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ON THE SHRINKABILITY OF DECOMPOSITIONS OF 3-MANIFOLDS(1)

BY
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Abstract. An upper semicontinuous decomposition G of a metric space M is said to be *shrinkable* in case for each covering $\mathscr{U}$ of the union of the nondegenerate elements, for each $\varepsilon > 0$, and for an arbitrary homeomorphism h from M onto M, there exists a homeomorphism f from M onto itself such that

(1) if $x \in M - (\bigcup \{U : U \in \mathscr{U}\})$, then $f(x) = h(x)$,

(2) for each $g \in G$, (a) $\operatorname{diam} f[g] < \varepsilon$ and (b) there exists $D \in \mathscr{U}$ such that $h[D] \supset h[g] \cup f[g]$.

Our main result is that if G is a cellular decomposition of a 3-manifold M, then $M/G = M$ if and only if G is shrinkable. We also define concepts of local and weak shrinkability, and we show the equivalence of the various types of shrinkability for certain cellular decompositions. Some applications of these notions are given, and extensions of theorems of Bing and Price are proved.

1. **Introduction.** In [3] Armentrout proved the following theorem: *Let G be a cellular 0-dimensional upper semicontinuous decomposition of E^3. Then E^3/G is homeomorphic to E^3 if and only if G is weakly shrinkable* (definition below). In this same paper he also raised the question as to whether a suitable definition of shrinkability could be found such that the following theorem is true: *Suppose G is a cellular decomposition of a 3-manifold, M. Then M/G is homeomorphic to M if and only if G is shrinkable.* By using a slightly modified version of a shrinkability definition due to McAuley [10], we obtain in §3 one solution to Armentrout's query.

§4 is devoted to an apparently weaker form of shrinkability. In the case of 0-dimensional decompositions of n-manifolds, the two notions of shrinkability are found to coincide, and this leads to the extension of a certain well-known result of Bing on the shrinkability of decompositions of E^3.

A local shrinkability criterion is investigated in §5, and its equivalence under certain conditions to the "global" versions mentioned above is demonstrated.

2. **Notation and definitions.** Let G be an upper semicontinuous decomposition (henceforth, referred to simply as a decomposition) of a space X. Then the decom-

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position space associated with G will be denoted by X/G, the natural projection mapping from X onto X/G by P, and H_G will denote the collection of nondegenerate elements of G. An open set, U, in X will be said to be *saturated with respect to G* (or simply *saturated* whenever the context is clear) in case $U=P^{-1}[P[U]]$.

An *n-manifold* is a separable metric space with the property that each point has a neighborhood which is an n-cell. If M is an n-manifold, a point p of M is an *interior point* of M if and only if p has an open neighborhood in M which is an open n-cell. The *interior* of M is the set of all interior points and the *boundary* of M is $M-$interior M. It should perhaps be noted that an n-manifold as defined in this paper is referred to as an n-manifold with boundary in the papers of Bing and Armentrout.

Suppose that M is an n-manifold. G is said to be a *monotone decomposition* of M in case each element of G is compact and connected. Furthermore, we require that each nondegenerate element of G lie in the interior of M. A subset K of M is said to be *cellular* in case there exists a sequence $C_1, C_2, \ldots$ of n-cells in M such that $C_i \subset$ interior C_{i-1}, and $\bigcap_{i=1}^{\infty} C_i = K$. Cellular subsets of a manifold must then lie in the interor of the manifold. A decomposition of an n-manifold is said to be cellular in case each nondegenerate element is cellular.

If M is a manifold, by a *triangulation* of M is meant a simplicial complex, T, such that (1) $M=\bigcup\{\sigma : \sigma \in T\}$ and (2) T is locally finite in the sense that each point of M has a neighborhood which intersects only finitely many sets of T. A *subdivision* of a triangulation, T, is a triangulation, S, of the simplicial complex, T, for which each simplex of S is contained in a simplex of T.

If T is a triangulation of M, and σ is a simplex of T, then

$$N(\sigma, T) = \bigcup\{\sigma' : \sigma' \in T \text{ and } \sigma \cap \sigma' \neq \varnothing\},$$
$$O(\sigma, T) = N(\sigma, T) - \bigcup\{\sigma' : \sigma' \cap \sigma = \varnothing\}.$$

Clearly $O(\sigma, T) \subset N(\sigma, T)$, and $O(\sigma, T)$ is an open set.

If C is a collection of subsets of a topological space, then C^* will denote $\bigcup\{c : c \in C\}$. If M is a set Cl M denotes the closure of M, Int M denotes the topological interior of M, and Bd M denotes the topological boundary of M. If ε is a positive number and A is a subset of a metric space, then $S(A, \varepsilon)$ denotes the ε-neighborhood of A.

3. **Shrinkability.** A decomposition, G, of a metric space, M, will be said to be *weakly shrinkable* in case for each open set, U, containing H_G^*, and each positive number ε, there is a homeomorphism, h, from M onto M such that (1) if $g \in H_G$, then diam $h[g] < \varepsilon$, and (2) if $x \in M-U$, then $h(x)=x$. A decomposition, G, is said to have a 0-dimensional set of nondegenerate elements in case $P[H_G^*]$ is 0-dimensional. In [6], Bing proved one of the most useful theorems concerning weakly shrinkable monotone decompositions of E^3 with 0-dimensional sets of nondegenerate elements: *If G is a monotone decomposition of E^3 such that $P[H_G^*]$*

is 0-dimensional and G is weakly shrinkable, then E^3/G is homeomorphic to E^3. We shall extend this result to 0-dimensional decompositions of n-manifolds.

