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FINITELY GENERATED IDEALS OF DIFFERENTIABLE FUNCTIONS(1)

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Abstract. In some spaces of differentiable functions, the finitely generated ideals which are closed are characterized in terms of the zeros of the generators. Applications are made to problems of division for distributions.

1. **Introduction.** Let $\mathscr{E}^m(\Omega)$ denote the algebra of real-valued m-times continuously differentiable functions on an open set Ω in R^n equipped with the topology of uniform convergence of all derivatives of order $\leqq m$ on all compact subsets of Ω. Here $0 \leqq m \leqq \infty$ and $\mathscr{E}^\infty(\Omega)$ will often be denoted $\mathscr{E}(\Omega)$. $\mathscr{E}^m(\Omega)$ is a Fréchet space, that is, a complete metrisable locally convex topological vector space.

Which finitely generated ideals in $\mathscr{E}^m(\Omega)$ are closed? Since $\mathscr{E}^m(\Omega)$ is topologically isomorphic to the cartesian product of the algebras $\mathscr{E}^m(\Omega_i)$ where the Ω_i are the components of Ω, we assume always that Ω is connected.

In §3 we show that for $m<\infty$, $I=(f_1,\ldots,f_p)$ is a closed ideal in $\mathscr{E}^m(\Omega)$ if and only if $f_1^2+\cdots+f_p^2$ never vanishes or $f_1^2+\cdots+f_p^2$ is identically zero. Thus there are no proper closed finitely generated ideals in $\mathscr{E}^m(\Omega)$ when m is finite.

In $\mathscr{E}(\Omega)$ the situation is not so simple and the question remains open for $\Omega \subset R^n$, $n>1$. However, in §4 we show that for $\Omega \subset R^1$, the following are equivalent.

(a) $I=(f_1,\ldots,f_p)$ is closed in $\mathscr{E}(\Omega)$.

(b) $f_1^2+\cdots+f_p^2$ satisfies the Lojasiewicz inequality.

(c) $f_1^2+\cdots+f_p^2$ has zeros of finite order or $f_1^2+\cdots+f_p^2$ is identically zero.

And in §5 we show that for $\Omega \subset R^n$, $n>1$, (a) implies (b), (a) implies (c), and (c) does not imply (a). Whether or not (b) implies (a) seems to be an open question.

Interest in closed principal ideals in $\mathscr{E}^m(\Omega)$ stems from their connection by duality with problems of division for distributions. In §6 information about certain finite systems of division problems is obtained from our study of closed finitely generated ideals.

The ideas involved in one of the methods of proof employed here are more

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(1) The results presented here are contained in the author's Ph.D. thesis written at Dartmouth College under the supervision of Professor Reese T. Prosser.

evident without the presence of differentiation; consequently, we begin with a brief look at closed finitely generated ideals in $C(X)$.

2. **Finitely generated ideals in** $C(X)$. If X is a topological space, let $C(X)$ be the algebra of continuous real-valued functions on X. Gillman and Henriksen [2] examine spaces X, called F-spaces, such that $C(X)$ has the property that all finitely generated ideals are principal. The class of F-spaces is not very extensive (for example, a metric space is an F-space if and only if it is discrete). However, if X is a compact Hausdorff space and $C(X)$ is given the sup norm topology, then the following theorem implies that all closed finitely generated ideals in $C(X)$ are principal.

THEOREM 2.1. *Let X be a compact Hausdorff space and provide $C(X)$ with the sup norm topology. Then $I=(f_1,\dots,f_p)$ is a closed ideal in $C(X)$ if and only if $Z(I)=\{x\in X: f(x)=0$ for all $f\in I\}$ is open.*

Proof. Suppose I is closed. Define $F\colon [C(X)]^p\to C(X)$ by $F(g_1,\dots,g_p)=f_1g_1+\cdots+f_pg_p$ and for $g=(g_1,\dots,g_p)$, define $\|g\|=\max\{\|g_1\|,\dots,\|g_p\|\}$. Then F is a continuous linear mapping of the Banach space $[C(X)]^p$ onto the Banach space $\operatorname{im}(F)=I$ and therefore F is an open mapping. Hence there exists $C>0$ such that if $h\in I$, then there exists $g\in F^{-1}(h)$ with $\|g\|\leqq C\|h\|$.

By Urysohn's lemma, for each $x\notin Z(I)$ there exists $h_x\in C(X)$ with $h_x(x)=1$, $h_x=0$ in a neighborhood of $Z(I)$, and $\|h_x\|=1$. Since $h_x\in I$ for each $x\notin Z(I)$, there exists $g=(g_1,\dots,g_p)\in F^{-1}(h_x)$ with $\|g\|\leqq C\|h_x\|=C$. And

$$F(g)(x) = f_1(x)g_1(x)+\cdots+f_p(x)g_p(x) = h_x(x) = 1$$

so for some i, $1\leqq i\leqq p$, we have $|f_i(x)g_i(x)|\geqq 1/p$. Thus

$$1/|f_i(x)| \leqq p|g_i(x)| \leqq p\|g_i\| \leqq p\|g\| \leqq pC.$$

Therefore for each $x\notin Z(I)$ there exists i, $1\leqq i\leqq p$, such that $|f_i(x)|\geqq 1/pC$, and hence

$$f_1^2(x)+\cdots+f_p^2(x) \geqq (1/pC)^2 \quad \text{for all } x\notin Z(I).$$

Clearly

$$Z(I) = \{x\in X: f_1^2(x)+\cdots+f_p^2(x) < (1/pC)^2\}$$

and therefore $Z(I)$ is open.

Conversely, suppose $Z(I)$ is open. Consider the function g which is equal to zero on $Z(I)$ and $1/(f_1^2+\cdots+f_p^2)$ on $X-Z(I)$. Since $Z(I)$ and $X-Z(I)$ are separated, $g\in C(X)$. Then $f=(f_1^2+\cdots+f_p^2)g\in I$ so $(f)\subset I$. Moreover, if $h\in \operatorname{cl}(I)$, then $h=fh\in(f)$. Therefore I is closed and we have proved

COROLLARY 2.1. *If I is a closed finitely generated ideal in $C(X)$, then I is principal and generated by an idempotent.*

COROLLARY 2.2. *If X is connected, then there are no proper closed finitely generated ideals in $C(X)$.*

Proof. Suppose $I=(f_1,\dots,f_p)$ is a closed ideal in $C(X)$. Then $Z(I)$ is both open and closed in the connected space X. If $Z(I)=\varnothing$, then $I=C(X)$ since $f_1^2+\cdots+f_p^2\in I$ is a unit, and if $Z(I)=X$, then I is the zero ideal.

