that $0 \leqq k(t) \leqq k(t_0) = 1$ for $t \in U$, $k \equiv 0$ outside U. Then $k \cdot h$ has support in U, $kh = khx \in B$. Hence

$$ky' = ky + kh \in B, \quad \text{and} \quad ky' \geqq 0.$$

Finally

$$\rho(ky') = \rho(ky) + \bar{\rho}(kh) = \rho(y) + h(t_0) = \rho'(y') \quad \text{by Lemma 3.}$$

Since $\rho|B$ is positive, $\rho'(y')$ is positive in this case also. The proof is complete.

Lemma 8. *Let γ' be a quasi-state on A' and suppose $\gamma'|A = \rho$. Then $\gamma' = \rho'$.*

Proof. Put $\sigma = \gamma' - \rho'$. Then $\sigma|A = 0$. We claim that $\sigma|\mathscr{C}^b(T)$ is $\geqq 0$. Let $h \geqq 0$; $h \in \mathscr{C}^b(T)$. Take an abelian C^*-subalgebra B_1 of A, closed under multiplication by $\mathscr{C}^b(T)$, and assume $\rho_1 = \rho|B_1 \neq 0$. Let $B_1' = B_1 + \mathscr{C}^b(T)$. ρ_1 extends uniquely to a positive linear functional ρ_1' on B_1' such that $\rho_1'(1) = \|\rho_1\|$. Moreover, if $\{u_i\} \subseteq B$ is a positive, increasing approximate unit for B, then $\rho_1'(h) = \lim_i \rho(hu_i)$. Clearly $hu_i \leqq h$ for all $i \in I$ so $\gamma'(h) \geqq \gamma'(hu_i) = \rho(hu_i)$ for all $i \in I$. Hence $\gamma'(h) \geqq \rho_1'(h) = \|\rho_1\| \cdot h(t_0) = \|\rho_1\| \cdot \rho'(h)$. Now B_1 may be chosen so that $\|\rho_1\|$ is arbitrarily close to 1; hence $\sigma(h) = (\gamma' - \rho')(h) \geqq 0$. But $\sigma(1) = \gamma'(1) - \rho'(1) = 0$. Since σ is a positive linear functional on $\mathscr{C}^b(T)$, $\sigma = 0$, i.e., $\gamma' = \rho'$. The proof is complete.

Lemma 9. *ρ' is a pure quasi-state.*

Proof. Let $0 \leqq \gamma' \leqq \rho'$; where γ' is a positive quasi-linear functional on A'. Let $\gamma = \gamma'|A$, so $0 \leqq \gamma \leqq \rho$. Since ρ is pure we get $\gamma = \lambda\rho$; $0 \leqq \lambda \leqq 1$. Put $\gamma'' = \lambda\rho'$, so γ'' becomes a positive quasi-linear functional on A', extending γ. As in the proof of Lemma 8, we obtain $\gamma' \geqq \gamma''$. Put $\sigma = \gamma' - \gamma''$. Then $0 \leqq \sigma \leqq \rho'$, and $\rho'|\mathscr{C}^b(T)$ is a pure state, so $\sigma|\mathscr{C}^b(T) = \mu(\rho'|\mathscr{C}^b(T))$; $0 \leqq \mu \leqq 1$. Pick a sequence $\{x_i\}_{i \in N} \subseteq A^+$, $\|x_i\| \leqq 1$, such that $\rho(x_i) \to 1$. Now $0 \leqq \rho(x_i) \leqq \rho'(k_{x_i}) \leqq 1$, so $\rho'(k_{x_i}) \to 1$. Let $y_i = -x_i + k_{x_i}$; so $y_i \geqq 0$ for all $i \in N$. $\sigma|A = 0$, so $\sigma(y_i) = \sigma(k_{x_i}) = \mu \cdot \rho'(k_{x_i})$. Hence $\sigma(y_i) \to \mu$. On the other hand $0 \leqq \sigma(y_i) \leqq \rho'(y_i) \to 0$, so $\mu = 0$. Consequently, $\sigma|\mathscr{C}^b(T) = 0$; and, since $\sigma|A = 0$, it follows that $\sigma = 0$. This implies that $\gamma' = \gamma'' = \lambda\rho'$, and the proof is finished.

