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UNION EXTENSIONS OF SEMIGROUPS(¹)

BY
L. A. M. VERBEEK

0. **Introduction.** The literature on algebraic semigroups contains publications concerning two distinct types of semigroup extensions. Ideal extensions of semigroups were introduced by Clifford in [1] whereas Rédei [9] introduced Schreier extensions of monoids (a monoid is a semigroup containing an identity element). Let A and S be two disjoint semigroups and let S contain a zero element o. A semigroup E is an ideal extension of A by S if it contains A as ideal and if the Rees factor semigroup E/A is isomorphic with S. In §§4.4 and 4.5 of [2] the early literature on ideal extensions of semigroups is discussed. Recent publications are [5], [7], [8], [11] and [12]. Schreier extensions of monoids are closely related to group extensions. Since Schreier extensions are somewhat complicated to describe and the notions involved are not used in this paper we only give the following references, [13], [3], [4] and [6].

Apart from ideal and Schreier extensions there are other methods of constructing a semigroup E from two given semigroups A and S such that it is appropriate to call E an extension of A by S. In §1 we give a general definition of semigroup extensions comprising both ideal and Schreier extensions. A particular class of semigroup extensions, which we call union extensions, is then introduced and considered in some detail. The class of union extensions of a semigroup A by a semigroup S contains the ideal extensions of A by S but is distinct from Schreier extensions of A by S. For the investigation of the existence and for the construction of union extensions of A by S the notion of composition of a semigroup is introduced and worked out in §2. In §3 we describe the method used for investigating union extensions. §4 gives solutions to the questions concerning existence and construction of union extensions of A by S in case S has a single composition. However, one of the cases considered in §4 involves ideal extensions and there we have nothing to add to the existing literature. In §5 the same questions are solved for two cases where S has double composition.

For the notational conventions and basic concepts used in this paper we refer to [2].

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1. Definitions and preliminary properties.

DEFINITION 1. A *semigroup extension of a semigroup A by a semigroup S* is an ordered pair, (E, θ), consisting of a semigroup E which contains a subsemigroup A' isomorphic with A, and a congruence θ on E such that A' is a θ-class and that the factor semigroup E/θ is isomorphic with S.

In the sequel we denote the isomorphism of A onto A' by α, the natural homomorphism of E onto E/θ by θ_{nat}, the isomorphism of E/θ onto S by β and the composite homomorphism $\theta_{\text{nat}}\beta$ of E onto S by γ. For brevity's sake we shall omit the qualification "semigroup" and write "(E, θ) is an extension of A by S".

Clearly, ideal and Schreier extensions are particular cases of extensions as defined above.

If (E, θ) is an extension of A by S, then $A'\gamma$ is an idempotent element of S. We shall call $A'\gamma$ the *extension idempotent* of the extension (E, θ). Moreover, if i is an idempotent element of a semigroup S and A is an arbitrary semigroup, then the direct product $A \otimes S$ of A and S, together with the congruence

$$\theta_S = \{((a, s), (b, s)) : a, b \in A \text{ and } s \in S\}$$

on $A \otimes S$ is an extension of A by S. In this extension $(A \otimes S, \theta_S)$ the subsemigroup $A' \cong A$ is $A' = \{(a, i) : a \in A\}$ and i is the extension idempotent. Therefore we have

PROPERTY 1. Let A and S be semigroups. Extensions of A by S exist if and only if S contains an idempotent element.

Notice that in a given extension (E, θ) of A by S the semigroup E may contain several (disjoint) subsemigroups, all isomorphic with A, such that each of them is a θ-class. Therefore the extension idempotent of an extension need not be unique. Several other general properties of extensions have been derived in [10], here we shall confine ourselves to a particular class of extensions, to wit union extensions.

DEFINITION 2. Let A and S be two semigroups, let i be an idempotent element of S and let $A \cap S^- = \varnothing$, where $S^- = S\backslash\{i\}$. An extension (E, θ) of A by S with extension idempotent i is a *union extension of A by S* if and only if

(i) the carrier of the semigroup E is the union, $A \cup S^-$, of the carriers of the semigroup A and the partial groupoid S^-, and

(ii) the homomorphism $\theta_{\text{nat}}\beta = \gamma : E \to S$ restricted to $E\backslash A'$ is the identity mapping of S^-.

