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TWO METHODS OF INTEGRATING MONGE-AMPÈRE'S EQUATIONS

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Abstract. Modifying Monge's method and Laplace's one respectively, we shall give two methods of integration of Monge-Ampère's equations. Although they seem quite different, the equivalence of our two methods will be shown.

The first method will be from a point of view different from that of Lewy. The second will present a solution to a problem of Goursat.

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1. **Introduction.** Limiting ourselves to Monge-Ampère's equation

$$\text{(M-A)} \qquad Hr+2Ks+Lt+M+N(rt-s^2) = 0,$$

we shall discuss the following problem:

What partial differential equation can be solved by integrating ordinary differential equations only?

Here H, K, L, M and N are functions of x, y, z, p and q. By p, q, r, s and t, we denote

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2}$$

respectively.

For a general equation $F(x, y, z, p, q, r, s, t)=0$ with a given initial value of hyperbolic type, Lewy [8] solved the Cauchy problem reducing it to solving difference equations. Here we shall try to reduce the problem to solving ordinary differential equations in the case of Monge-Ampère's equation where the hyperbolicity does not depend on the given initial value.

Let us try to solve the Cauchy problem of Monge-Ampère's equation, integrating ordinary equations only. Monge's method is as follows. For a given initial value, find such an intermediate integral equation of the first order that is satisfied by the

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initial value. This process requires us to solve a system of linear partial equations of the first order with one unknown function. In virtue of Lagrange's theorem it can be solved by integrating ordinary equations only, if it has a solution. Suppose that such an intermediate integral exists. Then we can obtain an integral surface of the original equation which satisfies the initial condition, integrating Lagrange-Charpit's system of the intermediate integral. Such an intermediate integral can be found for every initial value, if and only if the equation possesses two independent intermediate integrals of the same characteristics. See [4, Chapter II]. In this case let us say that the equation is Monge-integrable.

Lagrange-Charpit's system

$$\text{(L-C)}\qquad \begin{aligned} \frac{dx}{\partial V/\partial p} = \frac{dy}{\partial V/\partial q} = \frac{dz}{(\partial V/\partial p)p+(\partial V/\partial q)q} = \frac{-dp}{\partial V/\partial x+(\partial V/\partial z)p} \\ = \frac{-dq}{\partial V/\partial y+(\partial V/\partial z)q} \end{aligned}$$

of an equation $V=0$ of the first order has the following property:

If an initial value $(x=x_0(s),\ y=y_0(s),\ z=z_0(s),\ p=p_0(s),\ q=q_0(s))$ satisfies $dz-pdx-qdy=0$ and $dV=0$, then the surface obtained by integrating Lagrange-Charpit's system $=dt$ under the initial condition $x(0,s)=x_0(s)$, $y(0,s)=y_0(s)$, $z(0,s)=z_0(s)$, $p(0,s)=p_0(s)$, $q(0,s)=q_0(s)$ satisfies $dz-pdx-qdy=0$.

We shall say that a system

$$\frac{dx}{A}=\frac{dy}{B}=\frac{dz}{Ap+Bq}=\frac{dp}{C}=\frac{dq}{D}$$

is integrable, if for any initial value which satisfies $dz-pdx-qdy=0$ and $Cdx+Ddy-Adp-Bdq=0$, the surface obtained by integrating the system $=dt$ under the given initial condition satisfies $dz-pdx-qdy=0$. There exists such an integrable system that is not Lagrange-Charpit's system of any equation.

This notion of an integrable system allows us to generalize Monge's method of integration of Monge-Ampère's equation: For a given initial value, find an integrable system which satisfies $Cdx_0+Ddy_0-Adp_0-Bdq_0=0$ and the characteristic condition of the equation. This process requires us to solve a system of nonlinear partial equations of the first order with one unknown function. Hence in virtue of Jacobi's theorem it can be solved by integrating ordinary equations only, if it has a solution. Suppose that such an integrable system exists. Then the surface obtained by integrating the system $=dt$ under the given initial condition is an integral surface of the equation. If Lagrange-Charpit's system of an equation can be taken as such an integrable system for every initial value, then the equation is Monge-integrable.

Our method can be applied for integrating an equation which is not Monge-integrable. Let us try to integrate the linear hyperbolic equation of the second order by our method. Then it turns out that it is possible if and only if the second Laplace invariant of the equation vanishes.

Due to Laplace we have another method of integration. To integrate the linear hyperbolic equations of the second order, he tried to transform a given equation to a Monge-integrable equation by a Laplace transformation. It is possible if and only if the second Laplace invariant of the original equation vanishes. See [1, Chapter II].

Investigating transformations of a surface of constant negative curvature, Bäcklund found a class of transformations which is wider than that of contact transformations. A member of his class is called a Bäcklund transformation. See [2, Chapter XII]. The Laplace transformation is a Bäcklund transformation (see [7]).

Let us consider the following Bäcklund transformation:

$$\begin{aligned} x' &= f(x, y, z, p, q), & y' &= g(x, y, z, p, q), \\ z' &= h(x, y, z, p, q), & p' &= k(x, y, z, p, q). \end{aligned} \tag{1}$$

Here we assume that the four functions f, g, h and k are functionally independent, and that each of the quotients made from their Lagrange's brackets

$$[f, g],\ [f, h],\ [f, k],\ [g, h],\ [g, k],\ [h, k]$$

is a function of f, g, h, and k. Then in virtue of Bäcklund's theorem [3, p. 440], the equation

$$(g, h) - k(g, f) = 0 \tag{2}$$

is transformed to the equation

$$([f, h] - q'[f, g])r' + ([g, h] + p'[f, g])s' - q'[g, k] - p'[f, k] + [h, k] = 0. \tag{3}$$

This equation has its meaning by our assumption. Here

$$(g, h) = \frac{Dg}{Dx}\frac{Dh}{Dy} - \frac{Dg}{Dy}\frac{Dh}{Dx}$$

and

$$\frac{Dg}{Dx} = \frac{\partial g}{\partial x} + p\frac{\partial g}{\partial z} + r\frac{\partial g}{\partial p} + s\frac{\partial g}{\partial q}, \qquad \frac{Dg}{Dy} = \frac{\partial g}{\partial y} + q\frac{\partial g}{\partial z} + s\frac{\partial g}{\partial p} + t\frac{\partial g}{\partial q}.$$