The following partial converse to the above theorem of Bing is due to Armentrout [3]: *If G is a cellular decomposition of E^3 such that $P[H_G^*]$ is 0-dimensional and E^3/G is homeomorphic to E^3, then G is weakly shrinkable.* We note that the 0-dimensional restriction is not needed in order that the result hold, and, furthermore, that E^3 may be replaced by an arbitrary 3-manifold.

A decomposition, G, of a metric space, M, is said to be *shrinkable* in case for each covering $\mathscr{U}$ of H_G^* by saturated open sets of M, for each $\varepsilon > 0$, and for an arbitrary homeomorphism, h, from M onto M, there exists a homeomorphism, f, from M onto itself such that

(1) if $x \in M - \mathscr{U}^*$, then $f(x) = h(x)$,

(2) for each $g \in G$, (a) $\operatorname{diam} f[g] < \varepsilon$ and (b) there exists $D \in \mathscr{U}$ such that $h[D] \supset h[g] \cup f[g]$.

THEOREM 0 (MCAULEY [10]). *If G is a decomposition of a complete metric space, M, such that G is shrinkable and M is locally compact, then M/G is homeomorphic to M.*

It should be noted that in McAuley's definition of shrinkability [10], part (2) of our definition becomes for each $g \in H_G$, etc. Theorem 0 is also valid when shrinkability is so defined.

THEOREM 1. *Suppose G is a cellular decomposition of a 3-manifold, M. Then M/G is homeomorphic to M if and only if G is shrinkable.*

Proof. If G is shrinkable, then M/G is homeomorphic to M by the above theorem of McAuley. Suppose now that M/G is homeomorphic to M. The proofs of the first two lemmas stated below are essentially based on a "projection approximation" result of Armentrout. Armentrout showed in [4], that if G is a cellular decomposition of a 3-manifold, M, such that M/G is homeomorphic to M, then there exist homeomorphisms from M onto M/G arbitrarily close to the projection map. Lemma 1.1 follows directly from Armentrout's construction of these approximating homeomorphisms.

LEMMA 1.1. *Suppose G is a cellular decomposition of a 3-manifold, M, such that M/G is homeomorphic to M. Let U be an open set containing H_G^*. Let T be a triangulation of M/G. Let f be a positive real valued continuous function defined on M. Then there exists a homeomorphism, h, from M onto M/G such that*

(1) $d(P(x), h(x)) \leqq f(x)$, *for each* $x \in M$,

(2) $P(x) = h(x)$, *if* $x \in M - U$,

(3) $h^{-1}[\sigma] \subset P^{-1}[O(\sigma, T)]$, *for each* $\sigma \in T$.

The proof of the next lemma is almost identical to Price's proof of the principal theorem of [12], and is omitted here.

LEMMA 1.2. *Let G be a cellular decomposition of a 3-manifold, M, such that M is homeomorphic to M/G. Then G is weakly shrinkable.*

Techniques found in Borsuk [8] may be used to establish Lemam 1.3.

LEMMA 1.3. *Suppose $\mathscr{U}$ is an open covering of a 3-manifold, M. Then there exists a triangulation, T, of M with the property that for each simplex $\sigma \in T$, there is a set $U \in \mathscr{U}$ such that $N(\sigma, T) \subset U$.*

We first show that if M and M/G are homeomorphic, then a slightly weakened version of shrinkability is obtained. In the following lemma the arbitrary homeomorphism, h, of the definition of shrinkability is replaced by the identity homeomorphism. In the proof of Lemma 1.4 we shall use notation consistent with that of Price [12], and the reader may fill in details of the argument by referring to that paper.

LEMMA 1.4. *Suppose G is a cellular decomposition of a 3-manifold, M, such that M/G is homeomorphic to M. Let $\mathscr{U}$ be a covering of H_G^* by saturated open sets of M and suppose $\varepsilon > 0$. Then there exists a homeomorphism, f, from M onto itself such that*

(1) *$f(x) = x$, if $x \in M - \mathscr{U}^*$,*

(2) *for each $g \in G$* (a) $\operatorname{diam} f[g] < \varepsilon$ *and* (b) *there exists $D \in \mathscr{U}$ such that $D \supset g \cup f[g]$.*

Proof. By Lemma 1.2, G is weakly shrinkable, and, thus, we essentially need to find a homeomorphism which shrinks the nondegenerate elements of G, but does not move them too far. Consider the collection $\{P[U] : U \in \mathscr{U}\}$. Since M/G is a metric space, there exists a locally finite refinement, $\mathscr{Z}$, of this collection covering $P[H_G^*]$. Let $\mathscr{W} = \{P^{-1}[Z] : Z \in \mathscr{Z}\}$. Then it is easily seen that $\mathscr{W}$ is a locally finite refinement of $\mathscr{U}$ which covers H_G^* and consists of saturated open sets.

Since $\mathscr{Z}^*$ is an open subset of M/G, it is a 3-manifold, and, hence, by Lemma 1.3 there exists a triangulation, T, of $\mathscr{Z}^*$ with the property that for each simplex σ of T, there is a set $Z \in \mathscr{Z} N(\sigma, T) \subset Z$. Armentrout has shown in [2] that $\mathscr{W}^*$ and $P[\mathscr{W}^*] = \mathscr{Z}^*$ are homeomorphic, and it follows from his work in this paper that there exists a homeomorphism, h_0, from M/G onto M such that

(1) for each $\sigma \in T$, $h_0[\sigma] \subset P^{-1}[O(\sigma, T)]$,

(2) for $x \in M - \mathscr{W}^*$, $h_0^{-1}(x) = P(x)$.