Since (i) requires that $E = A \cup S^-$, it is necessary that A and S^- be disjoint. From (ii) follows immediately $E\backslash A' = S^-$ and because $E = A \cup S^-$, clearly $A' = A$ so that α is an automorphism of A. Moreover, the θ-classes of E are A and every one-element subset of S^-. Condition (ii) may seem to be unnecessarily restrictive and hence to make the notion of union extension rather limited. However, it facilitates the further investigation of union extensions. Moreover, weakening this condition in a natural way to the requirement that θ restricted to $E\backslash A'$ be a partial automorphism of S^-, i.e. that θ restricted to $E\backslash A'$ be the identity relation, does not enlarge the class of union extensions up to isomorphism. This is stated more precisely in Property 2, the straightforward proof of which we omit.

PROPERTY 2. Let A and S be two semigroups, let i be an idempotent element of S, and let $A \cap S^- = \varnothing$. If an extension (E, θ) of A by S with extension idempotent i has the property that θ restricted to $E\backslash A'$ is the identity relation, then there is a union extension (E', θ') of A by S such that $E \cong E'$. Conversely, if (E', θ') is a union extension of A by S and E is a semigroup such that $E \cong E'$, then there is a congruence θ on E such that (E, θ) is an extension of A by S with the property that θ restricted to $E\backslash A'$ is the identity relation.

The connection between union extensions and Schreier and ideal extensions can be described as follows. The semigroup E of a union extension (E, θ) of A by S is never a Schreier extension of A by S except in the rather trivial case that A consists of one element and the extension idempotent is the identity element of S. The semigroup E of an extension (E, θ) of A by S is an ideal extension of A by S if and only if S contains a zero element o and (E, θ) is a union extension of A by S with o as extension idempotent. Hence the notion of union extension is a veritable generalization of the notion of ideal extension. For the study of union extensions it is expedient to use the following property, easily derived from Definition 2, which is analogous to a remark in §4.4 of [2].

PROPERTY 3. Let A and S be semigroups, let S contain the idempotent element i, let $A \cap S^- = \varnothing$, and let $E(*)$ be a semigroup. Then there is a congruence θ on $E(*)$ such that (E, θ) is a union extension of A by S with extension idempotent i if and only if

(i) the carrier of the semigroup $E(*)$ is $A \cup S^-$, and

(ii) the operations of the semigroups A and S and the operation $(*)$ of the semigroup $E(*)$ are related as follows:

$$\begin{array}{llll} a*b = ab, & & s*a = si & \text{if } si \neq i, \\ & & \in A & \text{if } si = i, \\ a*s = is & \text{if } is \neq i, & s*t = st & \text{if } st \neq i, \\ \in A & \text{if } is = i, & \in A & \text{if } st = i, \end{array}$$

for all $a, b \in A$ and all $s, t \in S^-$.

Moreover, the union extension (E, θ) of A by S with extension idempotent i is determined by the semigroup $E(*)$ and either the semigroup A or the semigroup S.

Property 3 enables us to write, as we shall do in the sequel, $E(*)$ instead of (E, θ) for a union extension of A by S. Notice that this means that for union extensions, as is the case for ideal and Schreier extensions, the congruence θ is not absolutely necessary to determine the whole situation. This is the reason why it is possible to investigate union extensions simply in terms of associative binary operations as is done in the rest of this paper.

2. **Composition of semigroups.** Let A and S be two given semigroups, let S contain the idempotent element i, and let $A \cap S^- = \varnothing$. We wish to find conditions for A and/or S that determine the existence of union extensions of A by S with extension idempotent i, and also how all such union extensions can be constructed. Before starting our quest let us consider the role of the idempotent element i of S.