If we put $f=x$, $g=y$, $h=q+a(x, y)z$ and $k=-b(x, y)q+\{\partial a/\partial x - c(x, y)\}z$, then we have the Laplace transformation which is applied to the linear hyperbolic equation $s+ap+bq+cz=0$. The four functions f, g, h and k satisfy the following functional equations:

$$[f, g] = [f, h] = [f, k] = 0 \tag{4}$$

and

$$\begin{aligned} [k, h]/[g, h] &= (\partial \log H/\partial y - a)k + (\partial a/\partial x - \partial b/\partial y - c + b(\partial \log H/\partial y))h, \\ [g, k]/[g, h] &= -b, \end{aligned}$$

where H is the first Laplace invariant defined by $H=\partial a/\partial x+ab-c$. The equation $s+ap+bq+cz=0$ is Monge-integrable (with respect to the characteristics $dy=0$, $dq+(ap+bq+cz)dx=0$), if and only if the first invariant H vanishes. In this case the Laplace transformation cannot be defined. Suppose that $H\neq 0$. Then by this Laplace transformation the original equation is transformed to the equation

$$s'+(a-\partial\log H/\partial y)p'+bq'+(c-\partial a/\partial x+\partial b/\partial y-b(\partial\log H/\partial y))z' = 0.$$

The second invariant H_1 of the original equation is defined as the first invariant of the transformed equation

$$H_1 = 2(\partial a/\partial x)-\partial b/\partial y+ab-c-\partial^2\log H/\partial x\partial y.$$

The Bäcklund transformation (1) gives a correspondence between a solution of the original equation (2) and that of the transformed equation (3) in the following way. For a surface Ψ given by $z=\psi(x, y)$, we have a transformed surface $B\Psi$ defined by $x'=f(x, y; \psi)$, $y'=g(x, y; \psi)$ and $z'=h(x, y; \psi)$ which we obtain from the equality (1) putting $p=\partial\psi/\partial x$ and $q=\partial\psi/\partial y$. Then the other function $p'=k(x, y; \psi)$ thus obtained is the partial derivative of the transformed surface $B\Psi$ with respect to x', if and only if the surface Ψ is a solution of the original equation. In this case the transformed surface $B\Psi$ is a solution of the transformed equation. The converse correspondence can be given in the same way, if we assume that the four functions f, g, h and k satisfy the functional equation (4). In fact they are functions of x, y, z and q, if we transform the function f to x by a contact transformation. In virtue of a theorem of Lie such a contact transformation exists (see [9]). Since they are functionally independent, we can solve the equality (1) conversely with respect to x, y, z and q. We can take also a contact transformation that transforms the functions f and g to x and y respectively, because of the equality $[f, g]=0$. Hence the correspondence given above is bijective.

We shall say that the Bäcklund transformation (1) is of Laplace type, if the four functions f, g, h and k satisfy the functional equations (4) and also

$$(5)\qquad [k, h]/[g, h] = \alpha k+\beta, \qquad [g, k]/[g, h] = \gamma k+\delta,$$

where α, β, γ and δ are functions of f, g and h. Then the transformed equation (3) is expressible in the form

$$s'-(\gamma p'q'+\alpha p'+\delta q'+\beta) = 0.$$

In this case let us call the original equation (2) also an equation of Laplace type. The linear hyperbolic equation is of Laplace type, unless $H=0$. Any equation of Laplace type is not Monge-integrable (with respect to the characteristics $dg=0$, $dh-kdf=0$).

On this transformation Lagrange's bracket $[g, h]$ cannot vanish. For to the contrary if we suppose that $[g, h]=0$, then we have $[k, h]=[g, k]=0$ by the equality (5). This is impossible, because the four functions f, g, h and k are functionally independent. Hence our transformation is different from a contact transformation for which the equality $[g, h]=0$ holds.

Our transformation can be applied for integrating a nonlinear equation. For example, putting $f=x$, $g=y$, $h=q$ and $k=e^z$, we can transform Liouville's equation $s=e^z$ to the Monge-integrable equation $s'=p'z'$.

This transformation is well known in the classical theory. Goursat presented the following problem (see [5, Chapter IX], and [7]):

Construct a general class of transformations for integrating equations which are not Monge-integrable such that it contains the Laplace transformation and the transformation above-mentioned as the special cases.

Our Bäckland transformations of Laplace type give a solution to this problem of Goursat.

The following theorem will be proved.

An equation of Laplace type is integrable by our first method if and only if the transformed equation is Monge-integrable.

Every notion of integrability defined here is independent of the choice of the coordinates, if they are changed by a contact transformation.

We are always in the category of infinite differentiability.

A modern formulation of our problem will not be given here. The problem will be discussed in the classical style.

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2. **An integrable system.** Let us find a condition which the four functions A, B, C and D should satisfy in order that the system

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{pA+qB} = \frac{dp}{C} = \frac{dq}{D} \tag{6}$$

may be integrable.

Integration of the ordinary equations

$$\frac{\partial x}{\partial t} = A, \quad \frac{\partial y}{\partial t} = B, \quad \frac{\partial z}{\partial t} = pA+qB, \quad \frac{\partial p}{\partial t} = C, \quad \frac{\partial q}{\partial t} = D \tag{7}$$

under the initial condition

$$\begin{aligned} x(0, s) &= x_0(s), \quad & y(0, s) &= y_0(s), \quad & z(0, s) &= z_0(s), \\ p(0, s) &= p_0(s), \quad & q(0, s) &= q_0(s) \end{aligned} \tag{8}$$

gives a two dimensional manifold $x=x(t, s)$, $y=y(t, s)$, $z=z(t, s)$, $p=p(t, s)$, $q=q(t, s)$.