Suppose $x \in \mathscr{W}^*$. Then $P(x) \in \sigma$, for some $\sigma \in T$, and $O(\sigma, T)$ is contained in some set $Z \in \mathscr{Z}$. Since $h_0 P(x) \in P^{-1}[O(\sigma, T)]$, it is true that $h_0 P(x)$ and x will both lie in $P^{-1}[Z] \in \mathscr{W}$.

We now define a positive valued lower semicontinuous function on $\mathscr{W}^*$. Let $x \in \mathscr{W}^*$, and let $W_1^x, W_2^x, \ldots, W_m^x$ be those and only those sets of $\mathscr{W}$ which contain both x and $h_0 P(x)$. Define

$$R_x = \{R : R \text{ is open in } M/G,\ P(x) \in R, \text{ and } h_0[R] \text{ is contained in } W_i^x, \text{ for some } i = 1, 2, \ldots, m\}.$$

Let $\varepsilon_R = d(P(x), M/G - R)$, and let $\varepsilon_x = \sup\{\varepsilon_R : R \in R_x\}$. Define a function, $\hat{r}$, on $\mathscr{W}^*$ by $\hat{r}(x) = \varepsilon_x/2$. Since h_0 and h_0P are both continuous, a straightforward argument by contradiction will show that $\hat{r}$ is a positive valued lower semicontinuous function. Let $r(x) = \min\{\hat{r}(x), d(x, M - \mathscr{W}^*)\}$ for each $x \in \mathscr{W}^*$. Then r is also lower semicontinuous and positive, and, hence, there exists a continuous real valued function, h, defined on $\mathscr{W}^*$ such that $0 < h < r$. We extend h to all of M by setting $h(x) = 0$, if $x \in M - \mathscr{W}^*$.

Let T_1 be a subdivision of T such that if $\sigma \in T_1$, then diam $h_0[\sigma] < \varepsilon/5$. Let h_1 be a homeomorphism from M/G onto M such that

(1) $d(P(x), h_1^{-1}(x)) \leqq h(x)$, for each $x \in M$,

(2) $h_1[\sigma] \subset P^{-1}[O(\sigma, T_1)]$, for each $\sigma \in T_1$.

Let $f = h_0 h_1^{-1}$. It then follows as in the proof of Lemma 1.2 (Price, [12]) that f will shrink the nondegenerate elements of G to the required size. We now show that for each $g \in G$ and lying in $\mathscr{U}^*$ there exists a set $D \in \mathscr{U}$, such that $g \cup f[g] \subset D$. This is trivially true if g does not lie in $\mathscr{W}^*$ since in this case $h_0 h_1^{-1}[g] = g$.

Suppose $g \in G$ and $g \subset W_1^x \cap W_2^x \cap \cdots \cap W_m^x$, where $x \in g$ and W_i^x is as defined previously. Then $h_1^{-1}[g] \subset S(P(x), \varepsilon_x/2)$. But $S(P(x), \varepsilon_x/2)$ is contained in at least one set $R \in R_x$, and $h_0[R]$ lies in W_i^x for some $i = 1, 2, \ldots, m$. Therefore, $g \cup h_0 h_1^{-1}[g] \subset W_i^x$, and, of course, there exists a set $D \in \mathscr{U}$, such that $W_i^x \subset D$. To conclude the proof of Lemma 1.4 we need only note that if $x \in M - \mathscr{W}^*$, then $f(x) = h_0 h_1^{-1}(x) = x$, since outside of $\mathscr{W}^*$, $h_0^{-1}(x) = h_1^{-1}(x) = P(x)$.

We now consider the general case where h is an arbitrary homeomorphism mapping M onto itself, $\mathscr{U}$ is a saturated open cover of H_G^*, and $\varepsilon > 0$ is given. We form a decomposition of $h[M] = M$ whose elements are of the form $h[g]$, for $g \in G$. The resulting decomposition space is homeomorphic to M/G and, hence, to M. Therefore, by the previous lemma we may obtain a homeomorphism, f, from $h[M]$ onto itself such that

(1) $fh(x) = h(x)$, if $x \in M - \mathscr{U}^*$,

(2) diam $f[h[g]] < \varepsilon$, for each $g \in H_G$,

(3) there exists a set $D \in \mathscr{U}$ such that $h[D] \supset fh[g] \cup h[g]$.

The shrinking homeomorphism will then be fh, and Theorem 1 is proved.

4. **Weakly shrinkable decompositions.** The following question is of considerable interest: Suppose G is a monotone decomposition of E^3 such that (1) E^3/G is homeomorphic to E^3, and (2) $P[H_G^*]$ is 0-dimensional. Does it follow that each element of G is cellular? Some partial answers have been obtained—see, for example, Martin [11] or Kwun [9]. We show in Lemma 2.4 that if G is a shrinkable decomposition of an n-manifold, then each nondegenerate element of G is cellular. Lemma 2.5 notes that for 0-dimensional decompositions of a manifold the notions of weakly shrinkable and shrinkable coincide. We shall always assume for decompositions of n-manifolds that no nondegenerate element of the decomposition intersects the boundary of the manifold.