For each element $s \in S^-$, where S^- is the partial groupoid $S\backslash\{i\}$, exactly one of the following three statements holds:

$$is = s, \qquad is = i, \qquad is \neq s,\ is \neq i$$

and also exactly one of the following three:

$$is = s, \qquad si = i, \qquad si \neq s,\ si \neq i.$$

Therefore we define nine disjoint subsets of S^- and also four subsets of S.

DEFINITION 3. Let S be a semigroup containing the idempotent element i. We define subsets (1) through (9) of S^- as follows:

(1) $U^- = \{s \in S^- : is=s,\ si=s\}$,
(2) $V^- = \{s \in S^- : is=i,\ si=i\}$,
(3) $W_r^- = \{s \in S^- : is=s,\ si=i\}$,
(4) $W_1^- = \{s \in S^- : is=i,\ si=s\}$,
(5) $X = \{s \in S^- : is\neq s,\ is\neq i,\ si\neq s,\ si\neq i\}$,
(6) $Y_r = \{s \in S^- : is=s,\ si\neq s,\ si\neq i\}$,
(7) $Y_1 = \{s \in S^- : is\neq s,\ is\neq i,\ si=s\}$,
(8) $Z_r = \{s \in S^- : is\neq s,\ is\neq i,\ si=i\}$,
(9) $Z_1 = \{s \in S^- : is=i,\ si\neq s,\ si\neq i\}$.

Moreover, we define the subsets (10) through (13) of S as follows:

(10) $U = U^- \cup \{i\} = \{s \in S : is=s,\ si=s\}$,
(11) $V = V^- \cup \{i\} = \{s \in S : is=i,\ si=i\}$,
(12) $W_r = W_r^- \cup \{i\} = \{s \in S : is=s,\ si=i\}$,
(13) $W_1 = W_1^- \cup \{i\} = \{s \in S : is=i,\ si=s\}$.

We would draw attention to the fact that the notation U^-, V^-, etc. is rather arbitrary as to the choice of the letters: we simply use the last six letters of the alphabet. However, the superscript "$^-$" and the subscripts "$_r$" and "$_1$" have a significance which is obvious from Definition 3.

Clearly, the subsets (1) through (9) are pairwise disjoint and their union is S^-. One or more of these subsets may be empty. Moreover, the intersection of two or more of the subsets (10) through (13) is precisely the idempotent element i. Further properties of the subsets of Definition 3, which are easily verified, are given in

PROPERTY 4. Let S be a semigroup containing the idempotent element i. The subsets of S given in Definition 3 satisfy the following conditions (a) through (h).

(a) U is a subsemigroup of S with i as identity element, and V is a subsemigroup of S with i as zero element.

(b) W_r is a right zero subsemigroup of S, and W_1 is a left zero subsemigroup of S.

(c) If $s \in Y_r$ then $si \in U^-$, hence if $Y_r \neq \varnothing$ then $U^- \neq \varnothing$.

(d) If $s \in Y_1$ then $is \in U^-$, hence if $Y_1 \neq \varnothing$ then $U^- \neq \varnothing$.

(e) If $s \in Z_r$ then $is \in W_r^-$, hence if $Z_r \neq \varnothing$ then $W_r^- \neq \varnothing$.

(f) If $s \in Z_1$ then $si \in W_1^-$, hence if $Z_1 \neq \varnothing$ then $W_1^- \neq \varnothing$.

(g) If $s \in W_r^-$ and $t \in W_1^-$ then $ts \in X$, hence if $W_r^- \neq \varnothing$ and $W_1^- \neq \varnothing$ then $X \neq \varnothing$.

(h) If $s \in X$ then $is \in U^- \cup W_r^- \cup Y_r$ and $si \in U^- \cup W_1^- \cup Y_1$, hence if $X \neq \varnothing$ then $U^- \cup W_r^- \cup Y_r \neq \varnothing$ and $U^- \cup W_1^- \cup Y_1 \neq \varnothing$.

We wish to know all the ways in which it is possible to compose a semigroup S out of an idempotent element i together with zero or one or more nonempty subsets as given in Definition 3, (1) through (9). For the sake of simplicity we introduce the notion of composition.