Denoting infinitesimal displacements in the direction of s (t=constant) and t (s=constant) by d and δ respectively, we have

$$\delta(dz-pdx-qdy)-d(\delta z-p\delta x-q\delta y) = dp\delta x+dq\delta y-\delta q dx-\delta q dy.$$

This gives

$$(\delta/\delta t)(dz-pdx-qdy) = (Adp+Bdq-Cdx-Ddy),$$

because $\delta z-p\delta x-q\delta y=(pA+qB)\delta t-pA\delta t-qB\delta t=0$. Hence the initial curve should have the properties

$$dz_0/ds-p_0dx_0/ds-q_0dy_0/ds = 0 \tag{9}$$

and

$$Adp_0/ds+Bdq_0/ds-Cdx_0/ds-Ddy_0/ds = 0, \tag{10}$$

in order that we may have $\delta(dz-pdx-qdy)=0$.

Let us calculate $\delta(Adp+Bdq-Cdx-Ddy)$ in the following:

$$\begin{aligned}\delta(Adp+Bdq-Cdx-Ddy)&-d(A\delta p+B\delta q-C\delta x-D\delta y)\\ &= \delta Adp+\delta Bdq-\delta Cdx-\delta Ddy-dA\delta p-dB\delta q+dC\delta x+dD\delta y,\end{aligned}$$

where

$$\delta A = \left\{\frac{\partial A}{\partial x}A+\frac{\partial A}{\partial y}B+\frac{\partial A}{\partial z}(pA+qB)+\frac{\partial A}{\partial p}C+\frac{\partial A}{\partial q}D\right\}\delta t$$

and

$$dA = \frac{\partial A}{\partial x}dx+\frac{\partial A}{\partial y}dy+\frac{\partial A}{\partial z}dz+\frac{\partial A}{\partial p}dp+\frac{\partial A}{\partial q}dq.$$

Then we have

$$\begin{aligned}\frac{\delta}{\delta t}(Adp+Bdq-Cdx-Ddy) &= (Edx+Fdy+Gdp+Jdq)\\ &\quad-\left(\frac{\partial A}{\partial z}C+\frac{\partial B}{\partial z}D-\frac{\partial C}{\partial z}A-\frac{\partial D}{\partial z}B\right)(dz-pdx-qdy),\end{aligned}$$

because

$$A\delta p+B\delta q-C\delta x-D\delta y = (AC+BD-CA-DB)\,\delta t = 0.$$

Here E, F, G and J are the following functions:

$$\begin{aligned}E &= B\left(\frac{\partial D}{\partial x}+\frac{\partial D}{\partial z}p-\frac{\partial C}{\partial y}-\frac{\partial C}{\partial z}q\right)-C\left(\frac{\partial A}{\partial x}+\frac{\partial A}{\partial z}p+\frac{\partial C}{\partial p}\right)-D\left(\frac{\partial B}{\partial x}+\frac{\partial B}{\partial z}p+\frac{\partial C}{\partial q}\right),\\ F &= A\left(\frac{\partial C}{\partial y}+\frac{\partial C}{\partial z}q-\frac{\partial D}{\partial x}-\frac{\partial D}{\partial z}p\right)-D\left(\frac{\partial B}{\partial y}+\frac{\partial B}{\partial z}q+\frac{\partial D}{\partial q}\right)-C\left(\frac{\partial A}{\partial y}+\frac{\partial A}{\partial z}q+\frac{\partial D}{\partial p}\right),\\ G &= A\left(\frac{\partial A}{\partial x}+\frac{\partial A}{\partial z}p+\frac{\partial C}{\partial p}\right)+B\left(\frac{\partial A}{\partial y}+\frac{\partial A}{\partial z}q+\frac{\partial D}{\partial p}\right)+D\left(\frac{\partial A}{\partial q}-\frac{\partial B}{\partial p}\right)\end{aligned}$$

and

$$J = B\left(\frac{\partial B}{\partial y}+\frac{\partial B}{\partial z}q+\frac{\partial D}{\partial q}\right)+A\left(\frac{\partial B}{\partial x}+\frac{\partial B}{\partial z}p+\frac{\partial C}{\partial q}\right)+C\left(\frac{\partial B}{\partial p}-\frac{\partial A}{\partial q}\right).$$

Hence in order that the system (6) may be integrable, the four functions A, B, C and D should satisfy the three partial equations

$$E/C = F/D = G/-A = J/-B. \tag{11}$$

DEFINITION. A system (6) is said to be integrable, if the four functions A, B, C and D satisfy the three partial equations (11).

Lagrange-Charpit's system (L-C) is integrable. In fact we have $E=-C\partial V/\partial z$, $F=-D\partial V/\partial z$, $G=A\partial V/\partial z$ and $J=B\partial V/\partial z$.

PROPOSITION 1. *Suppose that the system* (6) *is integrable and that an initial curve with the properties* (9) *and* (10) *is given. Then the two dimensional manifold N obtained by integrating the ordinary equations* (7) *under the initial condition* (8) *is a surface which satisfies* $dz-pdx-qdy=0$.

Proof. Since we have $\partial z/\partial t-p\partial x/\partial t-q\partial y/\partial t=0$ on N, it is sufficient to show $\partial z/\partial s-p\partial x/\partial s-q\partial y/\partial s=0$ on N.

Put

$$\mu_1 = \partial z/\partial s-p\partial x/\partial s-q\partial y/\partial s$$

and

$$\mu_2 = A\partial p/\partial s+B\partial q/\partial s-C\partial x/\partial s-D\partial y/\partial s.$$

Then the functions $\mu_1(t, s)$ and $\mu_2(t, s)$ satisfy the following system of ordinary equations

$$\partial \mu_1/\partial t = \mu_2, \qquad \partial \mu_2/\partial t = -\omega\mu_1-\pi\mu_2,$$

where

$$\omega = \frac{\partial A}{\partial z}C+\frac{\partial B}{\partial z}D-\frac{\partial C}{\partial z}A-\frac{\partial D}{\partial z}B$$

and $\pi=$the function given by each term of (11). Since $\mu_1(0, s)$ and $\mu_2(0, s)$ vanish by our assumption, we obtain $\mu_1(t, s)=\mu_2(t, s)=0$ in virtue of the uniqueness theorem for a system of ordinary equations.