Let us summarize what we have obtained so far: If ρ is a pure quasi-state on A, then there is a unique quasi-state ρ' on A' which extends ρ. ρ' is pure and $\rho'|\mathscr{C}^b(T)$ is pure.

Proof of Theorem 4. By the preceding remark, (iv) is proved. Let $\{\rho_1, \ldots, \rho_n\}$ be a finite set of pure quasi-states on A. Each ρ_i extends to a pure quasi-state ρ_i' on A'. Let $\rho = \sum_{i=1}^n \lambda_i \rho_i$ be a convex combination of the ρ_i's, and define $\rho' = \sum_{i=1}^n \lambda_i \rho_i'$. Clearly ρ' becomes a quasi-state on A' which extends ρ. Now let $\rho \neq 0$ be an arbitrary positive quasi-linear functional on A. Put $\rho_1 = \|\rho\|^{-1} \cdot \rho$, so $\|\rho_1\| = 1$. If we can extend ρ_1 to a quasi-state on A', then we can also extend ρ to a positive quasi-linear form ρ' on A' with $\rho'(1) = \|\rho\|$. So we assume $\|\rho\| = 1$. By Proposition 1, §2 and the Krein-Milman theorem, there is a net $\{\rho_\nu\}\subseteq Q$, such that $\rho_\nu\to\rho$ pointwise on A, and where each ρ_ν is a convex combination of 0 and pure quasi-states on A. Let $Q'=\{$positive quasi-linear functionals on A' of norm $\leqq 1\}$. Each ρ_ν extends to an element $\rho'_\nu\in Q'$. Since Q' is compact, there is a subnet $\rho'_{\nu_i}\to\gamma'\in Q'$. Let $\gamma=\gamma'|A$ and let $x\in A$ be an arbitrary element. Then $\rho'_{\nu_i}(x)\to\gamma'(x)=\gamma(x)$. On the other hand $\rho'_{\nu_i}(x)=\rho_{\nu_i}(x)\to\rho(x)$, so $\gamma=\rho$. Now $1=\|\rho\|\leqq\|\gamma'\|=\gamma'(1)\leqq 1$, so γ' is a quasi-state on A' extending ρ. This proves (i).

Observe that if $x'=x+h\in A'$, with $x\in A$, $h\in\mathscr{C}^b(T)$, and ρ' is a positive quasi-linear functional on A', then

(5) $\rho'(x')=\rho(x)+\rho'(h)$.

Indeed, if x is selfadjoint, then x and h belong to the same abelian C^*-subalgebra, so (5) follows from Theorem 1, §2. For general x' in A', (5) then follows by a simple computation, based on property (ii) in the definition of positive quasi-linear functionals. From (5) we obtain directly that an extension ρ' of a $\rho\in Q$ is linear if and only if ρ is linear, and (iii) is proved. Finally, if ρ' is continuous so is ρ; suppose therefore that ρ is continuous on A. Since $\rho'|\mathscr{C}^b(T)$ is linear, we know that it is continuous. Let $\phi\colon A'\to A'/A$ be the canonical homomorphism, and let $\psi=\phi|\mathscr{C}^b(T)$. Since $A'=A+\mathscr{C}^b(T)$ we have

(6) $A'/A\approx\mathscr{C}^b(T)/A\cap\mathscr{C}^b(T)$.