DEFINITION 4. Let S be a semigroup containing the idempotent element i. The *composition of S with respect to i* is the union of the nonempty subsets (1) through (9) of Definition 3 which are contained in the partial groupoid S^-.

From the conditions (c) through (h) of Property 4 it is clear that not every combination of nonempty subsets taken from (1) through (9) of Definition 3 is a possible composition, by which we mean the composition of some semigroup. In order to find all possible compositions one can start by scrutinizing all combinations as to whether or not they satisfy conditions (c) through (h) of Property 4. There are $2^9=512$ combinations of nonempty subsets, 130 of which satisfy those conditions. If a given combination satisfies the conditions (c) through (h) of Property 4 it may be a possible composition. However in order to prove that it is a possible composition one has to give an explicit example of a semigroup S containing the idempotent element i such that S has this composition with respect to i. For 40 of the 130 combinations which may be possible compositions we have examples of semigroups with these combinations as composition. Due to lack of space we do not give here the list of the 130 possible compositions and the 40 examples. For the other $130-40=90$ combinations we conjecture that they are indeed possible compositions. The numerical results of the argument above are listed in Table 1, where the number of combinations and the number of possible compositions is given for each of zero through nine nonempty subsets of Definition 3, (1) through (9).

number of nonempty subsets	number of combinations	number of possible compositions
0	1	1
1	9	4
2	36	10
3	84	19
4	126	$\leqq 28$
5	126	$\leqq 29$
6	84	$\leqq 21$
7	36	$\leqq 12$
8	9	5
9	1	1
	+—	+—
	total 512	$\leqq 130$

TABLE 1. Numerical results concerning the composition of semigroups

3. **Method for investigating union extensions.** Equipped with the notion of composition of a semigroup we can work out the problems posed at the beginning of §2. We assume that a semigroup S contains an idempotent element i. By the composition of S we mean its composition with respect to i. We shall call the composition of S single, double, triple, etc. if S^- contains precisely one, two, three, etc. nonempty subsets of Definition 3, (1) through (9). The questions to be answered are: what conditions are necessary and sufficient in a semigroup A for the existence of union extensions of A by S, where S has a given composition. Furthermore, if A satisfies these conditions, how can one construct all union extensions of A by S?

In trying to answer these questions we use the following method. For a given composition of S and an arbitrary semigroup A we assume that we have a binary operation $(*)$ on the set $E=A \cup S^-$ satisfying condition (ii) of Property 3. The conditions necessary in A for associativity of $(*)$ are derived and, if necessary, enlarged upon until they are also sufficient. The construction of all union extensions of A by S is then described in terms of all possible associative binary operations $(*)$ on the set $S=A \cup S^-$ which satisfy condition (ii) of Property 3.

The problems concerning existence and construction of union extensions of a semigroup A by a semigroup S with single composition are solved along these lines in §4. However, in the case where we have to deal with ideal extensions no results are obtained. Hence we have no contribution to make to the existing literature on ideal extensions. In §5 union extensions of a semigroup A by a semigroup S with double composition are worked out for two cases. Herewith we indicate and demonstrate the way in which the problems concerning existence and construction of union extensions can be handled not only for semigroups S with double composition but also for semigroups S with threefold up to ninefold composition. However, we are not sure that satisfactory results can always be obtained in this way because we have not worked it out for all possible compositions.

For dealing with union extensions by a semigroup S with twofold up to ninefold composition the following theorem is very helpful.

THEOREM 1. *Let A be a semigroup, let S be a semigroup containing an idempotent element i, and let $A \cap S^- = \varnothing$. Let $E(*)$ be a union extension of A by S with extension idempotent i. Let T be a subsemigroup of S such that $i \in T$ and let $T^- = T\backslash\{i\}$. Then the subsemigroup $F(*)$ of $E(*)$, where the set $F=A \cup T^-$, is a union extension of A by T with extension idempotent i.*

Proof. Since T is a subsemigroup of S, clearly tt', ti and it are elements of T for all $t, t' \in T$. Therefore Property 3(ii) entails that for all $a \in A$ and all $t, t' \in T^-$, $a * t$, $t * a$ and $t * t'$ are elements of $A \cup T^- = F$. Hence $F(*)$ is indeed a subsemigroup of $E(*)$. Furthermore, $F(*)$ satisfies the conditions of Property 3. Hence $F(*)$ is a union extension of A by T.

