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A DIOPHANTINE PROBLEM ON GROUPS. I(1)

BY
R. C. BAKER

Abstract. The following theorem of H. Weyl is generalised to the context of locally compact abelian groups.

THEOREM. *Let* $\lambda_1<\lambda_2<\lambda_3\cdots$ *be a sequence such that, for some* $c>0$, $\varepsilon>0$, $\lambda_{n+k}-\lambda_n \geqq c$ *whenever* $k\geqq n/(\log n)^{1+\varepsilon}$ $(n=1,2,\ldots)$. *Then for almost all real* u *the sequence* $\lambda_1 u, \lambda_2 u, \ldots, \lambda_n u \pmod 1$ *is uniformly distributed.*

1. **Introduction.** In a famous paper of 1916, Hermann Weyl introduced and developed the concept of a uniformly distributed sequence. One of his results is as follows [1, §7]: if a Dirichlet sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots, \qquad \lambda_n \to \infty \tag{1}$$

does not grow too slowly, then the sequence

$$\lambda_1 u,\ \lambda_2 u,\ \lambda_3 u,\ \ldots \tag{2}$$

is uniformly distributed modulo one for almost all real u. Rephrasing, the sequence

$$\exp(2\pi i\lambda_1 u),\ \exp(2\pi i\lambda_2 u),\ \exp(2\pi i\lambda_3 u),\ \ldots \tag{3}$$

is uniformly distributed on the unit circle T in the complex plane: if A is an arc of T having length δ, and if n_A of the first n of the points (3) fall in A, then

$$\lim_{n\to\infty} (n_A/n) = \delta/2\pi \qquad \text{(almost all } u\text{)}. \tag{4}$$

If $z_1, z_2, \ldots$ is any sequence on T, then it is uniformly distributed if and only if for each integer $m\neq 0$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} (z_k)^m = 0 \tag{5}$$

(Weyl [1]). We shall frequently make use of this criterion.

In this paper we generalise the result of Weyl quoted above as follows: if G is a locally compact abelian group satisfying certain restrictions, and if

$$\chi_1, \chi_2, \chi_3, \ldots \tag{6}$$

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is a sequence in G, not too slowly growing (in a sense to be made precise in §2) then, for almost all u in G, the sequence

$$\chi_1(u),\ \chi_2(u),\ \chi_3(u),\ \dots \tag{7}$$

is uniformly distributed on T. In particular, this is true if G is a connected group. This result seems interesting in its own right and also underlines the essentially group-theoretic nature of Weyl's result. Let us now turn to more precise ideas.

2. **The compact case.**

We begin with a definition.

DEFINITION 2.1. Let $\hat{G}$ be the dual group of a locally compact abelian group G. A sequence $\chi_1, \chi_2, \dots$ in $\hat{G}$ is said to be *not too slowly growing* if, for sufficiently small neighbourhoods V of 0 in $\hat{G}$, and, for some $\varepsilon > 0$ and $c > 0$ (depending on V),

$$\text{card } \Phi_n \leqq cn/(\log n)^{1+\varepsilon} \qquad (n = 1, 2, \dots) \tag{8}$$

where Φ_n is the largest bunch of terms in $\{\chi_1, \dots, \chi_n\}$ for which $\chi_i - \chi_j \in V$ for all $\chi_i, \chi_j \in \Phi_n$, and card Φ_n is the number of elements in Φ_n.

Roughly speaking, this definition says that the χ_i may group together as $i \to \infty$, but must do so fairly slowly. For instance, if $\hat{G}$ is noncompact and W is a fixed compact symmetric neighbourhood of 0 in G, and χ_1 is chosen arbitrarily, $\chi_2 \notin \chi_1 + W, \dots, \chi_n \notin (\chi_1 + W) \cup \cdots \cup (\chi_{n-1} + W)$ for $n > 1$, then we plainly have a not too slowly growing sequence (*N-sequence for short*). This shows that N-sequences exist aplenty in noncompact $\hat{G}$. It is clear that if $\hat{G}$ is compact there are no N-sequences in $\hat{G}$.

Our definition does not coincide with Weyl's in case $\{\chi_i\}$ is a Dirichlet sequence in R, but seems a fairly good substitute.

DEFINITION 2.2. A locally compact abelian (LCA) group G is said to be a *Weyl group* if for any N-sequence $\{\chi_i\}$ in $\hat{G}$, $\{\chi_i(u)\}$ is uniformly distributed on the unit circle for almost all u in G.