THEOREM 2. *Suppose G is a decomposition of an n-manifold, M, into compact sets. Furthermore, assume that G is weakly shrinkable and $P[H_G^*]$ is 0-dimensional.*

Then (1) *G is cellular, and* (2) *M/G is homeomorphic to M.*

LEMMA 2.1. *Let G be a monotone decomposition of an n-manifold M, such that $P[H_G^*]$ is 0-dimensional. Then the following two statements are equivalent:*

(1) *If U is any open set containing H_G^* and ε is any positive number, then there exists a homeomorphism, h, from M onto M such that* (a) *if $x \in M-U$, $h(x)=x$, and* (b) *if $g \in H_G$,* diam $h[g]<\varepsilon$.

(2) *If U is any open set containing H_G^*, ε is any positive number, and f is any homeomorphism from M onto M, then there exists a homeomorphism, h, from M onto M such that* (a) *if $x \in M-U$, $h(x)=f(x)$, and* (b) *if $g \in H_G$,* diam $h[g]<\varepsilon$.

Proof. In [1], Armentrout has proved this lemma for $M=E^n$. With obvious small changes, a similar proof holds for arbitrary n-manifolds.

LEMMA 2.2. *Suppose G is a monotone decomposition of an n-manifold, M, such that $P[H_G^*]$ is 0-dimensional. Let $\mathscr{U}$ be an open covering in M of H_G^* such that each set of $\mathscr{U}$ is bounded and saturated.*

Then there exists an open covering, $\mathscr{V}$, in M of H_G^* such that

(1) *the sets of $\mathscr{V}$ are mutually disjoint,*

(2) *each set of $\mathscr{V}$ lies in some set of $\mathscr{U}$,*

(3) *if $V \in \mathscr{V}$, then* (Bd V) $\cap H_G^*=\varnothing$,

(4) *if h is a homeomorphism from M onto M with the property that for each $V \in \mathscr{V}$ and each $x \in$* Bd V, *$h(x)=x$, then for each $V' \in \mathscr{V}$, $h[V']=V'$.*

Proof. Armentrout has shown in [1] that there exists an open covering, $\mathscr{W}$, in M of H_G^* which satisfies properties (1), (2), and (3). For each $W \in \mathscr{W}$, let C_{W_1}, $C_{W_2}, \ldots$ be the components of W. Then for each C_{W_i}, let $V_{W_i}=C_{W_i}-\{p_{W_i}\}$, where p_{W_i} is a point lying in $C_{W_i}-H_G^*$. $\mathscr{V}$ is then defined to be the covering of H_G^* consisting of all the sets, V_{W_i}, for $W \in \mathscr{W}$.

Suppose h is a homeomorphism from M onto M which leaves the boundary of each $V \in \mathscr{V}$ pointwise fixed. Let $V=V_{W_i}$ be a member of $\mathscr{V}$. Then V is connected, and, hence, either $h[V]=V$ or $h[V] \cap V=\varnothing$. But $h(p_{W_i})=p_{W_i}$, and the desired conclusion follows.

LEMMA 2.3. *Suppose G is a 0-dimensional decomposition of an n-manifold, M, into compact sets. If G is weakly shrinkable, then G is monotone.*

Proof. Suppose $g' \in H_G$ and g' is not connected. Let C_1 and C_2 be two distinct components of g'. Since M is locally connected, we may obtain two connected open sets, U_1 and U_2, containing respectively C_1 and C_2 such that

(1) $d(\text{Cl } U_1, \text{Cl } U_2)=r>0$,

(2) (Bd U_i) $\cap H_G^*=\varnothing$, for $i=1, 2$,

(3) if h is a homeomorphism from M onto M such that for each $x \in \text{Bd } U_1 \cup \text{Bd } U_2$, $h(x)=x$, then $h[U_i]=U_i$, for $i=1, 2$ (see the proof of Lemma 2.2).

Let $U_3=M-(\text{Cl } U_1 \cup \text{Cl } U_2)$. Since G is weakly shrinkable, there exists a homeomorphism, h, from M onto M such that

(1) if $g \in H_G$, then diam $h[g]<r$,

(2) if $x \in M-(U_1 \cup U_2 \cup U_3)$, then $h(x)=x$.

Since $h[U_1]=U_1$ and $h[U_2]=U_2$, it is clear that diam $h[g']>r$, an impossibility.

LEMMA 2.4. *Let G be a shrinkable decomposition of an n-manifold, M, into compact sets. Then G is cellular.*

Proof. Let $g \in H_G$. Let W be any saturated open set which lies in the interior of M and contains g. We shall find an n-cell, C, such that $g \subset \text{Int } C \subset C \subset W$. Let V be an open set with compact closure such that $g \subset V \subset \text{Cl } V \subset W$. We cover Cl V with a finite number of n-cells, $C_1, C_2, \ldots, C_m$ where $C_i \subset W$, for each i. We now cover H_G^* with saturated open sets in the following manner. First choose U_g such that $g \subset U_g \subset V$. If $g' \neq g$, choose $U_{g'}$ such that $g' \subset U_{g'}$ and $g \cap U_{g'}=\varnothing$. Furthermore, the U's are chosen such that if $g' \subset W$, then $U_{g'} \subset W$, and if $g' \cap W =\varnothing$, then $U_{g'} \cap (\bigcup_{i=1}^m C_i)=\varnothing$.