From Property 4, it follows immediately that a semigroup S with nonsingle composition always contains one or more proper subsemigroups containing i. Therefore, Theorem 1 always provides the possibility of reducing problems concerning existence and construction of union extensions by a semigroup S with nonsingle composition. This idea is applied in §5.

Notice that if the composition of S is empty, i.e. if S consists only of the idempotent element i, then any semigroup A is itself the unique union extension of A by S.

4. **Union extension by semigroups with single composition.** Let S be a semigroup with a single composition. From Property 4 it follows immediately that S has one of the following four compositions: U^-, V^-, W_r^- or W_1^-. We consider these one by one in subsections 4.1 through 4.4.

4.1. *The composition* V^-. If S is a semigroup with composition V^-, then Property 4(a) ensures that $S=S^- \cup \{i\}$ is a semigroup with i as zero element. As already stated immediately after Property 2 this means that $E(*)$ is a union extension of a semigroup A by S if and only if it is an ideal extension of A by S. We have nothing to add to the existing literature concerning the existence and construction of ideal extensions.

4.2. *The composition* U^-. If S is a semigroup with composition U^-, then Property 4(a) states that $S=S^- \cup \{i\}$ is a monoid with i as identity element. Conversely, a monoid with identity element i has the composition U^-. The union extensions by S are characterized in Theorem 2.

THEOREM 2. *Let A be a semigroup, let S be a monoid with identity element i, and let $A \cap S^- = \varnothing$.*

(i) *If $i \notin S^-S^-$, then there is exactly one union extension $E(*)$ of A by S with extension idempotent i. Moreover, $E(*)$ is just the ideal extension of the semigroup S^- by the semigroup $A' = A \cup \{i\}$, obtained by adjoining the element i as zero element to A, determined by setting $a * s = s * a = s$ for all $s \in S^-$ and all $a \in A = A' \setminus \{i\}$.*

(ii) *If $i \in S^-S^-$, then there is at most one union extension $E(*)$ of A by S with extension idempotent i, $E(*)$ exists if and only if A contains a zero element o. Moreover, $E(*)$ is just the ideal extension of the semigroup $S' = S^- \cup \{o\}$, obtained by adjoining the element o as identity element to the partial groupoid S^- and putting $st = o$ in S' if $st = i$ in S, by the semigroup A, determined by setting $a * s = s * a = s$ for all $s \in S'$ and all $a \in A \setminus \{o\}$.*

Proof. (i) Assume that $i \notin S^-S^-$ and that $E(*)$ is a union extension of A by S with extension idempotent i. Property 3 ensures that $E = A \cup S^-$ and that for all $a, b \in A$ and all $s, t \in S^-$ we have $a * b = ab$, $s * t = st$ and also $a * s = is = s$ and $s * a = si = s$ because S is a monoid with i as identity element. Hence the operation $(*)$ on E is determined so that $E(*)$ is the unique union extension of A by S with extension idempotent i. Moreover, S^- is clearly an ideal of $E(*)$ and the Rees factor semigroup $E/S^- \cong A' = A \cup \{i\}$, where i is a zero element adjoined to A.

Hence $E(*)$ is just the ideal extension of S^- by A' determined by setting $a*s=s*a=s$ for all $s \in S^-$ and all $a \in A = A' \backslash \{i\}$.

Conversely, assume that $i \notin S^-S^-$ and let $E(*)$ be the ideal extension of S^- by A' determined by setting $a*s=s*a=s$ for all $s \in S^-$ and all $a \in A=A'\backslash\{i\}$. Clearly, $E(*)$ satisfies the conditions of Property 3 so that it is a union extension of A by S with extension idempotent i.