Now suppose that $x'_n\to x'$ in A'. Since $\phi(x'_n)\to\phi(x')$ in A'/A, and ψ is open; there is, by (6), an element $h\in\mathscr{C}^b(T)$, and a sequence $\{h_n\}\subseteq\mathscr{C}^b(T)$ such that $\psi(h_n)=\phi(x'_n)$ for all n, $\psi(h)=\phi(x')$ and $h_n\to h$. Let $x=x'-h$, $x_n=x'_n-h_n$; $n=1,2,\ldots$. Clearly $x, x_n\in A$ for all n, and $x-x_n=(x'-x'_n)-(h-h_n)\to 0$. Hence, by (5):

$$\rho'(x'_n) = \rho'(x_n)+\rho'(h_n)\to\rho'(x)+\rho'(h)=\rho'(x').$$

This shows that ρ' is continuous, and the proof is complete.

6. **Quasi-states on general C^*-algebras.** In this section A is a C^*-algebra with or without identity, Prim $A=T$ is assumed to be Hausdorff. We regard A' ($=A$ if A has an identity) as a C^*-algebra of vector-fields on T as in the previous section.

Let ρ be a pure quasi-state on A, and let ρ' be its canonical extension to A'. ρ' is pure and there is a unique point $t_0\in T$ such that $\rho'(h)=h(t_0)$ for all $h\in\mathscr{C}^b(T)$.

Lemma 1. *Let $x\in A$,*

(i) *if $x(t_0)=0$ then $\rho(x)=0$,*

(ii) *if $h=h^*\in\mathscr{C}^b(T)$ and $h(t_0)=1$, then $\rho(x)=\rho(hx)$.*

Proof. (i) $x=x_1+ix_2$ with x_1, x_2 selfadjoint. If $x(t_0)=0$; then $x_i(t_0)=0$, $i=1,2$. $\rho(x)=\rho(x_1)+i\rho(x_2)$, so it is sufficient to prove (i) when x is selfadjoint. Suppose this to be the case. Then $|\rho(x)|^2\leqq\|\rho\|\cdot\rho(x^2)$, and $y=x^2\geqq 0$, $y(t_0)=0$ so it suffices to prove (i) when $x\geqq 0$. In this case $0\leqq x\leqq k_x$, and $k_x(t_0)=0$ so $0\leqq\rho(x)=\rho'(x)\leqq\rho'(k_x)=k_x(t_0)=0$ so $\rho(x)=0$. This proves (i).

(ii) Let $x\in A$ be arbitrary, $h=h^*\in\mathscr{C}^b(T)$ and $h(t_0)=1$. Put $y=hx-x$. Then $y(t_0)=0$ so by (i) $\rho(y)=0$. Write $x=x_1+ix_2$; with x_1, x_2 selfadjoint. Then

$$\begin{aligned}0 = \rho(hx-x) &= \rho(h(x_1+ix_2)-(x_1+ix_2))\\&= \rho((hx_1-x_1)+i(hx_2-x_2))\\&= \rho(hx_1-x_1)+i\rho(hx_2-x_2)\\&= \rho(hx_1)-\rho(x_1)+i\rho(hx_2)-i\rho(x_2)\\&= \rho(hx_1+ihx_2)-\rho(x_1+ix_2)\\&= \rho(hx)-\rho(x).\end{aligned}$$

Hence $\rho(hx)=\rho(x)$.

REMARK. The difficulties in proving that ρ is linear on A are now very much in sight. The problems are the following: ρ vanishes on the kernel of the homomorphism $x\to x(t_0)$ of A into $A(t_0)$ (Lemma 1 (i)). This kernel may be identified with the primitive ideal $t_0\subseteq A$, so it is reasonable to try to lift ρ to the quotient algebra A/t_0. This amounts to the problem of showing that

(I) If $x(t_0)=y(t_0)$, then $\rho(x)=\rho(y)$, for $x, y\in A$. By Lemma 1 (i) we obtain that if $x(t_0)=y(t_0)$, so $(x-y)(t_0)=0$, then $\rho(x-y)=0$. However, since we lack linearity of ρ, we cannot conclude that $\rho(x)=\rho(y)$ at this point.