3. **Finitely generated ideals in $\mathscr{E}^m(\Omega)$, $m<\infty$.** We here show that there are no proper closed finitely generated ideals in $\mathscr{E}^m(\Omega)$, $m<\infty$. Our proof uses the space $\mathscr{E}'^m(\Omega)$ of distributions of order $\leqq m$ with compact support, the dual space of $\mathscr{E}^m(\Omega)$. The proof relies on a theorem of Whitney [9] which describes the closed ideals in $\mathscr{E}^m(\Omega)$ and the closed range theorem for Fréchet spaces due to Dieudonné and Schwartz [1].

We begin with two lemmas, the first of which will be needed again in §5 when $m=\infty$.

LEMMA 3.1. *Let $f_1,\dots,f_p\in\mathscr{E}^m(\Omega)$, $0\leqq m\leqq\infty$, and define F: $[\mathscr{E}^m(\Omega)]^p\to\mathscr{E}^m(\Omega)$ by $F(g_1,\dots,g_p)=f_1g_1+\cdots+f_pg_p$. Suppose there exists a sequence $\{x_k\}$ in Ω with $x_k\to a\in\Omega$ such that $(f_1(x_k),\dots,f_p(x_k))\neq(0,\dots,0)$ for all k. Then there exists*

$$T=(c_1\delta_a,\dots,c_p\delta_a)\in[\ker(F)]^\perp$$

where some c_i is nonzero.

Proof. For any $g_1,\dots,g_p$ with $g_i=(g_{i1},\dots,g_{ip})\in\ker(F)$ for $1\leqq i\leqq p$, we have $\det(g_{ij}(x_k))_{1\leqq i,j\leqq p}=0$ for all k since $(f_1(x_k),\dots,f_p(x_k))\neq(0,\dots,0)$ for all k. Hence $\det(g_{ij}(a))_{1\leqq i,j\leqq p}=0$.

If for all $g_i=(g_{i1},\dots,g_{ip})\in\ker(F)$ we have $g_{i1}(a)=0$, then $T=(\delta_a,0,\dots,0)\in[\ker(F)]^\perp$ and the proof is complete. Otherwise choose $g_1,\dots,g_l$ with $g_i=(g_{i1},\dots,g_{ip})\in\ker(F)$ for $1\leqq i\leqq l$ so that $d=\det(g_{ij}(a))_{1\leqq i,j\leqq l}\neq 0$ with l maximal. Clearly $1\leqq l\leqq p-1$. Then for any $h=(h_1,\dots,h_p)\in\ker(F)$, we have

$$\det\begin{pmatrix} g_{11}(a) & \cdots & g_{1l}(a) & g_{1(l+1)}(a)\\ \vdots & & \vdots & \vdots\\ g_{l1}(a) & \cdots & g_{ll}(a) & g_{l(l+1)}(a)\\ h_1(a) & \cdots & h_l(a) & h_{l+1}(a)\end{pmatrix}=0.$$

Hence for any $h=(h_1,\dots,h_p)\in\ker(F)$,

$$c_1h_1(a)+\cdots+c_lh_l(a)+c_{l+1}h_{l+1}(a)=0$$

where c_i is the cofactor of $h_i(a)$ and $c_{l+1}=d\neq 0$. Therefore

$$T=(c_1\delta_a,\dots,c_l\delta_a,c_{l+1}\delta_a,0,\dots,0)\in[\ker(F)]^\perp$$

and this completes the proof of Lemma 3.1.

LEMMA 3.2. *Let $f_1,\dots,f_p\in\mathscr{E}^m(\Omega)$, $m<\infty$, and define F: $[\mathscr{E}^m(\Omega)]^p\to\mathscr{E}^m(\Omega)$ by $F(g_1,\dots,g_p)=f_1g_1+\cdots+f_pg_p$. Suppose $I=(f_1,\dots,f_p)$ has a zero of order k,*

$0 \leqq k < m$, *at* $a=(a_1, \ldots, a_n) \in \Omega$, *that is,* $(Lf_i)(a)=0$ *for all* i, $1 \leqq i \leqq p$, *and all constant coefficient linear partial differential operators* L *of order* $\leqq k$, *but*

$$(\partial^\alpha f_i)(a) = (\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f_i)(a) \neq 0$$

for some i, *say* $i=j$, *and some* $\alpha=(\alpha_1, \ldots, \alpha_n)$ *with* $\alpha_1 + \cdots + \alpha_n = k+1$. *Then there exists*

$$T = (T_1, \ldots, T_p) \in [\ker(F)]^\perp$$

where order $(T_i) \leqq m-k$ *and* supp $(T_i) \subset \{a\}$ *for* $1 \leqq i \leqq p$, *and* order $(T_j) = m-k$.

Proof. We use induction on the number p of generators of I. Suppose $p=1$ and $g_1 \in \ker(F)$. Then $f_1 g_1 = 0$ so $g_1(a)=0$ since f_1 is not identically zero in any neighborhood of a. Since $\alpha_1 + \cdots + \alpha_n = k+1$, some $\alpha_i \neq 0$, say α_l. By Leibnitz's rule expand

$$(\partial_1^{\alpha_1} \cdots \partial_l^{m-\alpha_1-\cdots-\hat{\alpha}_l-\cdots-\alpha_n} \cdots \partial_n^{\alpha_n})(f_1 g_1).$$

Divide each term in the expansion by $\partial_1^{\alpha_1} \cdots \partial_l^{\alpha_l - 1} \cdots \partial_n^{\alpha_n} f_1$, which is nonzero at $(a_1, \ldots, x, \ldots, a_n)$ for all x in $(a_l, a_l+\varepsilon)$ for some $\varepsilon > 0$. Now let $x \to a_l^+$. The result is that a linear combination of the values at a of the partial derivatives of order $\leqq m-k$ of g_1 equals zero. The coefficients in the linear combination depend only on $\alpha_1, \ldots, \alpha_n$, m, and f_1. And $(\partial_l^{m-k} g_1)(a)$ has a nonzero coefficient, namely

$$\binom{m-\alpha_1-\cdots-\hat{\alpha}_l-\cdots-\alpha_n}{\alpha_l - 1} + \binom{m-\alpha_1-\cdots-\hat{\alpha}_l-\cdots-\alpha_n}{\alpha_l}.$$

Therefore there exists $T_1 \in [\ker(F)]^\perp$ with order $(T_1) = m-k$ and supp $(T_1) = \{a\}$.