Our task, then, is to show that there are lots of Weyl groups. Let us first establish a necessary condition.

LEMMA 2.3. *Let G be a Weyl group and let m be a positive integer. Set*

$$F_m = \{\chi \in \hat{G} : m\chi = \chi + \cdots + \chi = 0\} \quad (m \text{ summands}).$$

Then for each discrete subgroup D of $\hat{G}$, $D \cap F_m$ is finite.

REMARK. Recalling that a group is *torsion free* if the only element with finite order is the identity, we may say that "*D is almost torsion free*" if $D \cap F_m$ is finite for each integer $m \neq 0$.

Proof of lemma. Let V be a neighbourhood of 0 in $\hat{G}$ such that $V \cap D = \{0\}$. If $D \cap F_m$ were infinite, there would be a sequence $\chi_1, \chi_2, \dots$ of distinct elements of $D \cap F_m$. Now if $i \neq j$, $\chi_i - \chi_j \notin V$, so $\{\chi_i\}$ is an N-sequence. But for every $u \in G$, $\{\chi_i(u)\}$ is a sequence of mth roots of unity. Plainly in this case G is not a Weyl group. This proves the lemma.

In fact, as we shall see, the conclusion of the lemma is not far from being a necessary and sufficient condition for G to be a Weyl group. Let us begin with the case when G is compact.

THEOREM 2.4. *Let G be a compact abelian group. Then G is a Weyl group, if and only if, $\hat{G}$ is almost torsion free.*

Proof. Since $\hat{G}$ is discrete, Lemma 2.3 shows that the condition is necessary. Suppose now that $\hat{G}$ is almost torsion free. Notice that an N-sequence is now one which is "not too repetitive": that is, $\{\chi_i\}$ is an N-sequence if, and only if, there exist $c>0$ and $\varepsilon>0$ for which

$$\text{card } \Phi_n \leqq cn/(\log n)^{1+\varepsilon} \qquad (n = 1, 2, \ldots) \tag{9}$$

where Φ_n is the largest bunch of *identical* terms in $\{\chi_1, \ldots, \chi_n\}$. Let $\{\chi_i\}$ be an N-sequence, and let $c>0$ and $\varepsilon>0$ be as in (9). Let m be an integer $\neq 0$. We wish to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\chi_k(x))^m = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (m\chi_k)(x) = 0 \tag{10}$$

for almost all $x \in G$ (see (5)). We follow Weyl [1, §7]. Set

$$f_n(x) = \frac{1}{n} \sum_{k=1}^{n} (m\chi_k)(x). \tag{11}$$

Notice that $m(\chi_i-\chi_j)=0$ implies $\chi_i-\chi_j \in F_m$, which by hypothesis is finite. If Ψ_n is the largest group of identical terms in $\{m\chi_1, \ldots, m\chi_n\}$ then ($F_m=\{\Delta_1, \ldots, \Delta_p\}$ say) ($\Psi_n=\{\chi_{n_1}, \chi_{n_2}, \ldots, \chi_{n_k}\}$):

χ_{n_2} is one of $\{\chi_{n_1}+\Delta_1, \ldots, \chi_{n_1}+\Delta_p\}$,

χ_{n_3} is one of $\{\chi_{n_1}+\Delta_1, \ldots, \chi_{n_1}+\Delta_p\}$ and so on.

By the "pigeon hole principle" at least $(k-1)/p$ of $\chi_{n_2}, \ldots, \chi_{n_k}$ are identical. So $(n=1, 2, \ldots)$

$$\begin{aligned} \text{card } \Psi_n - 1 = k-1 &\leqq cpn/(\log n)^{1+\varepsilon} \\ &\leqq Cn/(\log n)^{1+\varepsilon}-1, \quad \text{some } C > 0, \end{aligned} \tag{12}$$

and plainly $m\chi_1, m\chi_2, \ldots$ is also an N-sequence.

We have to show that $\lim_{n\to\infty} f_n(x)=0$ a.e. w.r.t. Haar measure. We begin by showing this for a subsequence. Set

$$n_\nu = [\exp(\nu^{1-b})], \qquad b = \varepsilon/(2+\varepsilon) \quad (0 < \varepsilon < 1) \qquad \nu = 1, 2, \ldots \tag{13}$$

where $[a]$ means "the integer part of a."