(ii) Assume that $i \in S^-S^-$ and that $E(*)$ is a union extension of A by S with extension idempotent i. As in the proof of (i) we know that $E=A \cup S^-$ and that for all $a, b \in A$ and all $s, t \in S^-$ we have $a*b=ab$, $s*t=st$ if $st \neq i$ and $s*a=a*s=s$. Let $s, t \in S^-$ such that $st=i$. Then $s*t \in A$, say $s*t=o \in A$. Now $(a*s)*t=s*t=o$ and $a*(s*t)=a*o=ao$ for all $a \in A$. Since $(*)$ is associative we have $ao=o$, hence o is a right zero element of A. Analogously one can show that o is a left zero element of A. We conclude that A contains a zero element o and that $s*t=o$ if $st=i$. Herewith we have shown that the operation $(*)$ on E is determined so that $E(*)$ is the unique union extension of A by S with extension idempotent i. Moreover, the semigroup S', obtained by adjoining the element o as identity element to the partial groupoid S^- and putting $st=o$ in S' if $st=i$ in S, is an ideal of $E(*)$ and the Rees factor semigroup $E/S' \cong A$. Hence $E(*)$ is just the ideal extension of S' by A determined by setting $a*s=s*a=s$ for all $s \in S'$ and all $a \in A\backslash\{o\}$.

Conversely, assume that $i \in S^-S^-$ and let $E(*)$ be the ideal extension of S' by A determined by setting $a*s=s*a=s$ for all $s \in S'$ and all $a \in A\backslash\{o\}$. Clearly, $E(*)$ satisfies the conditions of Property 3 so that it is a union extension of A by S with extension idempotent i.

4.3. *The composition* W_r^-. If S is a semigroup with composition W_r^-, then Property 4(b) ensures that $S=S^- \cup \{i\}$ is a right zero semigroup. Conversely each element x of a right zero semigroup is an idempotent element and the composition of S with respect to x is W_r^-. In the sequel we need the following lemma which is easily verified.

LEMMA 1. (i) *Let A be a semigroup containing the right zero element p. Then the subset $P=pA$ of A consists of all right zero elements of A and P is a principal ideal of A.*

(ii) *Let A be a semigroup containing the left zero element q. Then the subset $Q=Aq$ of A consists of all left zero elements of A and Q is a principal ideal of A.*

(iii) *An element o of a semigroup A is the zero element of A if and only if o is the unique right zero element of A, or also, if and only if o is the unique left zero element of A.*

Using Lemma 1 we can now give necessary and sufficient conditions for the existence of union extensions of A by S and describe the construction of such extensions.

THEOREM 3. *Let A be a semigroup, let S be a right zero semigroup containing the element i, and let $A \cap S^- = \varnothing$.*

(i) *Union extensions of A by S with extension idempotent i exist if and only if A contains a right zero element.*

(ii) *Let P be the nonempty set of right zero elements of A.*

Then all union extensions of A by S with extension idempotent i are obtained by finding all ideal extensions $E(*)$ of the right zero semigroup on the set $K=P \cup S^-$ by the Rees factor semigroup A/P which satisfy the following conditions: $a * b=ab$, $a * k=k$, $p * a=pa$ and $s * a \in P$ hold for all $a, b \in A\backslash P$, all $k \in K$, all $p \in P$ and all $s \in S^-$.

Proof. (i) Assume that $E(*)$ is a union extension of A by S. Since $is=s$ and $si=i$ Property 3 entails that $a * s=s$ and $s * a \in A$ for all $a \in A$ and all $s \in S^-$. Consider arbitrary elements $a, b \in A$ and $s \in S^-$. Then $(b * s) * a=s * a \in A$ and the associativity of $(*)$ entails that $b * (s * a)=(b * s) * a=s * a \in A$. Hence $s * a$ is a right zero element of A. Conversely, assume that A contains the right zero element p. Define on the set $E=A \cup S^-$ the binary operation $(*)$ as follows: $a * b=ab$, $a * s=s$, $s * a=pa$ and $s * t=t$ for all $a, b \in A$ and all $s, t \in S^-$. Clearly, $(*)$ is determined for every pair of elements of E. We prove the associativity of $(*)$ by verifying it for all possible triples of elements of E which we indicate by AAA, AAS^-, etc. depending on where the elements are. We omit expressions like "for all $a, b \in A$ and all $s \in S^-$".