Now suppose the lemma holds for $p-1$ generators and consider generators $f_1, \ldots, f_p$ as in the statement of the lemma. For simplicity we assume $j=1$, that is, $(\partial^\alpha f_1)(a) \neq 0$.

Case 1. Suppose for some $g=(g_1, \ldots, g_p) \in \ker(F)$, $(g_1(a), \ldots, g_p(a)) \neq (0, \ldots, 0)$. If $g_1(a) \neq 0$, then

$$f_1 = -(g_2/g_1) f_2 - \cdots - (g_p/g_1) f_p$$

in some neighborhood of a. Hence

$$(\partial^\alpha f_1)(a) = -(g_2/g_1)(a)(\partial^\alpha f_2)(a) - \cdots - (g_p/g_1)(a)(\partial^\alpha f_p)(a) \neq 0.$$

Thus $g_i(a) \neq 0$ for some i with $1 < i \leqq p$, say for simplicity $i=p$. Hence

$$f_p = -(g_1/g_p) f_1 - \cdots - (g_{p-1}/g_p) f_{p-1}$$

in a neighborhood U of a.

Let $I' = (f_1, \ldots, f_{p-1})$ and define $F' : [\mathscr{E}^m(\Omega)]^{p-1} \to \mathscr{E}^m(\Omega)$ by $F'(h_1, \ldots, h_{p-1}) = f_1 h_1 + \cdots + f_{p-1} h_{p-1}$. By the induction hypothesis there exists

$$T = (T_1, \ldots, T_{p-1}) \in [\ker(F')]^\perp$$

where order $(T_i) \leqq m-k$ and supp $(T_i) \subset \{a\}$ for $1 \leqq i \leqq p-1$, and order $(T_1) = m-k$. Therefore

$$(T_1, \ldots, T_{p-1}, -(g_1/g_p)T_1 - \cdots - (g_{p-1}/g_p)T_{p-1}) \in [\ker(F)]^\perp.$$

For if $(h_1, \ldots, h_p) \in \ker(F)$, then by choosing $\varphi \in \mathscr{E}^m(\Omega)$ with $\operatorname{supp}(\varphi) \subset U$ and $\varphi = 1$ in a neighborhood of a, we have

$$\varphi[h_1 - (g_1/g_p)h_p, \ldots, h_{p-1} - (g_{p-1}/g_p)h_p] \in \ker(F')$$

and thus

$$\begin{aligned}[T_1, \ldots, T_{p-1}, &-(g_1/g_p)T_1 - \cdots - (g_{p-1}/g_p)T_{p-1}](h_1, \ldots, h_p) \\ &= [T_1, \ldots, T_{p-1}, -(g_1/g_p)T_1 - \cdots - (g_{p-1}/g_p)T_{p-1}](\varphi h_1, \ldots, \varphi h_p) \\ &= (T_1, \ldots, T_{p-1})[\varphi(h_1 - (g_1/g_p)h_p), \ldots, \varphi(h_{p-1} - (g_{p-1}/g_p)h_p)] = 0.\end{aligned}$$

Letting $T_p = -(g_1/g_p)T_1 - \cdots - (g_{p-1}/g_p)T_{p-1}$, we have the desired conclusion.

Case 2. Suppose $(g_1(a), \ldots, g_p(a)) = (0, \ldots, 0)$ for all $g = (g_1, \ldots, g_p) \in \ker(F)$. Since $\alpha_1 + \cdots + \alpha_n = k+1$, some $\alpha_i \neq 0$, say α_l. By Leibnitz's rule expand

$$(\partial_1^{\alpha_1} \cdots \partial_l^{m-\alpha_1-\cdots-\hat{\alpha}_l-\cdots-\alpha_n} \cdots \partial_n^{\alpha_n})(f_1 g_1 + \cdots + f_p g_p),$$

which is identically zero for $(g_1, \ldots, g_p) \in \ker(F)$. Divide each term in the expansion by

$$\partial_1^{\alpha_1} \cdots \partial_l^{\alpha_l - 1} \cdots \partial_n^{\alpha_n} f_1,$$

which is nonzero at $(a_1, \ldots, x, \ldots, a_n)$ for all x in $(a_l, a_l + \varepsilon)$ for some $\varepsilon > 0$. Now let $x \to a_l^+$. The result is that a linear combination of the values at a of the partial derivatives of order $\leqq m-k$ of $g_1, \ldots, g_p$ equals zero. The coefficients in the linear combination depend only on $\alpha_1, \ldots, \alpha_n$, m, and $f_1, \ldots, f_p$. And $(\partial_l^{m-k} g_1)(a)$ has a nonzero coefficient, namely

$$\binom{m - \alpha_1 - \cdots - \hat{\alpha}_l - \cdots - \alpha_n}{\alpha_l - 1} + \binom{m - \alpha_1 + \cdots - \hat{\alpha}_l - \cdots - \alpha_n}{\alpha_l}.$$

This implies that

$$T = (T_1, \ldots, T_p) \in [\ker(F)]^{\perp}$$

where $\operatorname{order}(T_i) \leqq m-k$ and $\operatorname{supp}(T_i) \subset \{a\}$ for $1 \leqq i \leqq p$, and $\operatorname{order}(T_1) = m-k$. This completes the proof of Lemma 3.2.