Let us disregard this for the moment, and assume that (I) is true. We then verify rapidly that ρ lifted to A/t_0 becomes a quasi-state (Lemma 4). The remaining problem can then be formulated as follows:

(II) If ρ is a quasi-state on a primitive C^*-algebra B, is ρ linear? (A C^*-algebra is called primitive if it has a faithful irreducible representation.) Although (II) remains very far from being settled in general, we know that special solutions exist: Theorem 2, Corollary 2, §4 and Proposition 3, §4.

We will now discuss (I).

LEMMA 2. *Let $x, y\in A$ and suppose there is a neighbourhood U of t_0 such that $x(t)=y(t)$ for all $t\in U$. Then $\rho(x)=\rho(y)$.*

Proof. Let $0\leqq h\in\mathscr{C}^b(T)$ be chosen such that $h(t_0)=1$ and the support of h is contained in U. Then $hx=hy$, so by Lemma 1 (ii): $\rho(x)=\rho(hx)=\rho(hy)=\rho(y)$. The proof is complete.

COROLLARY 1. *If* Prim *A is discrete, and $x(t_0)=y(t_0)$ for $x, y\in A$, then $\rho(x)=\rho(y)$.*

LEMMA 3. *Let $x, y\in A$ and suppose $x(t_0)=y(t_0)$. If ρ is continuous, then $\rho(x)=\rho(y)$.*

Proof. Let $x, y\in A$ be given; $x(t_0)=y(t_0)$. Let $\varepsilon>0$ be given. By continuity of ρ there is a $\delta>0$ such that $\|x-z\|\leqq\delta$, $z\in A$ implies $|\rho(x)-\rho(z)|<\varepsilon$. Since T is assumed to be Hausdorff, the function $t\to\|x(t)-y(t)\|$ is continuous on T and vanishes at t_0. Hence there is a neighbourhood U of t_0 such that $t\in U\Rightarrow\|x(t)-y(t)\|<\delta$. Now choose a compact neighbourhood C of t_0, $C\subseteq U$, and a function $0 \leqq h \leqq 1$; $h \in \mathscr{C}^b(T)$ such that $h(t) \equiv 1$ on C, $h(t) \equiv 0$ on $T \backslash U$. Let $z = hy + (1-h)x \in A$. We get

$$\begin{aligned} \|x(t)-z(t)\| &= \|x(t)-h(t)y(t)-(1-h(t))x(t)\| \\ &= h(t)\|x(t)-y(t)\| < \delta; \qquad t \in T \end{aligned}$$

by the choice of h. Hence $\|x-z\| \leqq \delta$. Moreover, $z(t) = y(t)$ on the neighbourhood C of t_0, so $\rho(y) = \rho(z)$ by Lemma 2. Hence $|\rho(x)-\rho(y)| = |\rho(x)-\rho(z)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\rho(x) = \rho(y)$. The proof is complete.

Let $\phi: x \to x(t_0)$ be the canonical homomorphism of A onto A/t_0.

LEMMA 4. *If $x(t_0) = y(t_0)$ implies that $\rho(x) = \rho(y)$; $x, y \in A$, then ρ lifts to a pure quasi-state $\tilde{\rho}$ on A/t_0 such that $\rho = \tilde{\rho} \cdot \phi$.*

Proof. For $a \in A/t_0$ we define $\tilde{\rho}(a) = \rho(x)$ if $x(t_0) = \phi(x) = a$. By assumption $\tilde{\rho}$ is well defined. To show that $\tilde{\rho}$ is a quasi-state on A/t_0, let first $a = \phi(x) \in A/t_0$ be an arbitrary element. If $x = x_1 + ix_2$ is the decomposition of x in selfadjoint parts x_1, x_2, then $a = a_1 + ia_2$; $a_1 = \phi(x_1)$, $a_2 = \phi(x_2)$ is the corresponding decomposition of a. Hence $\tilde{\rho}(a) = \tilde{\rho}(a_1) + i\tilde{\rho}(a_2)$, so (ii) in the definition of quasi-states is satisfied.