AAA	$(a * b) * c = (ab) * c = abc = a * (bc) = a * (b * c)$
AAS^-	$(a * b) * s = (ab) * s = s = a * s = a * (b * s)$
AS^-A	$(a * s) * b = s * b = pb = a(pb) = a * (pb) = a * (s * b)$
S^-AA	$(s * a) * b = (pa) * b = p(ab) = s * (ab) = s * (a * b)$
AS^-S^-	$(a * s) * t = s * t = t = a * t = a * (s * t)$
S^-AS^-	$(s * a) * t = (pa) * t = t = s * t = s * (a * t)$
S^-S^-A	$(s * t) * a = t * a = pa = ppa = s * (pa) = s * (t * a)$
$S^-S^-S^-$	$(r * s) * t = s * t = t = r * t = r * (s * t)$.

Hence $(*)$ is associative. Since $E(*)$ satisfies the conditions of Property 3, it is a union extension of A by S.

(ii) Assume that $E(*)$ is a union extension of A by S with extension idempotent i. From the proof of (i) we know that $s * a \in P$ for all $s \in S^-$ and all $a \in A$. If $s \in S^-$ and $p \in P$, then $s * p = s * (p * p) = (s * p) * p = p$. Also $a * p = p$ for all $a \in A$ and all $p \in P$. Hence every element of P is a right zero element of E. Likewise, $a * s = s$ and $t * s = s$ for all $a \in A$ and all $s, t \in S^-$, hence every element of S^- is a right zero element of E. Clearly, the set $K=P \cup S^-$ is just the set of all right zero elements of $E(*)$ so that $K(*)$ is an ideal of $E(*)$ according to Lemma 1. Moreover, it is obvious that the Rees factor semigroup $E/K \cong A/P$. Hence $E(*)$ is an ideal extension of the right zero semigroup on the set $K=P \cup S^-$ by the semigroup A/P.

Let $a, b \in A\backslash P$ and $p \in P$. Since $E(*)$ is a union extension of A by S it follows from Property 3 that $a * b = ab$ and $p * a = pa$. Above we found that K is the set of right zero elements of $E(*)$, hence $a * k = k$ for all $a \in A$ and all $k \in K$. From the proof of (i) we know that $s * a \in P$ for all $s \in S^-$ and all $a \in A\backslash P$.

Conversely, assume that $E(*)$ is an ideal extension of the right zero semigroup on the set $K = P \cup S^-$ by A/P satisfying the conditions of (ii). Then the set $E = K \cup (A\backslash P) = A \cup S^-$. Moreover, it is easily verified that also condition (ii) of Property 3 is satisfied. Hence $E(*)$ is a union extension of A by S with extension idempotent i.

By counterexamples one can show that not every ideal extension of the right zero semigroup on the set $K = P \cup S^-$ by the semigroup A/P is a union extension of A by S.

Notice that all freedom in the construction of the union extensions of A by S resides in the choice of the element $s * a \in P$ if $a \in A\backslash P$. Even this choice is restricted because $(*)$ has to be associative. The particular determination of $(*)$ used in the proof of Theorem 3(i) clearly satisfies this condition.

Theorem 3 yields the following two corollaries, which can be proved by straightforward verification using Lemma 1.

COROLLARY 3.1. *Let A be a right zero semigroup, let S be a right zero semigroup containing the element i, and let $A \cap S^- = \varnothing$. Then the unique union extension $E(*)$ of A by S with extension idempotent i is the right zero semigroup on the set*

$$E = A \cup S^-.$$

COROLLARY 3.2. *Let A be a semigroup containing the zero element o, let S be a right zero semigroup containing the element i, and let $A \cap S^- = \varnothing$. Then the unique union extension of A by S with extension idempotent i is $E(*)$, where the set $E = A \cup S^-$ and the operation $(*)$ is determined by $a * b = ab$, $a * s = s$, $s * a = o$ and $s * t = t$ for all $a, b \in A$ and all $s, t \in S^-$.*

4.4. *The composition W_1^-*. If S is a semigroup with composition W_1^-, then Property 4(b) ensures that $S = S^- \cup \{i\}$ is a left zero semigroup. Conversely, each element x of a left zero semigroup S is an idempotent element and the composition of S with respect to x is W_1^-. Clearly all properties given in subsection 4.3 for a right zero semigroup are also valid if we replace "right" by "left" appropriately.