The three equations (11) with respect to A, B, C and D are reducible to the following two equations with respect to A/B, C/B and D/B:

$$\frac{C}{B}\frac{d}{dy}\frac{A}{B}+\frac{C}{B}\frac{D}{B}\frac{\partial}{\partial q}\frac{A}{B}-\frac{A}{B}\frac{d}{dy}\frac{C}{B}-\frac{A}{B}\frac{D}{B}\frac{\partial}{\partial q}\frac{C}{B}+\frac{A}{B}\frac{d}{dx}\frac{D}{B}+\frac{C}{B}\frac{\partial}{\partial p}\frac{D}{B} = 0 \tag{12}$$

and

$$\frac{C}{B}\frac{d}{dx}\frac{A}{B}+\left(\frac{C}{B}\right)^2\frac{\partial}{\partial q}\frac{A}{B}+\frac{d}{dy}\frac{C}{B}+\frac{C}{B}\frac{\partial}{\partial p}\frac{C}{B}+\left(\frac{D}{B}-\frac{A}{B}\frac{C}{B}\right)\frac{\partial}{\partial q}\frac{C}{B}-\frac{d}{dx}\frac{D}{B}-\frac{C}{B}\frac{\partial}{\partial q}\frac{D}{B} = 0, \tag{13}$$

where $d/dx=\partial/\partial x+p\partial/\partial z$, $d/dy=\partial/\partial y+q\partial/\partial z$.

Here we assumed that $C\neq 0$. If $C=0$ and $D\neq 0$, then we have the following reduced equations:

$$\frac{d}{dx}\frac{D}{B} = 0, \qquad \frac{A}{B}\frac{d}{dx}\frac{A}{B}+\frac{d}{dy}\frac{A}{B}+\frac{D}{B}\frac{\partial}{\partial q}\frac{A}{B}+\frac{\partial}{\partial p}\frac{D}{B}-\frac{A}{B}\frac{\partial}{\partial q}\frac{D}{B} = 0.$$

Also if $C=D=0$, then we have the single reduced equation

$$\frac{A}{B}\frac{d}{dx}\frac{A}{B}+\frac{d}{dy}\frac{A}{B} = 0.$$

Hence if four functions A, B, C and D give an integrable system, then the four functions τA, τB, τC and τD multiplied by an arbitrary function τ also give an integrable system.

The existence of an integrable system which is not Lagrange-Charpit's system of any equation will be shown in the next section (see Remark 2).

3. **A method of integration by integrable systems.** Let us try to solve the Cauchy problem of Monge-Ampère's equation (M-A) by integrable systems.

One of the characteristics of the equation is given by

$$Ndp+Ldx+\lambda_1 dy = 0, \qquad Ndq+\lambda_2 dx+Hdy = 0, \tag{14}$$

where λ_1 and λ_2 are the roots of the quadratic equation $\lambda^2+2K\lambda+HL-MN=0$. Here we assumed that $N\neq 0$. Any equation the coefficient N of which is zero can be transformed to an equation for which $N\neq 0$ by a contact transformation.

The fundamental theorem of Monge and Ampère on their equations is as follows:

A two dimensional submanifold of the space (x, y, z, p, q) gives a solution of the equation (M-A), if and only if it is a surface generated by one-parameter family of characteristic curves of the given equation on which we have $dz-pdx-qdy=0$.

Here a characteristic curve of the equation (M-A) is a curve along which we have $dz-pdx-qdy=0$ and the identity (14).

Let the system (6) belong to the characteristics (14). Then the four functions A, B, C and D should satisfy

$$NC+LA+\lambda_1 B = 0, \qquad ND+\lambda_2 A+HB = 0. \tag{15}$$

Replacing C/B and D/B by $-1/N(LA/B+\lambda_1)$ and $-1/N(\lambda_2 A/B+H)$ respectively, we have a system S of the two partial equations with respect to A/B from the two equations (12) and (13).

For a given initial curve with the property that $dz_0-p_0dx_0-q_0dy_0=0$, let us find such a solution of the system S that satisfies the initial condition

$$\frac{1}{N}\left(L\frac{A}{B}+\lambda_1\right)\frac{dx_0}{ds}+\frac{1}{N}\left(\lambda_2\frac{A}{B}+H\right)\frac{dy_0}{ds}+\frac{A}{B}\frac{dp_0}{ds}+\frac{dq_0}{ds} = 0$$

on the given initial curve. Suppose that such a solution of S can be found. Then by Proposition 1 we can obtain an integral surface which satisfies the given initial condition, integrating

$$\frac{\partial x}{\partial t} = \frac{A}{B}, \qquad \frac{\partial y}{\partial t} = 1, \qquad \frac{\partial z}{\partial t} = p\frac{A}{B}+q, \qquad \frac{\partial p}{\partial t} = -\frac{1}{N}\left(L\frac{A}{B}+\lambda_1\right),$$
$$\frac{\partial q}{\partial t} = -\frac{1}{N}\left(\lambda_2\frac{A}{B}+H\right)$$

under the initial condition (8).

In order to solve the system S, let us recall here the theorems of Lagrange and Jacobi (see [6, Chapters II, VII and VIII]).

By $x_1, x_2, \ldots, x_n$ and u, we denote independent variables and only one dependent variable respectively.

A system $(X_1, \ldots, X_r)$ of independent vector fields

$$X_\lambda = \sum_{i=1}^{n} a_{\lambda i}(x_1, \ldots, x_n) \frac{\partial}{\partial x_i} \qquad (1 \leqq \lambda \leqq r)$$

is said to be complete, if each of their brackets

$$[X_\lambda, X_\mu] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{\lambda j} \frac{\partial a_{\mu i}}{\partial x_j} - a_{\mu j} \frac{\partial a_{\lambda i}}{\partial x_j} \right) \frac{\partial}{\partial x_i} \qquad (1 \leqq \lambda, \mu \leqq r)$$

is a linear combination of $X_1, \ldots, X_r$.

Any system $(Y_1, \ldots, Y_s)$ of vector fields can be prolonged to a complete system uniquely.