Now let B be a C^*-subalgebra of A/t_0 generated by a single selfadjoint element of $a \in A/t_0$. We will show that $\tilde{\rho}|B$ is a positive linear functional. This will imply condition (i) in the definition of quasi-states. Indeed, even in the case where A/t_0 has an identity this is sufficient; for if C is any C^*-subalgebra of A/t_0 generated by a selfadjoint element b and the identity $1(t_0)$, then it is easily seen that C is also generated by the element $b + 2\|b\| \cdot 1(t_0)$ which is invertible, so that C is singly generated in the traditional sense. With B as above, choose a selfadjoint element $x \in A$ such that $\phi(x) = a$. Letting $A(x)$ be the C^*-subalgebra of A generated by x, each polynomial in x is mapped into B by ϕ; so that $\phi(A(x)) \subseteq B$. Since $\phi(A(x))$ is a closed *-subalgebra of B containing a, we must have $\phi(A(x)) = B$. Now $\rho|A(x)$ is positive and linear by assumption, and it follows that $\tilde{\rho}|B$ is positive and linear. Hence $\tilde{\rho}$ is a positive quasi-linear functional of norm $\leqq 1$ on A/t_0. It remains to see that $\|\tilde{\rho}\| = 1$. Let $0 \leqq x_n \in A$; $n = 1, 2, \ldots$; $\|x_n\| \leqq 1$, be a sequence such that $\rho(x_n) \to 1$. Then $0 \leqq \phi(x_n) \in A/t_0$; $\|\phi(x_n)\| \leqq 1$ and $\tilde{\rho}(\phi(x_n)) = \rho(x_n) \to 1$. Hence $\|\tilde{\rho}\| = 1$, and $\tilde{\rho}$ is a quasi-state on A/t_0.

Finally, let $\tilde{\gamma}$ be a positive quasi-linear functional on A/t_0 such that $\tilde{\gamma} \leqq \tilde{\rho}$. Let $\gamma = \tilde{\gamma} \cdot \phi$, so γ becomes a positive quasi-linear functional on A such that $\gamma \leqq \rho$. Since ρ is pure, we have $\gamma = \lambda\rho$; $0 \leqq \lambda \leqq 1$. It follows that $\tilde{\gamma} = \lambda\tilde{\rho}$, so $\tilde{\rho}$ is pure on A/t_0. The proof is complete.

PROPOSITION 1. *If either* Prim A *is discrete, or ρ is continuous, then ρ lifts to a pure quasi-state $\tilde{\rho}$ on A/t_0 such that $\rho = \tilde{\rho} \cdot \phi$. If ρ is continuous then $\tilde{\rho}$ is continuous.*

Proof. If Prim A is discrete, apply Corollary 1 and Lemma 4. If ρ is continuous, apply Lemma 3 and Lemma 4. By definition $\rho = \tilde{\rho} \cdot \phi$, and ϕ is open, so $\tilde{\rho}$ is continuous if ρ is continuous. The proof is complete.

THEOREM 5. *Let A be a C^*-algebra with* Prim A *discrete. Suppose that each quasi-state on $\pi(A)$ is linear, for each irreducible representation π of A. Then all quasi-states on A are linear.*

Proof. By Proposition 1 each pure quasi-state ρ on A is of the form $\rho=\tilde{\rho}\cdot\pi$, for some irreducible representation π of A. By assumption $\tilde{\rho}$ is linear on $\pi(A)$, hence ρ is linear on A. The conclusion now follows from the Krein-Milman theorem applied to Q=the set of all positive quasi-linear functionals on A, of norm $\leqq 1$. The proof is complete.

THEOREM 6. *Let A be a C^*-algebra with Hausdorff primitive ideal-space, and suppose that each pure, continuous quasi-state on $\pi(A)$ is linear, for each irreducible representation π of A. Then*:

(1) *Each pure, continuous, quasi-state on A is linear.*

(2) *If all quasi-states on A are continuous, then they are all linear.*

Proof. The first part of the conclusion follows from Proposition 1, and the second part is again a consequence of the Krein-Milman theorem applied to Q. The proof is complete.