THEOREM 3.1. *Suppose $f_1, \ldots, f_p \in \mathscr{E}^m(\Omega)$, $m < \infty$. Then $I = (f_1, \ldots, f_p)$ is a closed ideal in $\mathscr{E}^m(\Omega)$ if and only if $f_1^2(x) + \cdots + f_p^2(x) > 0$ for all $x \in \Omega$ or $f_1^2(x) + \cdots + f_p^2(x) = 0$ for all $x \in \Omega$.*

Proof. Suppose $I = (f_1, \ldots, f_p)$ is a closed ideal in $\mathscr{E}^m(\Omega)$. We suppose $f_1^2 + \cdots + f_p^2$ takes on both zero and nonzero values in Ω and we obtain a contradiction. Let $Z = \{x \in \Omega : f_1^2(x) + \cdots + f_p^2(x) = 0\}$. Then Z is a nonempty proper closed subset of Ω. Therefore, since Ω is connected, $\operatorname{bd}(Z)$ is nonempty. Choose $a = (a_1, \ldots, a_n) \in \operatorname{bd}(Z)$ and let k be the order of the zero of I at a, that is, let k be the largest integer $\leqq m$ such that $(Lf_i)(a) = 0$ for all i, $1 \leqq i \leqq p$, and all constant coefficient linear partial differential operators L of order $\leqq k$.

Define F: $[\mathscr{E}^m(\Omega)]^p \to \mathscr{E}^m(\Omega)$ by $F(g_1, \ldots, g_p) = f_1 g_1 + \cdots + f_p g_p$. Then there exists

$$T = (T_1, \ldots, T_p) \in [\ker(F)]^\perp$$

where order $(T_i) \leqq m-k$ and supp $(T_i) \subset \{a\}$ for $1 \leqq i \leqq p$, and for some i, say $i=j$, order $(T_j) = m-k$. For if $k=m$, then Lemma 3.1 guarantees the existence of T, while if $k<m$, then Lemma 3.2 guarantees the existence of T.

Since $\operatorname{im}(F) = I$ is closed, the closed range theorem for Fréchet spaces [1, Theorem 7, p. 92] implies that

$$\operatorname{im}(F') = [\ker(F)]^\perp$$

where F': $\mathscr{E}'^m(\Omega) \to [\mathscr{E}'^m(\Omega)]^p$ is the transpose of F. Therefore there exists $S \in \mathscr{E}'^m(\Omega)$ with $F'(S) = T$.

Let $f(x_1, \ldots, x_n) = (x_1 - a_1)^l + \cdots + (x_n - a_n)^l$ where l is an even integer greater than $m-k$. Then

$$F'(fS) = fF'(S) = fT = (fT_1, \ldots, fT_p) = (0, \ldots, 0).$$

Thus $fS \in \ker(F') = [\operatorname{im}(F)]^\perp = I^\perp$ and hence $S \in (fI)^\perp$ where fI is the ideal $\{fg : g \in I\}$.

Let $g(x_1, \ldots, x_n) = (x_1 - a_1)^{\alpha_1} \cdots (x_n - a_n)^{\alpha_n}$ where $\alpha_1, \ldots, \alpha_n$ are chosen so that $\alpha_1 + \cdots + \alpha_n = m-k$ and $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \delta_a$ appears with a nonzero coefficient in T_j.

For $x \in \Omega$, let J_x^m denote the ideal in $\mathscr{E}^m(\Omega)$ consisting of all functions which vanish at x together with all derivatives of order $\leqq m$ and let T_x^m denote the natural mapping of $\mathscr{E}^m(\Omega)$ onto $\mathscr{E}^m(\Omega)/J_x^m$. Then the local ideals $T_a^m(fI)$ and $T_a^m(gI)$ are both the zero ideal in $\mathscr{E}^m(\Omega)/J_x^m$, since both fI and gI have zeros of order m at a. And for $x \in \Omega$, $x \neq a$, any function in gI is equal to a function in fI in a neighborhood of x since $f(x) \neq 0$ and therefore $T_x^m(gI) \subset T_x^m(fI)$. By Whitney's theorem [9, Theorem 1, p. 636], we conclude $\operatorname{cl}(gI) \subset \operatorname{cl}(fI)$ and hence $(fI)^\perp \subset (gI)^\perp$.

Therefore $S \in (gI)^\perp$ and hence $gS \in I^\perp = \ker(F')$. Thus

$$(0, \ldots, 0) = F'(gS) = gF'(S) = gT = (gT_1, \ldots, gT_p).$$

But $gT_j \neq 0$ since $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \delta_a$ appears with a nonzero coefficient in T_j. This is a contradiction.

Conversely, suppose $f_1^2 + \cdots + f_p^2$ never vanishes or $f_1^2 + \cdots + f_p^2$ is identically zero. If $f_1^2 + \cdots + f_p^2$ never vanishes, then $I = (f_1, \ldots, f_p) = \mathscr{E}^m(\Omega)$ since $f_1^2 + \cdots + f_p^2 \in I$ is a unit. And if $f_1^2 + \cdots + f_p^2$ is identically zero, then I is the zero ideal. This completes the proof of Theorem 3.1 and also proves

Corollary 3.1. *There are no proper closed finitely generated ideals in $\mathscr{E}^m(\Omega)$ when m is finite.*

Suppose $I = (f_1, \ldots, f_p)$ is a closed ideal in $\mathscr{E}^m(\Omega)$, $m < \infty$. Using the open mapping theorem as in Theorem 2.1 (and later in Theorem 4.1), one can show that for each compact set $K\subset\Omega$ there exists a constant $C>0$ such that

$$\max\{|f_1(x)|,\ldots,|f_p(x)|\} \geqq C[d(x,Z)]^m \quad \text{for all } x\in K,$$

where $Z=\{x\in\Omega : f_1^2(x)+\cdots+f_p^2(x)=0\}$. When $m=0$, a proof of Theorem 3.1 can be based on this inequality, but in general more than this inequality is needed in order to prove that $f_1^2+\cdots+f_p^2$ never vanishes or $f_1^2+\cdots+f_p^2$ is identically zero.