Lagrange's theorem is as follows:

Suppose that a system $(X_1, \ldots, X_r)$ of independent vector fields is complete. Then the following linear system

$$\sum_{i=1}^{n} a_{\lambda i}(x) \frac{\partial u}{\partial x_i} = 0 \qquad (1 \leqq \lambda \leqq r)$$

of partial equations possesses $n-r$ functionally independent solutions $f_1, \ldots, f_{n-r}$. Every solution is expressed in the form $\psi(f_1, \ldots, f_{n-r})$, where ψ is a function of $n-r$ variables.

Let us consider a nonlinear system $(F_1, \ldots, F_r)$ of partial equations

$$F_\lambda = F_\lambda(x_1, \ldots, x_n, u, p_1, \ldots, p_n) = 0 \qquad (1 \leqq \lambda \leqq r),$$

where $p_1 = \partial u/\partial x_1, \ldots, p_n = \partial u/\partial x_n$.

A system $(F_1, \ldots, F_r)$ of independent equations F_λ $(1 \leqq \lambda \leqq r)$ is said to be complete, if each of their Lagrange's brackets $(1 \leqq \lambda, \mu \leqq r)$

$$[F_\lambda, F_\mu] = \sum_{i=1}^{n} \frac{\partial F_\lambda}{\partial p_i} \left(\frac{\partial F_\mu}{\partial x_i} + \frac{\partial F_\mu}{\partial u} p_i \right) - \sum_{i=1}^{n} \frac{\partial F_\mu}{\partial p_i} \left(\frac{\partial F_\lambda}{\partial x_i} + \frac{\partial F_\lambda}{\partial u} p_i \right)$$

is a linear combination of $F_1, \ldots, F_r$.

Any system $(G_1, \ldots, G_s)$ can be prolonged to a complete system $(F_1, \ldots, F_r)$ uniquely. Let us call $n-r$ the rank of the original system. Then the linear complete system $(X_1, \ldots, X_r)$ has its rank $n-r$.

Jacobi's theorem is as follows:

Suppose that a nonlinear complete system F of rank m is given. Then we can solve the following Cauchy problem for every m-dimensional surface N, integrating a system of ordinary differential equations:

For a given initial value $u_0(s_1, \ldots, s_m)$ on the surface N defined by $x_1 = x_1^0(s_1, \ldots, s_m), \ldots, x_n = x_n^0\,(s_1, \ldots, s_m)$, find a solution $u(x_1, \ldots, x_n)$ such that

$$u(x_1^0(s_1, \ldots, s_m), \ldots, x_n^0(s_1, \ldots, s_m)) = u_0(s_1, \ldots, s_m).$$

In virtue of this theorem of Jacobi, we have the following

PROPOSITION 2. *The equation* (M-A) *is integrable by our method, if and only if the system S has its rank greater than zero.*

As to Monge's method, he tried to solve the following linear system M:

$$-N\frac{dV}{dx}+L\frac{\partial V}{\partial p}+\lambda_1\frac{\partial V}{\partial q} = 0, \qquad -N\frac{dV}{dy}+\lambda_2\frac{\partial V}{\partial p}+H\frac{\partial V}{\partial q} = 0.$$

The equation is Monge-integrable if and only if the system M has its rank greater than one. In this case it is said to be Monge-integrable with respect to the other characteristics

$$Ndp+Ldx+\lambda_2 dy = 0, \qquad Ndq+\lambda_1 dx+Hdy = 0, \tag{16}$$

because the differential dV of a solution of M can be expressed in a linear combination of (16) and $dz-pdx-qdy$. See [4, Chapter II].

Hence we shall say that the equation is integrable by our method with respect to the other characteristics (16), if the system S has its rank greater than zero.

If the equation is Monge-integrable with respect to the characteristics (16), then it is integrable by our method with respect to the characteristics (16).

REMARK 1. An involutive system of two Monge-Ampère's equations has a system of the form (6) which belongs to the characteristics of both equations. It is integrable in our sense (see [5, Chapter VI]). From this point of view, our method can be explained in the following way:

For a given initial value, find such Monge-Ampère's equation that forms with the original equation an involutive system which is compatible with the given initial value.

EXAMPLE 1. Let us consider a linear parabolic equation $r+ap+bq+cz=0$, where a, b and c are functions of x and y. Its two characteristics are the same and given by

$$dy = 0, \qquad dp+(ap+bq+cz)dx = 0.$$

Put $A=1$, $B=0$ and $C=-(ap+bq+cz)$. Then we have

$$\begin{aligned} E &= -(ap+bq+cz)a+Db, \\ F &= -\left(\frac{\partial a}{\partial y}p+\frac{\partial b}{\partial y}q+\frac{\partial c}{\partial y}z\right)-cq-\frac{dD}{dx}-D\frac{\partial D}{\partial q}+(ap+bq+cz)\frac{\partial D}{\partial p}, \\ G &= -a, \quad \text{and} \quad J = -b. \end{aligned}$$

Hence the three equations (11) are reducible to

$$b = 0, \qquad \frac{dD}{dx} - (ap+cz)\frac{\partial D}{\partial p} + D\frac{\partial D}{\partial q} + aD - \left(\frac{\partial a}{\partial y}p + cq + \frac{\partial c}{\partial y}z\right) = 0.$$

The parabolic equation is integrable by our method if and only if $b=0$.

The system M is given by

$$\frac{\partial V}{\partial q} = 0, \qquad -\frac{\partial V}{\partial x} - p\frac{\partial V}{\partial z} + (ap+bq+cz)\frac{\partial V}{\partial p} = 0.$$

Since $\partial V/\partial q=0$, we have $b\partial V/\partial p=0$ from the second equation. If $b\neq 0$, then $\partial V/\partial p=0$. This gives $\partial V/\partial z=0$ and $\partial V/\partial x=0$. Hence the equation is Monge-integrable if and only if $b=0$.