To prove that $\lim_{\nu\to\infty} f_{n_\nu}(x)=0$ a.e., it is enough to show that

$$\sum_{\nu=1}^{\infty} \int_G |f_{n_\nu}(x)|^2 \, dx < \infty \tag{14}$$

where "dx" denotes Haar measure. To see this, observe that

$$|f_n(x)|^2 = \frac{1}{n^2}|h_1\gamma_1(x)+\cdots+h_p\gamma_p(x)|^2$$
$$= \frac{1}{n^2}\sum_{i,j=1}^{p} h_ih_j(\gamma_i-\gamma_j)(x)$$

where $\gamma_1, \gamma_2, \ldots, \gamma_p$ are the distinct characters among $m\chi_1, m\chi_2, m\chi_n$, and $h_1, \ldots, h_p$ the numbers of them occurring, so that $h_1+\cdots+h_p=n$. We have, using orthogonality of distinct characters,

$$\int_G |f_n(x)|^2\,dx = \frac{1}{n^2}(h_1^2+\cdots+h_p^2) \leqq \frac{\text{card }\Phi_n}{n^2}(h_1+\cdots+h_p)$$
$$\leqq \frac{\text{card }\Phi_n}{n}.$$

Now $\log n_\nu \leqq \nu^{1-b} < \log(n_\nu+1)$. So

$$\int_G |f_{n_\nu}(x)|^2\,dx \leqq \frac{\text{card }\Phi_{n_\nu}}{n_\nu} \leqq \frac{C}{(\log n_\nu)^{1+\varepsilon}}$$
$$\leqq \frac{C}{(\nu^{1-b})^{1+\varepsilon}}\left(\frac{\log(n_\nu+1)}{\log n_\nu}\right)^{1+\varepsilon} \leqq \frac{D}{\nu^{1+b}}$$

where D is independent of ν. We find that

$$\sum_{\nu=1}^{\infty}\int_G |f_{n_\nu}(x)|^2\,dx \leqq \sum_{\nu=1}^{\infty}\frac{D}{\nu^{1+b}} < \infty$$

as desired. So $\lim_{\nu\to\infty} f_{n_\nu}(x)=0$ a.e. in G.

Now let n be any integer $\geqq 2$. For some $\nu \geqq 1$, $n_\nu \leqq n < n_{\nu+1}$ so

$$\left|f_n(x)-\frac{n_\nu}{n}f_{n_\nu}(x)\right| \leqq \frac{n_{\nu+1}-n_\nu}{n} \leqq \frac{n_{\nu+1}}{n_\nu}-1 \tag{15}$$

(consulting (11)). Also one can easily show that

$$(\nu+1)^{1-b}-\nu^{1-b} < \nu^{-b} \qquad (\nu \geqq 1).$$

Thus

$$0 \leqq \frac{n_{\nu+1}}{n_\nu}-1 < \frac{e^{\nu^{-b}}}{1-\bar{e}^{\nu^{1-b}}}-1.$$

Letting $\nu\to\infty$, $n_{\nu+1}/n_\nu-1\to 0$ and by (15) $\lim_{n\to\infty} f_n(x)=0$ almost everywhere in G. This completes the proof of Theorem 2.4.

COROLLARY. *If G is a compact connected abelian group and $\{\chi_n\}$ an N-sequence in $\hat{G}$, then $\{\chi_n(u)\}$ is uniformly distributed on the circle for almost all u in G.*

Proof. The hypothesis implies that $\hat{G}$ is torsion-free [2, Theorem 24.25].

We shall use the case of a discrete dual as a stepping stone to the more general case in Theorem 3.2. Again, we use Weyl's original proof as a guide, as the careful reader will observe.

3. **The compactly generated case.**

LEMMA 3.1. *Let G be a compactly generated, locally compact abelian group. Then given any neighbourhood V of 0 in $\hat{G}$, there is a closed subgroup H of G having compact index, and such that*

(i) *ϕ_H^{-1} preserves sets of measure zero,*

(ii) *$(G/H)^\wedge$ is V-dense in $\hat{G}$.*

Some explanation of the terminology is called for here. A subgroup H of G has compact index if G/H is compact; $\phi_H: G \to G/H$ denotes the natural mapping. Let m_0, m_1 denote Haar measure on G, G/H respectively. Then (i) requires that if $E \subset G/H$, $m_1(E)=0$, then $m_0(\phi_H^{-1}(E))=0$. Now for (ii). A set F is said to be V-dense in $\hat{G}$ if for each $\chi \in \hat{G}$ there is a $\gamma \in F$ such that $\gamma-\chi \in V$. Also, $(G/H)^\wedge$ is here identified with the closed subgroup of $\hat{G}$ consisting of characters that take the value 1 on H. This is justified by the standard results 24.10 and 24.11 of [2].