4. **Finitely generated ideals in $\mathscr{E}(\Omega)$, $\Omega\subset R^1$.** Consider $\mathscr{E}(\Omega)$, $\Omega\subset R^n$. For each compact set $K\subset\Omega$ and integer $l\geqq 0$, let

$$|f|_{K,l} = \sup\{|\partial_1^{\alpha_1}\ldots\partial_n^{\alpha_n}f(x)| : x\in K, \alpha_1+\cdots+\alpha_n \leqq l\}$$

for $f\in\mathscr{E}(\Omega)$. The seminorms $|\cdot|_{K,l}$ define the topology of $\mathscr{E}(\Omega)$.

We here return to the method of proof demonstrated in §2. However, due to the complications introduced by the presence of differentiation, the following fact is needed (Urysohn's lemma played an analogous role in the proof of Theorem 2.1). For any compact set $K\subset\Omega$, integer $l\geqq 0$, and nonempty closed set Z in Ω there exists a constant $C>0$ such that if $x\in K-Z$, then there exists $h_x\in\mathscr{E}(\Omega)$ with $h_x(x)=1$, $h_x=0$ in a neighborhood of Z, and

$$|h_x|_{K,l} \leqq C/[d(x,Z)]^l.$$

Here $d(x,Z)=\inf\{d(x,z) : z\in Z\}$ where d is the Euclidean metric. Merely let φ be an infinitely differentiable function on R^n with support in the unit ball and $\varphi(0)=1$. Then $h_x(y)=\varphi((y-x)/\delta)$ has the desired properties when $\delta=d(x,Z)/2$.

Suppose $f\in\mathscr{E}(\Omega)$, $\Omega\subset R^1$. We say that *f has zeros of finite order* if at every point $x\in\Omega$ where $f(x)=0$ some derivative of f is nonzero.

THEOREM 4.1. *Suppose $f_1,\ldots,f_p\in\mathscr{E}(\Omega)$, $\Omega\subset R^1$. Then the following are equivalent.*

(a) *$I=(f_1,\ldots,f_p)$ is closed in $\mathscr{E}(\Omega)$.*

(b) *$f_1^2+\cdots+f_p^2$ satisfies the Lojasiewicz inequality, that is, for each compact set $K\subset\Omega$ there exists a constant $C>0$ and an integer $l\geqq 0$ such that*

$$f_1^2(x)+\cdots+f_p^2(x) \geqq C[d(x,Z)]^l \quad \textit{for all } x\in K$$

where $Z=\{x\in\Omega : f_1^2(x)+\cdots+f_p^2(x)=0\}$. (Here $d(x,Z)=1$ for all x if $Z=\varnothing$.)

(c) *$f_1^2+\cdots+f_p^2$ has zeros of finite order or $f_1^2+\cdots+f_p^2$ is identically zero.*

Proof. (a) implies (b). Suppose $I=(f_1,\ldots,f_p)$ is closed and Z is nonempty. Let $K\subset\Omega$ be a compact set. Define F: $[\mathscr{E}(\Omega)]^p\to\mathscr{E}(\Omega)$ by $F(g_1,\ldots,g_p)=f_1g_1+\cdots+f_pg_p$ and for $g=(g_1,\ldots,g_p)\in[\mathscr{E}(\Omega)]^p$, define $|g|_{K,l}=\max\{|g_1|_{K,l},\ldots,|g_p|_{K,l}\}$. Then F is a continuous linear mapping of the Fréchet space $[\mathscr{E}(\Omega)]^p$ onto the Fréchet space im $(F)=I$ and therefore F is an open mapping. Hence there exists a constant $C>0$, an integer $l\geqq 0$, and a compact set $K'\subset\Omega$ such that if $h\in I$, then there exists $g\in F^{-1}(h)$ with $|g|_{K,0}\leqq C|h|_{K',l}$.

For the compact set $K''=K\cup K'\subset\Omega$, the integer $l\geqq 0$, and the nonempty closed set $Z\subset\Omega$ there exists a constant $C'>0$ such that if $x\in K''-Z$, then there exists $h_x\in\mathscr{E}(\Omega)$ with $h_x(x)=1$, $h_x=0$ in a neighborhood of Z, and

$$|h_x|_{K'',l} \leqq C'/[d(x,Z)]^l.$$

For each $x\in K-Z$, we have $h_x\in I$ since $h_x=0$ in a neighborhood of Z. Therefore there exists $g=(g_1,\ldots,g_p)\in F^{-1}(h_x)$ with

$$|g|_{K,0} \leqq C|h_x|_{K',l} \leqq C|h_x|_{K'',l} \leqq CC'/[d(x,Z)]^l.$$

But

$$F(g)(x) = f_1(x)g_1(x)+\cdots+f_p(x)g_p(x) = h_x(x) = 1$$

so for some i, $1\leqq i\leqq p$, we have $|f_i(x)g_i(x)|\geqq 1/p$. Thus

$$1/|f_i(x)| \leqq p|g_i(x)| \leqq p|g_i|_{K,0} \leqq p|g|_{K,0} \leqq pCC'/[d(x,Z)]^l.$$

Therefore for each $x\in K-Z$ there exists i, $1\leqq i\leqq p$, such that

$$|f_i(x)| \geqq C''[d(x,Z)]^l.$$

Hence $f_1^2+\cdots+f_p^2$ satisfies the Lojasiewicz inequality.

(b) implies (c). Suppose $f=f_1^2+\cdots+f_p^2$ satisfies the Lojasiewicz inequality. We suppose that f is not identically zero but has zeros of infinite order and we obtain a contradiction. Consider $b\in\Omega$ with $f(b)\neq 0$. Let a be a zero of infinite order of f closest to b, say $a<b$.

Case 1. Suppose f has no zeros in $(a, a+\varepsilon)$ for some $\varepsilon>0$. By Taylor's theorem, for each integer k there exists $C_k>0$ such that

$$|f(x)| \leqq C_k|x-a|^k = C_k[d(x,Z)]^k$$

for all $x\in[a, a+\varepsilon/2]$, which contradicts the Lojasiewicz inequality.

Case 2. Suppose a is a limit point of zeros of f in (a, b). The zeros of f in (a, b) can have no other limit points, for such a limit point would be a zero of infinite order of f, contradicting the choice of a. Therefore the zeros of f in (a, b) are a decreasing sequence $\{z_k\}$ with $z_k\to a$.