All the main results of this paper are now more or less immediate consequences of the last two theorems.

COROLLARY 2. *If A is* CCR *with* Prim A *Hausdorff, and* $\dim \pi\neq 2$ *for each irreducible representation π of A, then* (1) *and* (2) *of Theorem* 9 *hold.*

Proof. If A is CCR, then $\pi(A)\approx\mathscr{L}\mathscr{C}(H)$ for some Hilbert-space H, for each irreducible representation π of A [6, 4.1.11]. The desired result now follows from Corollary 2, §4.

Before we can prove the next result, we need the following simple lemma.

LEMMA 5. *Let ρ be a quasi-state on a finite direct sum $A=A_1\oplus\cdots\oplus A_n$ of the C^*-algebras A_i; $i=1,\ldots,n$. If $\rho|A_i$ is linear for each i, then ρ is linear.*

Proof. It is sufficient to show that ρ is linear on A_h. Each $x\in A_h$ can be written uniquely $x=x_1+\cdots+x_n$, where $x_i\in A_i$ is selfadjoint; $i=1,\ldots,n$. The elements x_i all commute with each other, and will therefore belong to a common abelian C^*-subalgebra of A. Hence, by Theorem 1, §2 we have $\rho(x)=\sum_{i=1}^{n}\rho(x_i)$, from which the linearity of ρ follows easily. The proof is complete.

COROLLARY 3. *If A is a type* I *von Neumann algebra of finite type, with no type* I_2 *part, or in particular if A is of the form $\mathscr{C}(X)\otimes M_n$ $(n\neq 2)$; X a compact Hausdorff-space, then* (1) *and* (2) *of Theorem* 9 *hold.*

Proof. Each finite type I von Neumann algebra with no type I_2 part is of the form $\sum_{n=1;n\neq 2}^{m}\bigoplus B_n\otimes M_n$, where B_n is an abelian C^*-algebra, and M_n is the $n\times n$ matrix algebra [5]. By Lemma 5 we may therefore restrict attention to a single summand $B_n\otimes M_n$. By [18], $B_n\otimes M_n$ is isomorphic to the C^*-algebra A_n of all continuous functions on a compact Hausdorff space X with values in M_n. Hence A_n is CCR with Prim A_n homeomorphic to X ([6, 10.4.3 and 10.4.4], or Proposition 2 below) so Corollary 2 applies. The proof is complete.

Before we can proceed we need a generalized version of the last result quoted in the proof above. This will follow from standard methods, and the proof is included for the convenience of the reader. The terminology is from [6, Chapter 10].

PROPOSITION 2. *Let $\mathscr{F}=((A(t), \theta)$ be a continuous field of simple C^*-algebras on a locally compact Hausdorff-space T. Let A be the C^*-algebra determined by $\mathscr{F}$. For each $t \in T$, let*

$$\tilde{t} = \{x \in A : x(t) = 0\}.$$

Then $t \to \tilde{t}$ is a homeomorphism of T onto Prim A.

Proof. Let $t \in T$, and let $\phi_t: x \to x(t)$; $x \in A$, be the homomorphism of A onto $A(t)$. Let π be any irreducible representation of $A(t)$. Since $A(t)$ is simple, ker $\pi=0$. $\pi\cdot\phi_t$ is an irreducible representation of A, and ker $\pi\cdot\phi_t=\tilde{t}$. Hence $\tilde{t}$ is a primitive ideal of A. We show next that the map $t \to \tilde{t}$ is injective. Suppose $t_1 \neq t_2$; $t_1, t_2 \in T$. There is a bounded complex continuous function h on T such that $h(t_1)=1$, $h(t_2)=0$. Choose an element $x \in A$; $x \notin \tilde{t}_1$. By [6, 10.1.9(ii)], $hx \in A$. $(hx)(t_1) =x(t_1)\neq 0$, and $(hx)(t_2)=h(t_2)x(t_2)=0$, so $hx \notin \tilde{t}_1$, but $hx \in \tilde{t}_2$. Hence $\tilde{t}_1 \neq \tilde{t}_2$, which shows that the map $t \to \tilde{t}$ is injective. It is also surjective: Let $I \in$ Prim A, and define $I(t)=\phi_t(I)$ for all $t \in T$. Let $Y=\{t \in T : I(t) \neq A(t)\}$. By [6, Lemma 10.4.2]; $Y \neq \varnothing$. Suppose $t_1, t_2 \in Y$; $t_1 \neq t_2$. Let U_1 and U_2 be disjoint neighbourhoods of t_1 and t_2 respectively, and let