Let δ be a positive number such that if $A \subset V$ and diam $A<\delta$, then for some i, $A \subset C_i$. Since G is shrinkable, there exists a homeomorphism, h, from M onto itself such that for each $g' \in H_G$, diam $h[g']<\delta$. Furthermore, it must be the case that $h[g] \subset U_g$, since U_g is the only element of the cover containing g. $h[g]$ lies in some C_i, say C_1. Let $C=h^{-1}[C_1]$. Since whenever $g' \cap W=\varnothing$, $U_{g'} \cap C_1=\varnothing$, and if $x \in M-\bigcup_{g' \in H_G} U_{g'}$, $h(x)=x$, it follows that $C \subset W$. It is now easy to construct a decreasing sequence of n-cells whose intersection is g.

LEMMA 2.5. *Suppose G is a 0-dimensional decomposition of an n-manifold, M, into compact sets. Then G is weakly shrinkable if and only if G is shrinkable.*

Proof. Clearly, if G is shrinkable, then G is weakly shrinkable. Suppose now that G is weakly shrinkable and, hence, by Lemma 2.3, monotone. Let h be a homeomorphism from M onto itself, $\mathscr{U}$ a saturated open cover of H_G^*, and $\varepsilon>0$. Let $\mathscr{V}$ be a refinement of $\mathscr{U}$ satisfying all the conclusions of Lemma 2.2. Since G is weakly shrinkable, we may obtain (using Lemma 2.1) a homeomorphism, f, from M onto itself such that

(1) for each $g \in H_G$, diam $f[g] < \varepsilon$,

(2) if $x \in M-\mathscr{V}^*$, then $f(x)=h(x)$.

Suppose $g \in G$. We must show that $f[g] \cup h[g] \subset h[D]$, for some set $D \in \mathscr{U}$. g lies in some set $V \in \mathscr{V}$. Consider the space homeomorphism, $h^{-1}f$. This homeomorphism is the identity on Bd V, and, thus, by Lemma 2.2, $h^{-1}f[V]=V$ or $f[V]=h[V]$. Therefore, $f[g] \cup h[g] \subset h[V] \subset h[D]$, for some $D \in \mathscr{U}$, which completes the proof.

Proof of Theorem 2. G is cellular by Lemma 2.5 and Lemma 2.4. Since by Lemma 2.5, G is shrinkable, it follows from Theorem 0 that M/G is homeomorphic to M.

THEOREM 3. *Let G be a 0-dimensional cellular decomposition of a 3-manifold, M, such that M/G is homeomorphic to M. Let W be a saturated open subset of M. Let G' be the decomposition of M where $H_{G'}=\{g \in H_G : g \cap W=\varnothing\}$. Then M/G' is homeomorphic to M.*

LEMMA 3.1 (ARMENTROUT [1]). *Under the hypotheses of Lemma 2.2 we may assume that $\mathscr{V}$ has the following additional property:*

If for each $V \in \mathscr{V}$, f_V is a homeomorphism from M onto itself such that for $x \in M - V$, $f_V(x)=x$, then the function, f, defined as follows is a homeomorphism

(1) *if $x \in V$, then $f(x)=f_V(x)$,*

(2) *if $x \in M-\mathscr{V}^*$, then $f(x)=x$.*

Proof of Theorem 3. We show that G' is weakly shrinkable. Let $\varepsilon>0$, and suppose $H_{G'}^*$ is contained in an open set, U. Let V be an open set such that (1) $H_{G'}^* \subset V \subset U$ and (2) V is saturated with respect to G. Then the sets V and W form a saturated open covering of H_G^*. Let $\mathscr{Z}$ be a refinement of this cover which satisfies all the conclusions of Lemma 2.2 and Lemma 3.1.

Since G is weakly shrinkable (Lemma 1.2), there exists a homeomorphism, h, from M onto M such that

(1) for each $g \in H_G$, diam $h[g]<\varepsilon$,

(2) if $x \in M-\mathscr{Z}^*$, $h(x)=x$,

(3) if $Z \in \mathscr{Z}$, $h[Z]=Z$.

Let $\hat{\mathscr{Z}}=\{Z \in \mathscr{Z}$: there exists $g \in H_{G'}$ such that $g \subset Z\}$. Note that $\hat{\mathscr{Z}}^* \subset V \subset U$. Define a function, $\hat{h}$, by

(1) if $x \in \hat{\mathscr{Z}}^*$, $\hat{h}(x)=h(x)$,

(2) $\hat{h}(x)=x$, otherwise.

By Lemma 3.1, $\hat{h}$ is a homeomorphism, and it is not difficult to verify that $\hat{h}$ shrinks elements of $H_{G'}$ to a diameter of less than ε and is the identity outside of U.

COROLLARY 3.1. *Let G be a 0-dimensional cellular decomposition of a 3-manifold, M, such that M/G is homeomorphic to M. Suppose G' is a decomposition of M such that* (1) *$H_{G'} \subset H_G$ and* (2) *$H_{G'}^*$ is closed in M. Then M/G' is homeomorphic to M.*

COROLLARY 3.2. *Let G be a 0-dimensional cellular decomposition of a 3-manifold, M, such that M/G is homeomorphic to M. Let δ be a positive number. Let G' be the decomposition of M where $H_{G'}=\{g \in H_G$: diam $g \geqq \delta\}$. Then M/G' is homeomorphic to M.*

5. **Locally shrinkable decompositions.** A potentially useful plan of attack for determining the nature of a decomposition space is to consider local properties of the decomposition. In general, however, simple consideration of the nature of the

nondegenerate elements does not prove to be of much benefit. For instance, Bing [7] has shown that there exists a countable collection of cellular continua in E^3 such that the resulting decomposition space is not E^3. In addition, Bing [5] has exhibited an uncountable collection of tame arcs in E^3 such that the associated decomposition fails to be E^3. If we consider, however, the manner in which the elements "fit together" as well as their particular characteristics (cellularity, ANR, etc.) various results may be obtained. For example, Price [13] has proved this theorem:

Let G be a 0-dimensional monotone decomposition of E^n. Assume that for each $g \in H_G$, there exists a sequence $\{B_i^g\}$ of n-cells such that for each positive integer i, (1) $B_{i+1}^g \subset \operatorname{Int} B_i^g$, (2) $\bigcap_{i=1}^{\infty} B_i^g = g$, *and* (3) $(\operatorname{Bd} B_i^g) \cap H_G^* = \varnothing$. *Then E^n/G is homeomorphic to E^n.*

We obtain a somewhat stronger result if H_G^* is a G_δ set and the decomposition satisfies the following local shrinkability condition. Suppose G is a 0-dimensional of a metric space, M. Then G is said to be *locally shrinkable* in case for each $g \in G$, and for each open set, U, containing g, there exists an open set, U_g, such that

(1) $g \subset U_g \subset U$,

(2) $(\operatorname{Bd} U_g) \cap H_G^* = \varnothing$,

(3) if K is a compact set, $g \subset K \subset U_g$, and $\varepsilon > 0$, then there exists a homeomorphism, h, from M onto itself such that (a) for each $g' \in H_G$ which intersects K, $\operatorname{diam} h[g'] < \varepsilon$, (b) if $x \in M - U_g$, $h(x) = x$.

THEOREM 4. *Let G be a 0-dimensional monotone decomposition of an n-manifold, M, such that H_G^* is a G_δ. Then G is locally shrinkable if and only if G is weakly shrinkable.*

LEMMA 4.1. *Let G be a 0-dimensional monotone decomposition of an n-manifold, M. Suppose C is a compact set contained in a saturated open set, U. Then there exists an open set, V, with compact closure such that $C \subset V \subset \operatorname{Cl} V \subset U$ and* $(\operatorname{Bd} V) \cap H_G^* = \varnothing$.

LEMMA 4.2 (BING AND MCAULEY [10]). *Suppose that M is a metric space. Furthermore, $\{f_i\}$ is a sequence of homeomorphisms from M onto M and $\{U_i\}$ is a sequence of open sets such that*

(1) $U_i \supset \operatorname{Cl} U_{i+1}$,

(2) $f_i = f_{i-1}$ *on* $M - U_i$, *where f_0 is the identity map,*

(3) $\bigcap_{i=1}^{\infty} U_i = \varnothing$,

(4) *for p in M, $\bigcup_{i=1}^{\infty} f_i^{-1}(p)$ lies in a compact subset of M.*

Then $\operatorname{Limit} f_i = f$ *is a homeomorphism from M onto M.*

Proof of Theorem 4. First, suppose that G is weakly shrinkable. Let $g \in H_G$ and suppose g is contained in an open set, U. Let U_g be an open set containing g such that $(\operatorname{Bd} U_g) \cap H_G^* = \varnothing$ and $U_g \subset U$. Cover H_G^* as follows. If $g' \in H_G$ and $g' \cap U_g$

$=\varnothing$, then choose $U_{g'}$ such that $U_{g'} \cap U_g = \varnothing$. If $g' \in H_G$ and $g' \subset U_g$ choose $U_{g'} = U_g$. Let $\mathscr{W}$ be a refinement of this cover which satisfies all of the conclusions of Lemma 2.2. If $\varepsilon > 0$, there exists a homeomorphism, $\hat{h}$, which shrinks the nondegenerate elements to a diameter less than ε and is the identity outside of $\mathscr{W}^*$. Let h be a homeomorphism from M onto M which is equal to $\hat{h}$ on U_g and to the identity otherwise. The conditions of local shrinkability are now easily seen to be satisfied.

Suppose that G is locally shrinkable. We shall divide the remainder of the proof into two parts. In Part 1, we show the equivalence (under the hypotheses of Theorem 4) of local shrinkability and a condition which we designate as Property S, and in Part 2 we prove that Property S will imply weak shrinkability. Suppose G is a 0-dimensional monotone decomposition of an n-manifold, M. Then G is said to satisfy *Property* S in case for each saturated compact set, K, contained in a saturated open set, U, and for $\varepsilon > 0$, there exists a space homeomorphism, h (dependent on K, U, and ε) which shrinks the nondegenerate elements lying in K to diameter less than ε, and is the identity outside of U.

Part 1. Clearly Property S implies local shrinkability. We are assuming that H_G^* is a G_δ. Hence, let $D_1, D_2, \ldots$ be a sequence of open sets such that $D_i \supset D_{i+1}$ and $\bigcap_{i=1}^{\infty} D_i = H_G^*$. Suppose K is a saturated compact set contained in a saturated open set, U, and let ε be a positive number. Let $B_1 = (\bigcup \{g \in H_G : \operatorname{diam} g \geqq \varepsilon\}) \cap K$. For each $g \subset B_1$ let U_g be an open set which satisfies the local shrinkability condition and is contained in $U \cap D_1$. The collection $\{U_g : g \subset B_1\}$ is then an open cover of the compact set, B_1. Select a minimal subcover, $U_{11}, U_{12}, \ldots, U_{1n_1}$, of this collection, where U_{1i} denotes $U_{g_{1i}}$.