Since f satisfies the Lojasiewicz inequality there exists a constant $C>0$ and an integer $l\geqq 0$ such that

$$|f(x)| \geqq C[d(x,Z)]^l \quad \text{for all } x\in[a,b].$$

Since $f(a)=0$, we have $l\geqq 1$.

Suppose $l=1$. Let $y_k=(z_k+z_{k+1})/2$. Expanding f about z_k by Taylor's theorem, we obtain

$$C|y_k-z_k| = C[d(y_k,Z)] \leqq |f(y_k)| = |y_k-z_k|\,|f^{(1)}(u_k)|$$

where $y_k<u_k<z_k$. Letting $k\to\infty$, we obtain $0<C\leqq|f^{(1)}(a)|$, contradicting the fact that f has a zero of infinite order at a.

For $l \geqq 2$, the argument is similar but somewhat more complicated. Choose a subsequence $\{z_{k_i}\}$ of $\{z_k\}$ such that for all i,

$$|z_{k_i} - z_{k_i+1}| \geqq |z_{k_i+j} - z_{k_i+j+1}|$$

for all j, $0 \leqq j \leqq l-2$. And for each i, choose

$$\begin{array}{lll} x_i^1 \in (z_{k_i+1}, z_{k_i}) & \text{with} & f^{(1)}(x_i^1) = 0, \\ x_i^2 \in (z_{k_i+2}, z_{k_i}) & \text{with} & f^{(2)}(x_i^2) = 0, \\ \cdot & \cdot & , \\ \cdot & \cdot & , \\ \cdot & \cdot & , \\ x_i^{l-1} \in (z_{k_i+l-1}, z_{k_i}) & \text{with} & f^{(l-1)}(x_i^{l-1}) = 0. \end{array}$$

Now let $x_i = z_{k_i}$ and $y_i = (z_{k_i} + z_{k_i+1})/2$.

For each i, expanding f by Taylor's theorem about x_i, we obtain

$$\begin{aligned} C|y_i - x_i|^l &= C[d(y_i, Z)]^l \leqq |f(y_i)| \\ &\leqq \sum_{j=1}^{l-1} (1/j!)|y_i - x_i|^j |f^{(j)}(x_i)| + (1/l!)|y_i - x_i|^l |f^{(l)}(v_i)| \end{aligned}$$

where $y_i < v_i < x_i$.

For each i, expanding $f^{(1)}$ about x_i^1, we obtain

$$f^{(1)}(x_i) = f^{(1)}(x_i^1) + (x_i - x_i^1) f^{(2)}(r_i)$$

where $x_i^1 < r_i < x_i$. Since $f^{(1)}(x_i^1) = 0$ and $|x_i - x_i^1| \leqq 2|y_i - x_i|$, we have $|f^{(1)}(x_i)| \leqq 2|y_i - x_i|\ |f^{(2)}(r_i)|$ for each i.

Expanding $f^{(2)}$ about x_i^2, we obtain for each i

$$|f^{(2)}(r_i)| = |r_i - x_i^2|\ |f^{(3)}(s_i)| \leqq 4|y_i - x_i|\ |f^{(3)}(s_i)|$$

where $x_i^2 < s_i < r_i$, and

$$|f^{(2)}(x_i)| = |x_i - x_i^2|\ |f^{(3)}(t_i)| \leqq 4|y_i - x_i|\ |f^{(3)}(t_i)|$$

where $x_i^2 < t_i < x_i$.

Continuing in this manner (the last step consists of expanding $f^{(l-1)}$ about x_i^{l-1}), we find that for all i

$$C|y_i - x_i|^l \leqq C_1|y_i - x_i|^l |f^{(l)}(u_i)| + \cdots + C_l |y_i - x_i|^l |f^{(l)}(v_i)|$$

where $z_{k_i+l-1} < u_i, \ldots, v_i < z_{k_i} = x_i$. Therefore for all i

$$0 < C \leqq C_1 |f^{(l)}(u_i)| + \cdots + C_l |f^{(l)}(v_i)|.$$

Clearly $u_i \to a, \ldots, v_i \to a$ as $i \to \infty$. Therefore

$$0 < C \leqq C_1 |f^{(l)}(a)| + \cdots + C_l |f^{(l)}(a)|,$$

contradicting the fact that f has a zero of infinite order at a. Thus either $f = f_1^2 + \cdots + f_p^2$ has zeros of finite order or $f = f_1^2 + \cdots + f_p^2$ is identically zero.

(c) implies (a). Suppose $f_1^2+\cdots+f_p^2$ has zeros of finite order or $f_1^2+\cdots+f_p^2$ is identically zero. If $f_1^2+\cdots+f_p^2$ is identically zero, then $I=(f_1,\ldots,f_p)$ is the zero ideal and hence closed. If $f_1^2+\cdots+f_p^2$ has zeros of finite order, then $I=(f_1,\ldots,f_p)$ has zeros of finite order. (The order of a zero x of I is the largest integer k such that $f_i^{(j)}(x)=0$ for $1\leqq i\leqq p$ and all $j\leqq k$.) Since zeros of finite order are isolated, there exist functions $g_0, g_1,\ldots, g_p\in\mathscr{E}(\Omega)$ such that the function

$$f_0 = (f_1^2+\cdots+f_p^2)g_0+f_1g_1+\cdots+f_pg_p\in I$$

has precisely the zeros of I (with the correct orders). Since $f_0\in I$, we have $(f_0)\subset I$. And if $h\in\operatorname{cl}(I)$, then $h/f_0\in\mathscr{E}(\Omega)$ so there exists $g\in\mathscr{E}(\Omega)$ with $h=f_0g\in(f_0)$. Therefore I is closed and we have also proved

COROLLARY 4.1. *If I is a closed finitely generated ideal in $\mathscr{E}(\Omega)$, $\Omega\subset R^1$, then I is principal.*

For $p=1$, Malgrange [6, p. 88] shows that (a) implies (b), and our proof is essentially the same as his.