$$J_i = \{x \in A : x \equiv 0 \text{ on } T\backslash U_i\}; \qquad i = 1, 2.$$

J_i is a closed two-sided ideal of A. We have $J_1 \cap J_2=(0)$, so by [6, Lemma 2.11.4], $I\supseteq J_1$ (or $I\supseteq J_2$). Hence $I(t_1)=\phi_{t_1}(I)\supseteq\phi_{t_1}(J_1)=A(t_1)$. This contradicts the assumption that $t_1 \in Y$. Hence $Y=\{t_0\}$. Now $I(t_0)$ is a closed two-sided ideal of $A(t_0)$. Since $A(t_0)$ is simple, and $I(t_0)\neq A(t_0)$, we must have $I(t_0)=(0)$. By [6, Lemma 10.4.2] it then follows that

$$I = \{x \in A : x(t_0) = 0\}$$

i.e., $I=\tilde{t}_0$. This shows that the map $t \to \tilde{t}$ is a bijection of T onto Prim A.

Next, let $Y\subseteq T$; $t_1 \in T$. Put

$$\tilde{Y} = \{x \in A : x(t) \equiv 0 \text{ on } Y\}.$$

Clearly $\tilde{Y}=\bigcap\{\tilde{t} : t \in Y\}$. By the definition of the hull-kernel topology for Prim A, we know that $\tilde{t}_1$ is in the closure of $\{\tilde{t} : t \in Y\}$ if and only if $\tilde{t}_1\supseteq \tilde{Y}$. If $t_1 \in \overline{Y}$, then each $x \in \tilde{Y}$ vanishes on t_1, since $t \to \|x(t)\|$ is continuous for each $x \in A$. Hence $\tilde{t}_1\supseteq \tilde{Y}$. On the other hand, if $t_1 \notin \overline{Y}$, then there is a bounded complex continuous function h on T such that $h(t_1)=1$, $h(t)\equiv 0$ on Y. Hence there is an $x \in A$ such that

$x(t_1)\neq 0$ but $x(t)\equiv 0$ on Y; i.e., $x\in \tilde{Y}$, but $x\notin t_1$. Consequently $\tilde{Y}\nsubseteq \tilde{t}_1$, and the proof is complete.

Recall that A is a *dual* C^*-algebra if A has a faithful representation as a C^*-subalgebra of $\mathscr{L}\mathscr{C}(H)$, for some Hilbert-space H [14].

COROLLARY 4. *Any quasi-state on a dual C^*-algebra A is linear if* $\dim \pi\neq 2$ *for all irreducible representations π of A.*

Proof. If A is dual, then A may be written as the *restricted* C^*-algebra product of C^*-algebras $A_i\approx \mathscr{L}\mathscr{C}(H_i)$; $i\in I$ [14, Lemma 2.3]; i.e., A is the set of all families $\{x_i\}_{i\in I}$ in $\prod_{i\in I} A_i$, such that, for each $\varepsilon>0$, the set of indices for which $\|x_i\|>\varepsilon$, is finite. If we give I the discrete topology, it is easily seen that $\mathscr{F}=((A_i)_{i\in I}, \prod_{i\in I} A_i)$ is a continuous field of (simple) C^*-algebras, and that A is the C^*-algebra defined by $\mathscr{F}$. So Prim A is discrete by Proposition 2, and Theorem 5 applies. The desired conclusion now follows from Corollary 2, §4.