Proof of the lemma. G is topologically isomorphic to a group $G_0 \times R^a \times Z^b$ where G_0 is compact, R denotes the real line and Z the integers [2, 9.8], and a, b are nonnegative integers, so it is enough to prove the lemma for a group of this form. We have $\hat{G}=\Gamma_0 \times R^a \times T^b$ where T denotes the circle group and Γ_0 is the discrete dual of G_0. Assume $a>0$ and $b>0$ (the cases $a=0$ or $b=0$ or both are even more straightforward). A basic neighbourhood of 0 in G takes the form $\{0\} \times B_a(0, \varepsilon) \times D_b(0, \varepsilon)$ where $\varepsilon>0$; $B_a(0, \varepsilon)$ is the ε-sphere about 0 in R^a; and $D_b(0, \varepsilon)=\{(x_1, \dots, x_b) \in T^b : \{x_1\}^2+\cdots+\{x_b\}^2<\varepsilon^2\}$. Here the x_i are real (modulo 2π) and $\{x_i\}$ the distance from x_i to the nearest multiple of 2π.

Take $H=\{0\} \times mZ^a \times mZ^b$ where m is a large positive integer. Thus $H=\{(0; r_1, \dots, r_a; x_1, \dots, x_b),\ r_i \text{ and } x_j \text{ all integer multiples of } m\}$. Then it is not hard to prove that G/H is topologically isomorphic with $G_0 \times (R/mZ)^a \times (Z(m))^b$ where $Z(m)$ is the cyclic group of order m. In turn, $(R/mZ)^a$ is topologically isomorphic with T^a. We find that G/H is compact. Also as a subgroup of $\hat{G}$, $(G/H)^\wedge$ is $\Gamma_0 \times (1/m)Z^a \times (D_m)^b$ where D_m is the group of mth roots of unity. This may plainly be made V-dense in $\hat{G}$ if m is sufficiently large; so (ii) is satisfied. As for (i), let $E \subset G/H$, $m_1(E)=0$. It is clear that E is a finite union of sets of the type $B \times \{\underline{u}\}$ where $\underline{u}$ is a fixed element of $(Z(m))^b$ and B is a subset of $G_0 \times (R/mZ)^a$ having measure zero. It is enough to show that $\phi_H^{-1}(B \times \{\underline{u}\})$ is of zero measure in G.

Now $\phi_H^{-1}(B \times \{\underline{u}\})$ is a countable union of sets of the form $P=\{(g, \underline{r}, \underline{v}): \underline{v} \text{ fixed in } Z^b, (g, \underline{r}) \text{ lies in the image of } B \text{ in } G_0 \times [0, m)^a\}$. Here we mean the image of B under the obvious measure-wise identification of $G_0 \times (R/mZ)^a$ with $G \times [0, m)^a$. Plainly a set of type P has measure zero. So $\phi_H^{-1}(E)$ has Haar measure zero in G proving (i); and this completes the proof of the lemma.

As will be seen in a moment, the *conclusion* of Lemma 3.1 is all that is needed in Theorem 3.2 on compactly generated groups. The question is thus raised as to whether other LCA groups, indeed whether all LCA groups, satisfy this conclusion. We have been unable to settle this, but notice that the conclusion does apply to $G=G_0\times R^a\times Z^\infty$, where Z^∞ denotes the weak direct product of countably many copies of Z. This is not hard to prove, using the above methods.

REMARK. Let G be a locally compact, compactly generated abelian group. Then $\hat{G}$ is almost torsion free, if and only if each discrete subgroup of $\hat{G}$ is almost torsion free.

Proof. Again, we can assume $G=G_0\times R^a\times Z^b$ where G_0 is compact. Then $\hat{G}=\Gamma_0\times R^a\times T^b$ where Γ_0 is the discrete dual of G_0. If $\hat{G}$ is almost torsion free so are all its discrete subgroups. Conversely if each discrete subgroup of $\hat{G}$ is almost torsion free, Γ_0 is almost torsion free. From the fact that T^b is almost torsion free it is clear that G is almost torsion free also. This proves the remark.