5. **Finitely generated ideals in $\mathscr{E}(\Omega)$, $\Omega\subset R^n$.** For $\Omega\subset R^n$, $n>1$, the question of which finitely generated ideals in $\mathscr{E}(\Omega)$ are closed is more difficult and remains unsolved. The results of Hörmander [3, Theorem 4, p. 568] on the division of distributions by polynomials imply that polynomials generate closed principal ideals in $\mathscr{E}(\Omega)$; more generally, the work of Lojasiewicz [4, p. 130] on the division of distributions by real analytic functions shows that real analytic functions generate closed principal ideals in $\mathscr{E}(\Omega)$. And Malgrange [5, No. 25, p. 1] shows that $I=(f_1,\ldots,f_p)$ is a closed ideal in $\mathscr{E}(\Omega)$ if $f_1,\ldots,f_p$ are real analytic.

Several necessary conditions that $I=(f_1,\ldots,f_p)$ be closed in $\mathscr{E}(\Omega)$ can be given. The first of these is that $f_1^2+\cdots+f_p^2$ satisfy the Lojasiewicz inequality. In fact, we have already proved this since the "(a) implies (b)" part of the proof of Theorem 4.1 made no use of the hypothesis that $\Omega\subset R^1$.

PROPOSITION 5.1. *If $I=(f_1,\ldots,f_p)$ is a closed ideal in $\mathscr{E}(\Omega)$, then for each compact set $K\subset\Omega$ there exists a constant $C>0$ and an integer $l\geqq 0$ such that*

$$f_1^2(x)+\cdots+f_p^2(x) \geqq C[d(x,Z)]^l \quad \textit{for all } x\in K$$

where $Z=\{x\in\Omega : f_1^2(x)+\cdots+f_p^2(x)=0\}$. (Here $d(x,Z)=1$ for all x if $Z=\varnothing$.)

Whether or not the converse of Proposition 5.1 holds for $\Omega\subset R^n$, $n>1$, seems to be an open question.

Suppose $f\in\mathscr{E}(\Omega)$, $\Omega\subset R^n$. We say that *f has zeros of finite order* if at every point $x\in\Omega$ where $f(x)=0$ some partial derivative of f is nonzero.

PROPOSITION 5.2. *If $I=(f_1,\ldots,f_p)$ is a closed ideal in $\mathscr{E}(\Omega)$, then $f_1^2+\cdots+f_p^2$ has zeros of finite order or $f_1^2+\cdots+f_p^2$ is identically zero.*

Proof. Suppose $I=(f_1,\dots,f_p)$ is a closed ideal in $\mathscr{E}(\Omega)$. We suppose $f_1^2+\cdots+f_p^2$ is not identically zero but has zeros of infinite order and we obtain a contradiction. Let $Z_\infty=\{x\in\Omega : x \text{ is a zero of infinite order of } f_1^2+\cdots+f_p^2\}$. Then

$$\begin{aligned} Z_\infty &= \{x\in\Omega : x \text{ is a zero of infinite order of } f_1,\dots,f_p\} \\ &= \{x\in\Omega : \text{for all } f\in I,\ x \text{ is a zero of infinite order of } f\}. \end{aligned}$$

And Z_∞ is a nonempty proper closed subset of Ω. Therefore, since Ω is connected, $\operatorname{bd}(Z_\infty)$ is nonempty. Choose $a=(a_1,\dots,a_n)\in\operatorname{bd}(Z_\infty)$. Then there exists a sequence $\{x_k\}$ in Ω such that $x_k\to a$ and for all k

$$(f_1(x_k),\dots,f_p(x_k))\neq(0,\dots,0).$$

Hence, by Lemma 3.1, there exists

$$T=(c_1\delta_a,\dots,c_p\delta_a)\in[\ker(F)]^\perp$$

where some c_i is nonzero and $F\colon[\mathscr{E}(\Omega)]^p\to\mathscr{E}(\Omega)$ is defined by $F(g_1,\dots,g_p)=f_1g_1+\cdots+f_pg_p$.

Since $\operatorname{im}(F)=I$ is closed, the closed range theorem for Fréchet spaces [1, Theorem 7, p. 92] implies that $\operatorname{im}(F')=[\ker(F)]^\perp$ where $F'\colon\mathscr{E}'(\Omega)\to[\mathscr{E}'(\Omega)]^p$ is the transpose of F. Therefore there exists $S\in\mathscr{E}'(\Omega)$ with $F'(S)=T$.

Let $f(x_1,\dots,x_n)=(x_1-a_1)^2+\cdots+(x_n-a_n)^2$. Then

$$F'(fS)=fF'(S)=fT=(c_1f\delta_a,\dots,c_pf\delta_a)=(0,\dots,0).$$

Thus $fS\in\ker(F')=[\operatorname{im}(F)]^\perp=I^\perp$ and hence $S\in(fI)^\perp$.

For $x\in\Omega$, let J_x denote the ideal in $\mathscr{E}(\Omega)$ consisting of all functions which vanish at x together with all derivatives and let T_x denote the natural mapping of $\mathscr{E}(\Omega)$ onto $\mathscr{E}(\Omega)/J_x$. Then the local ideals $T_x(fI)$ and $T_x(I)$ are equal for any $x\in\Omega$. By Whitney's theorem for infinitely differentiable functions [7, p. 506], we conclude that $\operatorname{cl}(fI)=\operatorname{cl}(I)$ and hence $(fI)^\perp=I^\perp$. Therefore $S\in I^\perp=\ker(F')$ and hence $F'(S)=T=0$. But $T=(c_1\delta_a,\dots,c_p\delta_a)\neq(0,\dots,0)$ since some c_i is nonzero. This contradiction completes the proof of Proposition 5.2.

An example due to Malgrange [6, p. 89] shows that the converse of Proposition 5.2 is false for $\Omega\subset R^n$, $n>1$.

EXAMPLE 5.1. Let

$$\begin{aligned} f(x,y) &= e^{-1/x^2}+y^2, && x\neq 0, \\ &= y^2, && x=0. \end{aligned}$$

Then the zero of f^2 is of finite order. However, by Proposition 5.1, we see that f does not generate a closed ideal in $\mathscr{E}(R^2)$ because f^2 fails to satisfy the Lojasiewicz inequality in any compact neighborhood of the origin.