COROLLARY 5. *If $A=\mathscr{C}(T)\otimes B$, where T is a compact Hausdorff-space, and B is a* UHF-*algebra, then* (1) *and* (2) *of Theorem* 9 *hold.*

Proof. B is simple [10], and A is isomorphic to the C^*-algebra of all continuous functions from T into B [18]. Hence Prim $A\approx T$ (Proposition 2) is a compact Hausdorff-space. By Proposition 2 we also obtain that A/t is isomorphic to B for each primitive ideal t in A. Combining Proposition 3, §4 with Theorem 6, the proof is complete.

REMARK. Since the condition that Prim A is Hausdorff has been in force throughout the last two sections, it seems appropriate to mention under what circumstances this condition is fulfilled. A C^*-algebra A with center Z is *central* if the map $I\to I\cap Z$ is an injection of Prim A into Prim Z. If A has an identity, then it is central if and only if Prim A is Hausdorff. If A has no identity, then it is central if and only if Prim A is Hausdorff, and no primitive ideal of A contains Z ([2] and [4]).

REFERENCES

1. J. F. Aarnes, *Physical states on a C^*-algebra*, Acta Math. **122** (1969), 161–172.

2. R. Busby, *On structure spaces and extensions of C^*-algebras*, J. Functional Analysis **1** (1967), 370–377. MR **37** #771.

3. J. Dauns and K. H. Hofmann, *Representations of rings by continuous sections*, Mem. Amer. Math. Soc. No. 83 (1968), 180 pp.

4. C. Delaroche, *Sur les centres des C^*-algèbres*, C. R. Acad. Sci. Paris **265** (1967), 465–466.

5. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Cahiers Sci., Fasc. XXV, Gauthier-Villars, Paris, 1957. MR **20** #1234.

6. J. Dixmier, *Les C^*-algèbres et leurs représentations*, Cahiers Sci., Fasc. XXIX, Gauthier-Villars, Paris, 1964. MR **30** #1404.

7. ———, *Ideal center of a C^*-algebra*, Duke Math. J. **35** (1968), 375–382. MR **37** #5703.

8. J. M. G. Fell, *The structure of algebras of operator fields*, Acta. Math. **106** (1961), 233–280. MR **29** #1547.

9. A. M. Gleason, *Measures on the closed subspaces of a Hilbert space*, J. Math. Mech. **6** (1957), 885–893. MR **20** #2609.

10. J. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340. MR **22** #2915.

11. R. V. Kadison, *Transformations of states in operator theory and dynamics*, Topology **3** (1965), 177–198; Suppl. 2, 1–22. MR **29** #6328.

12. ———, *On the additivity of the trace in finite factors*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 385–387. MR **16** #1033.

13. ———, *Irreducible operator algebras*, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 273–276. MR **19**, 47.

14. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255. MR **13**, 48.

15. J. L. Kelley and I. Namioka, *Linear topological spaces*, The University Series in Higher Math., Van Nostrand, Princeton, N. J., 1963. MR **29** #3851.

16. G. W. Mackey, *The mathematical foundations of quantum mechanics*, Benjamin, New York, 1963. MR **27** #5501.

17. F. J. Murray and J. von Neumann, *On rings of operators*. II, Trans. Amer. Math. Soc. **41** (1937), 208–248.

18. S. Sakai, *The theory of W^*-algebras*, Mimeographed, Yale University, New Haven, Conn., 1962.

19. I. Segal, *Postulates for general quantum mechanics*, Ann. of Math. (2) **48** (1947), 930–948. MR **9**, 241.

20. M. Takesaki, *On the conjugate space of operator algebra*, Tôhoku Math. (2) **10** (1958), 194–203. MR **20** #7227.

21. V. S. Varadarajan, *Probability in physics and a theorem on simultaneous observability*, Comm. Pure Appl. Math. **15** (1962), 189–217. MR **29** #917.

University of Oslo,
Oslo, Norway
University of Pennsylvania,
Philadelphia, Pennsylvania 19104