EXAMPLE 2. Let us consider a linear hyperbolic equation $s+ap+bq+cz=0$, where a, b and c are functions of x and y. Its two characteristics are given by

$$dx = 0, \qquad dp + (ap+bq+cz)dy = 0, \tag{17}$$

and

$$dy = 0, \qquad dq + (ap+bq+cz)dx = 0, \tag{18}$$

respectively. Put $A=0$, $B=1$ and $C=-(ap+bq+cz)$. Then we have

$$E = \frac{dD}{dx} + \left(\frac{\partial a}{\partial y}p + \frac{\partial b}{\partial y}q + \frac{\partial c}{\partial y}z\right) + cq - (ap+bq+cz)a + bD,$$

$$F = -D\frac{\partial D}{\partial q} + (ap+bq+cz)\frac{\partial D}{\partial p},$$

$$G = \frac{\partial D}{\partial p}, \quad \text{and} \quad J = \frac{\partial D}{\partial q}.$$

The three equations (11) are reducible to

$$\frac{\partial D}{\partial p} = 0,$$

$$\frac{dD}{dx} - (ap+bq+cz)\frac{\partial D}{\partial q} + bD + \left(\frac{\partial a}{\partial y}p + \frac{\partial b}{\partial y}q + \frac{\partial c}{\partial y}z\right) + cq - a(ap+bq+cz) = 0.$$

This system S has its rank two, if and only if $H=\partial a/\partial x+ab-c=0$. Suppose that $H\neq 0$. Then the system S has its rank one, if and only if $H_1=2\partial a/\partial x-\partial b/\partial y+ab-c-\partial^2 \log H/\partial x\partial y=0$.

The equation is Monge-integrable with respect to the characteristics (18), if and only if $H=0$.

REMARK 2. Take a linear hyperbolic equation for which $H\neq 0$ and $H_1=0$. Then there exists such a solution of S that is not Lagrange-Charpit's system of any equation. In fact if every solution of S is Lagrange-Charpit's system of an equation, the equation is Monge-integrable. This is a contradiction.

4. A Bäcklund transformation of Laplace type. One of the two characteristics of the equation

$$(g, h)-k(g, f) = 0 \tag{19}$$

is given by

$$dg = 0, \qquad dh-kdf = 0. \tag{20}$$

We shall prove the following

PROPOSITION 3. *An equation* (19) *of Laplace type is not Monge-integrable with respect to the characteristics* (20).

Proof. We may assume that $f=x$, $g=y$ and $\partial h/\partial q \neq 0$, without losing the generality. Then the system M is generated by

$$\frac{\partial V}{\partial p} = 0, \qquad \frac{\partial h}{\partial q}\frac{dV}{dx}-\left(\frac{dh}{dx}-k\right)\frac{\partial V}{\partial q} = 0.$$

Since $\partial h/\partial p=\partial k/\partial p=0$, the second equation gives

$$\frac{\partial h}{\partial q}\frac{\partial V}{\partial z}-\frac{\partial h}{\partial z}\frac{\partial V}{\partial q} = 0, \qquad \frac{\partial h}{\partial q}\frac{\partial V}{\partial x}-\left(\frac{\partial h}{\partial x}-k\right)\frac{\partial V}{\partial q} = 0. \tag{21}$$

The first equation of (21) shows that V is a function $U(x, y, h)$ of x, y and h. Replacing

$$\frac{\partial V}{\partial x} \text{ and } \frac{\partial V}{\partial q} \text{ by } \frac{\partial U}{\partial x}+\frac{\partial U}{\partial h}\frac{\partial h}{\partial x} \text{ and } \frac{\partial U}{\partial h}\frac{\partial h}{\partial q}$$

respectively, we have

$$\frac{\partial h}{\partial q}\left(\frac{\partial U}{\partial x}-k\frac{\partial U}{\partial h}\right) = 0$$

from the second equation of (21). Since k is functionally independent of x, y and h, this equation is reducible to $\partial U/\partial x=\partial U/\partial h=0$. Hence the system M has its rank one.

Let us find a condition which the three functions g, h and k should satisfy in order that the equation

$$[g, h]s'-q'[g, k]+[h, k] = 0 \tag{22}$$

may be Monge-integrable. Here we assume that $[g, h]\neq 0$ and that $[g, k]/[g, h]$ and $[k, h]/[g, h]$ are functions of x $(=x')$, g, h and k.

The two characteristics of the equation (22) are given by

$$dy' = 0, \qquad dq'-\left(\frac{[k, h]}{[g, h]}+\frac{[g, k]}{[g, h]}q'\right)dx' = 0 \tag{23}$$

and

$$dx' = 0, \qquad dp'-\frac{[k, h]}{[g, h]}dy'-\frac{[g, k]}{[g, h]}dz' = 0 \tag{24}$$

respectively. We are interested in the system M with respect to (23). It is generated by

$$\frac{\partial V'}{\partial p'} = 0, \qquad \frac{dV'}{dx'}+\left(\frac{[k, h]}{[g, h]}+\frac{[g, k]}{[g, h]}q'\right)\frac{\partial V'}{\partial q'} = 0.$$

Applying Lagrange's method we have

$$\frac{\partial V'}{\partial z'}+\frac{\partial}{\partial p'}\left(\frac{[k, h]}{[g, h]}+\frac{[g, k]}{[g, h]}q'\right)\frac{\partial V'}{\partial q'} = 0$$

and

$$\frac{\partial^2}{\partial p'^2}\left(\frac{[k, h]}{[g, h]}+\frac{[g, k]}{[g, h]}q'\right)\frac{\partial V'}{\partial q'} = 0.$$

Hence the two functions $[k, h]/[g, h]$ and $[g, k]/[g, h]$ should be linear with respect to k, in order that the system M may have its rank two.

Let us put $[k, h]/[g, h]=\alpha k+\beta$ and $[g, k]/[g, h]=\gamma k+\delta$, where α, β, γ and δ are functions of x, g and h. Then the equation (22) is expressed in the form

$$s'-(\gamma p'q'+\alpha p'+\delta q'+\beta) = 0. \tag{25}$$

The system M is generated by

$$\frac{\partial V'}{\partial p'} = 0, \qquad \frac{\partial V'}{\partial z'}+(\alpha+\gamma q')\frac{\partial V'}{\partial q'} = 0, \qquad \frac{\partial V'}{\partial x'}+(\beta+\delta q')\frac{\partial V'}{\partial q'} = 0.$$

This system has its rank two if and only if we have

$$\frac{\partial \beta}{\partial h}+\alpha\delta = \frac{\partial \alpha}{\partial x}+\beta\gamma, \qquad \frac{\partial \delta}{\partial h} = \frac{\partial \gamma}{\partial x},$$

(see [4, p. 87]).