6. **Applications to problems of division for distributions.** Let $\mathscr{D}_K^m(\Omega)$ denote the subspace of $\mathscr{E}^m(\Omega)$ consisting of all functions with support in the compact set $K\subset\Omega$ equipped with the relative topology and let $\mathscr{D}^m(\Omega)$ denote the inductive

limit of the $\mathscr{D}^m_K(\Omega)$, K a compact subset of Ω. Here $0 \leqq m \leqq \infty$ and $\mathscr{D}^\infty(\Omega)$ will often be denoted $\mathscr{D}(\Omega)$.

Suppose $f_1, \ldots, f_p \in \mathscr{E}^m(\Omega)$, $0 \leqq m \leqq \infty$, $\Omega \subset R^n$. Define F_c: $[\mathscr{D}^m(\Omega)]^p \to \mathscr{D}^m(\Omega)$ by $F_c(g_1, \ldots, g_p) = f_1 g_1 + \cdots + f_p g_p$ and let F_c': $\mathscr{D}'^m(\Omega) \to [\mathscr{D}'^m(\Omega)]^p$ be the transpose of F_c.

Consider the system of equations

$$(*) \qquad f_1 S = T_1, \ldots, f_p S = T_p$$

where $T_1, \ldots, T_p \in \mathscr{D}'^m(\Omega)$. In order that there exist a solution $S \in \mathscr{D}'^m(\Omega)$ to (*), it is necessary that

$$(T_1, \ldots, T_p) \in [\ker(F_c)]^\perp.$$

For if $S \in \mathscr{D}'^m(\Omega)$ satisfies (*), then

$$F_c'(S) = (f_1 S, \ldots, f_p S) = (T_1, \ldots, T_p)$$

and $\operatorname{im}(F_c') \subset [\ker(F_c)]^\perp$.

We are interested in those cases in which there exists a solution $S \in \mathscr{D}'^m(\Omega)$ to (*) for every $(T_1, \ldots, T_p) \in [\ker(F_c)]^\perp$ and thus are led to the following definition.

Definition 6.1. $[\mathscr{D}'^m(\Omega)]^p$ *is divisible by* $f_1, \ldots, f_p$ if for every $(T_1, \ldots, T_p) \in [\ker(F_c)]^\perp$ there exists $S \in \mathscr{D}'^m(\Omega)$ with $f_1 S = T_1, \ldots, f_p S = T_p$.

Then $[\mathscr{D}'^m(\Omega)]^p$ is divisible by $f_1, \ldots, f_p$ if and only if $\operatorname{im}(F_c') = [\ker(F_c)]^\perp$.

Define F: $[\mathscr{E}^m(\Omega)]^p \to \mathscr{E}^m(\Omega)$ by $F(g_1, \ldots, g_p) = f_1 g_1 + \cdots + f_p g_p$ and let

$$F': \mathscr{E}'^m(\Omega) \to [\mathscr{E}'^m(\Omega)]^p$$

be the transpose of F. A simple partition of unity argument establishes that $\operatorname{im}(F_c') = [\ker(F_c)]^\perp$ if and only if $\operatorname{im}(F') = [\ker(F)]^\perp$. The closed range theorem for Fréchet spaces now provides the link between problems of division and closed finitely generated ideals. For it says that $\operatorname{im}(F') = [\ker(F)]^\perp$ if and only if $\operatorname{im}(F)$, which is the ideal generated by $f_1, \ldots, f_p$, is closed. Therefore $[\mathscr{D}'^m(\Omega)]^p$ is divisible by $f_1, \ldots, f_p$ if and only if the ideal $I = (f_1, \ldots, f_p)$ is closed in $\mathscr{E}^m(\Omega)$.

All our results on closed finitely generated ideals now translate into results about problems of division. The most interesting are the following.

Proposition 6.1. *Suppose* $f_1, \ldots, f_p \in \mathscr{E}^m(\Omega)$, $m < \infty$, $\Omega \subset R^n$. *Then* $[\mathscr{D}'^m(\Omega)]^p$ *is divisible by* $f_1, \ldots, f_p$ *if and only if* $f_1^2 + \cdots + f_p^2$ *never vanishes or* $f_1^2 + \cdots + f_p^2$ *is identically zero.*

Corollary 6.1. *Suppose* $f \in \mathscr{E}^m(\Omega)$, $m < \infty$, $\Omega \subset R^n$. *Then* $f\mathscr{D}'^m(\Omega) = \mathscr{D}'^m(\Omega)$ *if and only if* f *never vanishes.*

Proposition 6.2. *Suppose* $f_1, \ldots, f_p \in \mathscr{E}(\Omega)$, $\Omega \subset R^1$. *Then the following are equivalent.*

(a) $[\mathscr{D}'^m(\Omega)]^p$ *is divisible by* $f_1, \ldots, f_p$.

(b) $f_1^2 + \cdots + f_p^2$ *satisfies the Lojasiewicz inequality.*

(c) $f_1^2 + \cdots + f_p^2$ *has zeros of finite order or* $f_1^2 + \cdots + f_p^2$ *is identically zero.*

COROLLARY 6.2. *Suppose* $f \in \mathscr{E}(\Omega)$, $\Omega \subset R^1$. *Then the following are equivalent.*

(a) $f\mathscr{D}'(\Omega) = \mathscr{D}'(\Omega)$.

(b) *f satisfies the Lojasiewicz inequality and f is not identically zero.*

(c) *f has zeros of finite order.*

Proof. If $f\mathscr{D}'(\Omega) = \mathscr{D}'(\Omega)$, then $\mathscr{D}'(\Omega)$ is divisible by f and f is not identically zero. Hence, by Proposition 6.2, f has zeros of finite order. Conversely, if f has zeros of finite order, then $\mathscr{D}'(\Omega)$ is divisible by f and F_c is one-to-one, where $F_c: \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ is multiplication by f. Therefore

$$f\mathscr{D}'(\Omega) = \operatorname{im}(F_c') = [\ker(F_c)]^{\perp} = \mathscr{D}'(\Omega).$$

This establishes the equivalence of (a) and (c); the equivalence of (b) and (c) is a consequence of Proposition 6.2.

It was L. Schwartz [8, p. 125] who first observed that if f has zeros of finite order, then $f\mathscr{D}'(\Omega) = \mathscr{D}'(\Omega)$ for $\Omega \subset R^1$.

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