Let $A_{11} = B_1 \cap U_{11}$. Then A_{11} is a compact set containing g_{11} and lying in U_{11}. Hence, there exists a homeomorphism, h_{11}, from M onto itself such that for $g' \in H_G$ and $g' \subset A_{11}$, $\operatorname{diam} h_{11}[g'] < \varepsilon$, and for $x \in M - U_{11}$, $h_{11}(x) = x$. h_{11} is uniformly continuous, and, thus corresponding to ε, there exists $\delta_{11} > 0$, such that if F is a subset of M with diameter less than δ_{11}, then $\operatorname{diam} h_{11}[F] < \varepsilon$.

Let $A_{12} = B_1 \cap U_{12}$. Then A_{12} is a compact set containing g_{12} and lying in U_{12}. Hence, there exists a space homeomorphism, h_{12}, such that for $g' \subset A_{12}$, $\operatorname{diam} h_{12}[g'] < \delta_{11}$, and if $x \in M - U_{12}$, $h_{12}(x) = x$. h_{12} is uniformly continuous, and thus corresponding to δ_{11}, there exists a positive number, δ_{12}, such that if F is a subset of M with diameter less than δ_{12}, then $\operatorname{diam} h_{12}[F] < \delta_{11}$. Continuing this procedure, we obtain a space homeomorphism, $h_1 = h_{11} h_{12} \cdots h_{1n_1}$, where if $g' \subset \bigcup_{j=1}^{n_1} A_{1j}$, then $\operatorname{diam} h_1[g'] < \varepsilon$ and if $x \in M - \bigcup_{j=1}^{n_1} U_{1j}$, $h_1(x) = x$. Let $U_1 = \bigcup_{j=1}^{n_1} U_{1j}$.

In general, we let $B_m = (\bigcup \{g : \operatorname{diam} h_{m-1}[g] \geqq \varepsilon\}) \cap K$. Note that B_m is compact, $B_m \subset U_{m-1}$, and $B_m \cap B_{m-1} = \varnothing$. If $g \subset B_m$, let U_g be an open set containing g such that

(1) U_g satisfies the local shrinkability condition,

(2) $\operatorname{Cl} U_g \subset U_{m-1} \cap D_m$,

(3) $(\operatorname{Cl} U_g) \cap B_{m-1} = \varnothing$.

Let $U_{m1}, U_{m2}, \ldots, U_{mn_m}$ be a minimal cover of B_m by sets from the collection $\{U_g : g \subset B_m\}$, where $U_{mi} = U_{g_{mi}}$.

For $i = 1, 2, \ldots, n_m$ let $C_{mi} = (\bigcup \{g \in H_G : \operatorname{diam} g \geqq 1/m\}) \cap U_{mi}$. (For $m = 1$, C_{1i} is simply $B_1 \cap U_{1i}$.) Let $A_{mi} = (B_m \cap U_{mi}) \cup C_{mi}$. A procedure similar to that employed in finding h_1 may be followed to obtain a space homeomorphism, h_m, which shrinks nondegenerate elements in $\bigcup_{j=1}^{n_m} A_{mj}$ to a diameter less than ε, and is equal to h_{m-1} outside of $\bigcup_{j=1}^{n_m} U_{mj}$. Let $U_m = \bigcup_{j=1}^{n_m} U_{mj}$.

We thus form a sequence, $\{U_i\}$, of open sets and a sequence, $\{h_i\}$, of space homeomorphisms which satisfy hypotheses (1), (2), (4) of Lemma 4.2. Let $h = \operatorname{Limit} h_i$. In order to see that (3) of the hypothesis of Lemma 4.2 is also satisfied, we observe the following. We have that $\bigcap_{i=1}^{\infty} U_i \subset \bigcap_{i=1}^{\infty} D_i = H_G^*$. Since for each i, $(\operatorname{Bd} U_i) \cap H_G^* = \varnothing$, it follows that if $\bigcap_{i=1}^{\infty} U_i \neq \varnothing$, the intersection could only contain nondegenerate elements of the decomposition. Suppose g is one such element. There exists a positive integer, i, such that $\operatorname{diam} g > 1/i$. Then, for some $j = 1, 2, \ldots, n_i$, g will be contained in A_{ij}, and, hence, g will not be contained in U_{i+1}, a contradiction. Thus, $\bigcap_{i=1}^{\infty} U_i = \varnothing$. Since it is clear that if $g \in G$, $\operatorname{diam} h[g] < \varepsilon$, Part 1 is established.

Part 2. We now show that under the hypotheses of Theorem 4, Property S implies weak shrinkability. Suppose then that G is a 0-dimensional monotone decomposition of an n-manifold, M. Let U be an open set containing H_G^*, and ε a positive number. We may break U up into a countable number of mutually disjoint open sets with compact closure which cover H_G^* and satisfy all of the conclusions of Lemma 2.2 and Lemma 3.1. Let $\mathscr{W}$ be this cover.

Although we shall work with a particular element, $W \in \mathscr{W}$, the same procedure is to be used for each set of $\mathscr{W}$. From the hypotheses of Theorem 4, there exists a sequence of open sets, $D_1', D_2', \ldots$ such that $D_i' \supset D_{i+1}'$ and $\bigcap_{i=1}^{\infty} D_i' = H_G^*$. We let $D_i = D_i' \cap W$. We shall obtain a homeomorphism, h, from M onto M such that if $g \in H_G$ and $g \subset W$, then $\operatorname{diam} h[g] < \varepsilon$, and if $x \in M - W$, $h(x) = x$.