We shall prove the following

THEOREM. *An equation* (19) *of Laplace type is integrable by our method developed in the previous section* (*with respect to the characteristics* (20)), *if and only if the transformed equation* (25) *is Monge-integrable* (*with respect to the characteristics* (23)).

Proof. We may assume that $f=x$ and $g=y$. Then the two functions h and k depend only on x, y, z and q. The other characteristic which is different from (20) is given by

$$dx = 0, \qquad \frac{\partial h}{\partial q}dp+\left(\frac{dh}{dx}-k\right)dy = 0.$$

Put $A=0$, $B=\partial h/\partial q$ and $-C=dh/dx-k$. Then we have

$$E = \frac{\partial h}{\partial q}\left(\frac{dD}{dx}+\frac{d^2h}{dxdy}-\frac{dk}{dy}\right)-\left(\frac{dh}{dx}-k\right)\frac{\partial h}{\partial z}-D\frac{\partial k}{\partial q},$$

$$F = -D\left(\frac{\partial^2 h}{\partial y\partial q}+\frac{\partial^2 h}{\partial z\partial q}q+\frac{\partial D}{\partial q}\right)+\left(\frac{dh}{dx}-k\right)\frac{\partial D}{\partial p},$$

$$G = \frac{\partial h}{\partial q}\frac{\partial D}{\partial p} \quad\text{and}\quad J = \frac{\partial h}{\partial q}\left(\frac{\partial^2 h}{\partial y\partial q}+\frac{\partial^2 h}{\partial z\partial q}q+\frac{\partial D}{\partial q}\right).$$

The three equations (11) are reducible to

$$\frac{\partial D}{\partial p} = 0,$$

$$\frac{\partial h}{\partial q}\left(\frac{dD}{dx}+\frac{d^2 h}{dxdy}-\frac{dk}{dy}\right)-\frac{\partial k}{\partial q}D-\left(\frac{dh}{dx}-k\right)\left(\frac{\partial D}{\partial q}+\frac{\partial^2 h}{\partial y\partial q}+\frac{\partial^2 h}{\partial z\partial q}q+\frac{\partial h}{\partial z}\right) = 0.$$

In this system S, let us replace D by $u-dh/dy$: $-D=dh/dy-u$. Then we get

$$\frac{\partial u}{\partial p} = 0, \qquad \frac{\partial h}{\partial q}\frac{du}{dx}-\left(\frac{dh}{dx}-k\right)\frac{\partial u}{\partial q}-\frac{\partial k}{\partial q}u+[k, h] = 0.$$

Since $\partial h/\partial p=\partial k/\partial p=0$, the second equation gives

$$\tag{26} \begin{gathered} \frac{\partial h}{\partial q}\frac{\partial u}{\partial z}-\frac{\partial h}{\partial z}\frac{\partial u}{\partial q} = 0, \\ \frac{\partial h}{\partial q}\frac{\partial u}{\partial x}-\left(\frac{\partial h}{\partial x}-k\right)\frac{\partial u}{\partial q}-\frac{\partial k}{\partial q}u+[k, h] = 0. \end{gathered}$$

The first equation of (26) shows that u is a function $v(x, y, h)$ of x, y and h. Replacing

$$\frac{\partial u}{\partial x} \quad\text{and}\quad \frac{\partial u}{\partial q} \quad\text{by}\quad \frac{\partial v}{\partial x}+\frac{\partial v}{\partial h}\frac{\partial h}{\partial x} \quad\text{and}\quad \frac{\partial v}{\partial h}\frac{\partial h}{\partial q}$$

respectively, we obtain

$$\tag{27} \frac{\partial v}{\partial x}+k\frac{\partial v}{\partial h}-\left(\frac{\partial k}{\partial q}\Big/\frac{\partial h}{\partial q}\right)v+[k, h]\Big/\frac{\partial h}{\partial q} = 0$$

from the second equation of (26).

Here our assumption allows us the following replacement:

$$\frac{\partial k}{\partial q}\Big/\frac{\partial h}{\partial q} = \gamma k+\delta \quad\text{and}\quad [k, h]\Big/\frac{\partial h}{\partial q} = -\alpha k-\beta,$$

where the four functions α, β, γ and δ depend only on x, y and h. Since k is functionally independent of x, y and h, the equation (27) is reducible to

$$\frac{\partial v}{\partial x}-\delta v-\beta = 0, \qquad \frac{\partial v}{\partial h}-\gamma v-\alpha = 0.$$

Hence the system S has its rank one if and only if

$$\frac{\partial \beta}{\partial h}+\alpha\delta = \frac{\partial \alpha}{\partial x}+\beta\gamma, \qquad \frac{\partial \delta}{\partial h} = \frac{\partial \gamma}{\partial x}.$$

This is the necessary and sufficient condition in order that the transformed equation (25) may be Monge-integrable with respect to the characteristics (23).

5. **Examples.**

EXAMPLE 3. Put $f=x$, $g=y$, $h=\int dq/\phi(q)$, $k=e^z$. Then we have

$$[g, h] = -1/\phi(q), \qquad [g, k] = 0, \qquad [k, h] = -(q/\phi)e^z = qk[g, h].$$

Hence it is a Bäcklund transformation of Laplace type. By this transformation the equation $s-e^z\phi(q)=0$ is transformed to $s'-\psi(z')p'=0$, where $q=\psi(h)$ is the inverse function of $h=\int dq/\phi(q)$. The transformed equation is Monge-integrable, because $\alpha=\psi(h)$ and $\beta=\gamma=\delta=0$.