THEOREM 3.2. *Let G be a locally compact, compactly generated abelian group. Then G is a Weyl group if and only if $\hat{G}$ is almost torsion free.*

Proof. The necessity of the condition is clear from Lemma 2.3 and the above remark. Suppose now that G is almost torsion free. Let $\{\chi_i\}$ be any N-sequence in G. Let V_0 be a nbd of 0 such that if $V\subset V_0$, the χ_i behave relative to V in the way described in Definition 2.1. Let m be any positive integer.

Since G is σ-compact, by (5) it is enough to show that for a fixed compact subset K of G,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(m\chi_k)(u) = 0 \tag{16}$$

for almost all u in K. Let $\delta>0$ be any positive number. Let

$$V_1 = V(0, K, \delta) = \{\chi\in\hat{G} : |\chi(x)-1| < \delta \text{ for all } x\in K\}.$$

This is a neighbourhood of 0 in $\hat{G}$. Let W be any neighbourhood of 0 in $\hat{G}$, $W\subset V_0\cap V_1$. Let V be a symmetric neighbourhood of 0 in G such that $V+\cdots+V\subset W$ ($m+1$ summands).

Let H be a closed subgroup of G with compact index, such that (i) and (ii) of Lemma 3.1 are satisfied relative to this V. Then for $i=1,2,\ldots$ we can choose $\gamma_i\in(G/H)^\wedge$, $\gamma_i\in\chi_i+V$.

If $i, j\leqq n$ and $\gamma_i=\gamma_j$ then $\chi_i\in\gamma_i+V=\gamma_j+V\subset\chi_j+W$. It is clear from this and the fact that $W\subset V_0$ that $\{\gamma_i\}$, regarded as a sequence of characters of the compact group G/H, is an N-sequence. Consequently, since $(G/H)^\wedge$ is almost torsion free, we can apply Theorem 2.4 to conclude that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(m\gamma_k)(u+H) = 0 \tag{17}$$

for almost all $u+H$ in G/H. Now let

$$E = \{u+H \in G/H : (17) \text{ fails}\}.$$

Then $m_1 E=0$. By property (ii) of H,

$$\phi_H^{-1}(E) = \left\{u \in G : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (m\gamma_k)(u) = 0 \text{ fails}\right\}$$

has Haar measure zero in G.

In particular,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (m\gamma_k)(u) = 0 \quad \text{for } u \in K \sim F$$

where F is a subset of K having measure zero. Now, for $u \in K$,

$$\left|\frac{1}{n} \sum_{k=1}^{n} (m\gamma_k)(u) - \frac{1}{n} \sum_{k=1}^{n} (m\chi_k)(u)\right| \leqq \sup_{\substack{\chi \in V+V+\cdots+V, u \in K \\ (m \text{ summands})}} |\chi(u)-1| < \delta$$

since $V+V+\cdots+V \subset V_1$ (m summands).

If $u \in K \sim F$, we see that

$$\limsup_{n\to\infty} \left|\frac{1}{n} \sum_{k=1}^{n} (m\chi_k)(u)\right| \leqq \delta.$$

Now F depends on δ, $F=F(\delta)$. Set $D=\bigcup_{j=1}^{\infty} F(1/j)$. D has measure zero, and if $u \in K \sim D$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (m\chi_k)(u) = 0. \tag{18}$$

Again D depends on m, but here again we take a countable union and we find that (18) holds, almost everywhere in G, for all m. This proves that $\{\chi_k(u)\}$ is uniformly distributed on the circle for almost all u. Q.E.D.

Notice that Theorem 2.4 is a special case of Theorem 3.2. Whether the hypothesis that G is compactly generated is really essential in 3.2 is open to conjecture.

A natural question is: Can we say anything more about the size of the exceptional sets where (18) fails? Are these for example countable in general? In Part II, which we hope to publish elsewhere, we use Riesz product methods to show that this is not the case, for example, when the $\{\chi_i\}$ is a suitable independent set.

Let us briefly notice an important special case of 3.2. Any connected LCA group has torsion free dual (see [2, 24.35]) and so is a Weyl group, as we promised to show in §1.

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Birkbeck College, London University,
London, England