Let $B_1 = \bigcup \{g \in H_G : \operatorname{diam} g \geqq \varepsilon \text{ and } g \subset W\}$. Then B_1 is compact. By Lemma 4.1, there exists an open set, V_1, containing B_1 such that $(\operatorname{Bd} V_1) \cap H_G^* = \varnothing$, and $\operatorname{Cl} V_1 \subset D_1$. $\operatorname{Cl} V_1$ is compact, and, thus, Property S implies there exists a space homeomorphism, h_1, shrinking the nondegenerate elements lying in $\operatorname{Cl} V_1$ to diameter less than ε, and which is the identity outside of D_1.

Let $B_2 = \bigcup \{g \in H_G : \operatorname{diam} h_1[g] \geqq \varepsilon \text{ and } g \subset W\}$. Apply Lemma 4.1 to obtain an open set, Z_2, such that (1) $Z_2 \supset B_2$, (2) $\operatorname{Cl} Z_2 \subset D_2 - \operatorname{Cl} V_1$, (3) $(\operatorname{Bd} Z_2) \cap H_G^* = \varnothing$.

Let $C_2 = \bigcup \{g \in H_G : g \subset Z_2 \text{ and } \operatorname{diam} g \geqq 1/2\}$. Let $A_2 = B_2 \cup C_2$.

Let V_2 be an open set with boundary disjoint from H_G^* such that $A_2 \subset V_2 \subset \operatorname{Cl} V_2 \subset Z_2$. Since h_1 is uniformly continuous, there exists a space homeomorphism, h_2', such that

(1) for each $g \subset \operatorname{Cl} V_2$, $\operatorname{diam} h_2'[g] < \gamma_2$, where γ_2 is chosen small enough so that $\operatorname{diam} h_1 h_2'[g] < \varepsilon$,

(2) if $x \in M - Z_2$, $h_2'(x) = x$.

Denote h_1h_2' by h_2.

Continuing this process we obtain two sequences of open sets, $\{V_i\}$ and $\{Z_i\}$, and a sequence of homeomorphisms, $\{h_i\}$, with the property that for each positive integer i:

(1) $(\mathrm{Bd}\, V_i) \cap H_G^* = \varnothing$ and $(\mathrm{Bd}\, Z_i) \cap H_G^* = \varnothing$,

(2) $Z_i \supset \mathrm{Cl}\, Z_{i+1}$ and $Z_i \subset D_i$,

(3) $V_i \supset B_i \cup C_i$, where

$$B_i = \bigcup \{g \in H_G : \operatorname{diam} h_{i-1}[g] \geqq \varepsilon \text{ and } g \subset W\} \qquad (h_0 = \text{identity}),$$
$$C_i = \bigcup \{g \in H_G : g \subset Z_i \text{ and } \operatorname{diam} g \geqq 1/i\},$$

(4) for $g \subset \bigcup_{j=1}^{i} \mathrm{Cl}\, V_j$, $\operatorname{diam} h_i[g] < \varepsilon$,

(5) if $x \in M - Z_i$, $h_i(x) = h_{i-1}(x)$.

Let $h = \mathrm{Limit}\, h_i$. Let $U_1 = D_1$, and for $i = 2, 3, \ldots$ let $U_i = Z_i$. That the hypotheses of Lemma 4.2 are satisfied by the sequences, $\{U_i\}$ and $\{h_i\}$, may now easily be verified by duplicating the latter stages of the proof of Part 1. Thus, h is a homeomorphism from M onto itself which shrinks nondegenerate elements lying in W to diameter of less than ε, and is the identity outside of W. For each $W' \in \mathscr{W}$, we find similar homeomorphisms, and then an application of Lemma 3.1 yields the desired result.

COROLLARY 4.1. *Suppose G is a locally shrinkable monotone decomposition of an n-manifold, M, such that $P[H_G^*]$ is 0-dimensional. Furthermore, assume that H_G^* is a G_δ. Then M/G is homeomorphic to M.*

COROLLARY 4.2. *Suppose G is a 0-dimensional monotone decomposition of an n-manifold, M. Furthermore, assume that H_G^* is a G_δ, and that for each $g \in H_G$, there exists a sequence, $\{B_i^g\}$, of n-cells such that for each positive integer i,* (1) $B_{i+1}^g \subset \mathrm{Int}\, B_i^g$, (2) $\bigcap_{i=1}^{\infty} B_i^g = g$, *and* (3) $(\mathrm{Bd}\, B_i^g) \cap H_G^* = \varnothing$.

Then M/G is homeomorphic to M.

THEOREM 5. *Suppose G is a monotone locally shrinkable decomposition of an n-manifold, M, such that $P[H_G^*]$ is 0-dimensional. Let G' be a decomposition of M such that* (1) $H_{G'} \subset H_G$ *and* (2) $H_{G'}^*$ *is a G_δ. Then M/G' is homeomorphic to M.*

Proof. By Corollary 4.1, we need only show that G' is locally shrinkable. Suppose $g \in H_{G'}$ and g is contained in an open set, U. Since G is locally shrinkable and $g \in H_G$, there exists an open set U_g such that $g \subset U_g \subset U$, and U_g satisfies the necessary local shrinkability properties with respect to the decomposition, G. It is then trivial to verify that U_g also satisfies the required shrinkability properties with respect to G', and Theorem 5 is established.

COROLLARY 5.1. *Suppose G is a 0-dimensional cellular decomposition of a 3-manifold, M, such that M/G is homeomorphic to M. Let G' be a decomposition of M such that* (1) $H_{G'} \subset H_G$ *and* (2) $H_{G'}^*$ *is a G_δ. Then M/G' is homeomorphic to M.*

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