Suppose that $\phi(q)=1$. Then the original equation is Liouville's one (see [4, p. 97]) $s-e^z=0$. The infinitesimal contact transformation which leaves Liouville's equation invariant is given by the following generating function

$$\psi = \psi_1(x)p+\psi_2(y)q+d\psi_1/dx+d\psi_2/dy.$$

From this function the infinitesimal transformation

$$X\partial/\partial x+ Y\partial/\partial y+Z\partial/\partial z+P\partial/\partial p+Q\partial/\partial q$$

is generated by

$$X = -\frac{\partial\psi}{\partial p}, \qquad Y = -\frac{\partial\psi}{\partial q}, \qquad Z = \psi-\frac{\partial\psi}{\partial p}p-\frac{\partial\psi}{\partial q}q,$$

$$P = \frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial z}p, \qquad Q = \frac{\partial\psi}{\partial y}+\frac{\partial\psi}{\partial z}q.$$

Hence Liouville's equation admits an infinite Lie group of parameters $\psi_1(x)$ and $\psi_2(y)$. However, it can be proved that Liouville's equation is not transformed to a linear equation by any contact transformation.

EXAMPLE 4. Put $f=x$, $g=y$, $h=q+a(x, y)e^{-z}$, $k=e^z$. Then we have

$$[g, h] = -1, \qquad [g, k] = 0$$

and

$$[k, h] = -e^zq = -e^z(h-ae^{-z}) = -hk+a.$$

Hence it is a Bäcklund transformation of Laplace type. By this transformation the equation

$$s-ae^{-z}p+(\partial a/\partial x)e^{-z}-e^z = 0$$

is transformed to $s'-z'p'+a=0$. This equation is Monge-integrable, because $\alpha=h$, $\beta=-a$ and $\gamma=\delta=0$.

Suppose that $a(x, y)=1$. Then the infinitesimal contact transformation which leaves the equation $s-e^{-z}p-e^z=0$ invariant is given by the generating function

$\psi=(2c_1x+c_2)p-(c_1y+c_3)q+c_1$, where c_1, c_2 and c_3 are constants. Hence this equation admits a (finite) Lie group of three parameters.

Finally we shall give an example of a Bäcklund transformation which is not of Laplace type. It will be shown that to this Bäcklund transformation our theorem cannot be applied.

EXAMPLE 5. (Levy's transformation, see [3, p. 94].) Put

$$f = x, \qquad g = y, \qquad h = q+(a(x, y)+\alpha(x, y))z,$$
$$k = \alpha p-b(x, y)q+\left(\frac{\partial a}{\partial x}+\frac{\partial \alpha}{\partial x}-c(x, y)\right)z.$$

Here α is a function which satisfies Levy's condition

$$\frac{\partial^2 \log \alpha}{\partial x\partial y}-\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial y}\frac{H}{\alpha}+K-H = 0,$$

where $H=\partial a/\partial x+ab-c$ and $K=\partial b/\partial y+ab-c$. Then we have

$$[f, g] = [f, h] = 0, \qquad [f, k] = -\alpha, \qquad [g, h] = -1, \qquad [g, k] = b$$

and

$$[k, h] = \left(\alpha a+\alpha^2-\frac{\partial \alpha}{\partial y}\right)p+\left(\frac{\partial b}{\partial y}-ab-\alpha b-\frac{\partial a}{\partial x}-\frac{\partial \alpha}{\partial x}+c\right)q$$
$$+\left(\alpha\frac{\partial a}{\partial x}+\alpha\frac{\partial \alpha}{\partial x}-b\frac{\partial a}{\partial y}-b\frac{\partial \alpha}{\partial y}-\frac{\partial^2 a}{\partial x\partial y}-\frac{\partial^2 \alpha}{\partial x\partial y}+\frac{\partial c}{\partial y}\right)z.$$

Levy's condition is a necessary and sufficient condition in order that $[k, h]$ may be a function of x, y, h and k.

By Levy's transformation the equation $s+ap+bq+cz=0$ is transformed to $s'+a'p'+b'q'+c'z'=0$, where

$$a' = a-\frac{1}{\alpha}\frac{\partial \alpha}{\partial y}, \qquad b' = b \quad \text{and} \quad c' = c+K-H-\frac{\partial \alpha}{\partial x}-\frac{b}{\alpha}\frac{\partial \alpha}{\partial y}.$$

Let H' be the first Laplace invariant of the transformed equation. Then the following identities are known:

$$H' = H+\frac{\partial}{\partial y}\frac{H}{\alpha} \quad \text{and} \quad H' = H_1+\frac{\partial}{\partial x}\frac{\alpha H'}{H},$$

where H_1 is the second Laplace invariant of the original equation. Hence $H'=0$ gives $H_1=0$. However, the converse is not true. For example, let us take a, b, c and α as follows:

$$a = x, \qquad b = 2y, \qquad c = 2xy, \qquad \alpha = x.$$

Then we get $H=1$, $K=2$, and $H_1=0$. The function α satisfies Levy's condition. However, we obtain $H'=1\neq 0$.

BIBLIOGRAPHY

1. G. Darboux, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, deuxième partie, Gauthier-Villars, Paris, 1889.

2. ———, *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, troisième partie, Gauthier-Villars, Paris, 1894.

3. A. R. Forsyth, *Theory of differential equations.* Part IV: *Partial differential equations.* Vol. VI, Cambridge Univ. Press, New York, 1906.

4. E. Goursat, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes.* Tome I, Hermann, Paris, 1896.

5. ———, *Leçons sur l' intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes.* Tome II, Hermann, Paris, 1898.

6. ———, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, Hermann, Paris, 1921.

7. ———, *Le problème de Bäcklund*, Memor. Sci. Math., fasc. VI, Gauthier-Villars, Paris, 1925.

8. H. Lewy, *Über das Anfangswertproblem einer hyperbolischen nichtlinearen partiellen Differentialgleichung zweiter Ordnung mit zwei unabhängigen Veränderlichen*, Math. Ann. **98** (1928), 179–191.

9. S. Lie, *Theorie der Transformationsgruppen*, Zweiter Abschnitt, Teubner, Leipzig, 1890.

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