5. **Union extensions by semigroups with double composition.** Let S be a semigroup with double composition. This means that the composition of S is one of the following ten (see Table 1):

$$U^- \cup V^-, \quad U^- \cup X, \quad V^- \cup W_r^-,$$
$$U^- \cup W_r^-, \quad U^- \cup Y_r, \quad V^- \cup W_1^-,$$
$$U^- \cup W_1^-, \quad U^- \cup Y_1, \quad W_r^- \cup Z_r,$$
$$W_1^- \cup Z_1.$$

We consider only two of these compositions in detail, viz. $U^- \cup V^-$ and $U^- \cup X$. In both cases U is an ideal of S as is easily verified from Definition 3. Therefore we consider first a more general situation in the following theorem.

THEOREM 4. *Let A be a semigroup, let S be a semigroup containing the idempotent element i, and let $A \cap S^- = \varnothing$. Furthermore, let U be an ideal of S and let i be a two-sided identity element of U. Then a semigroup $E(*)$ is a union extension of A by S with extension idempotent i if and only if $E(*)$ is an ideal extension of the unique union extension $F(*)$ of A by U with extension idempotent i by the Rees factor semigroup S/U and this ideal extension satisfies the following conditions:*
For all $a \in A$, $u \in U^-$ and $s \in S\backslash U$ hold

$$\begin{aligned} a * s &= is \in U^- && \text{if } is \neq i, & u * s &= us \in U^- && \text{if } us \neq i,\\ &\in A && \text{if } is = i, & &\in A && \text{if } us = i,\\ s * a &= si \in U^- && \text{if } si \neq i, & s * u &= su \in U^- && \text{if } su \neq i,\\ &\in A && \text{if } si = i, & &\in A && \text{if } su = i.\end{aligned}$$

Proof. Assume that $E(*)$ is a union extension of A by S with extension idempotent i. Since U is a subsemigroup of S and $i \in U$ we know from Theorem 1 that the subsemigroup $F(*)$ of $E(*)$, where the set $F = A \cup U^-$, is a union extension of A by U with extension idempotent i. Since U is a monoid with i as identity element Theorem 2 entails that $F(*)$ is the unique union extension of A by U. Furthermore, since U is an ideal of S clearly $F(*)$ is an ideal of $E(*)$ and the Rees factor semigroup $E/F \simeq S/U$. Hence $E(*)$ is an ideal extension of $F(*)$ by S/U. Since $E(*)$ is a union extension of A by S with extension idempotent i it follows from Property 3 that for all $a \in A$ and all $s \in S\backslash U$ we have $a * s = is$ if $is \neq i$, $a * s \in A$ if $is = i$, $s * a = si$ if $si \neq i$ and $s * a \in A$ if $si = i$. Property 3 also ensures that for all $u \in U^-$ and all $s \in S\backslash U$ we have $u * s = us$ if $us \neq i$, $u * s \in A$ if $us = i$, $s * u = su$ if $su \neq i$ and $s * u \in A$ if $su = i$. Clearly is, si, us and su are all in U because $i \in U$ and U is an ideal of S.

Conversely, assume that the unique union extension $F(*)$ of A by U with extension idempotent i exists and that $E(*)$ is an ideal extension of $F(*)$ by S/U satisfying the conditions. Clearly, the set $E = F \cup (S\backslash U) = A \cup S^-$. From the conditions of the theorem together with Theorem 2 it is easily verified that the conditions of Property 3(ii) are satisfied. Hence $E(*)$ is a union extension of A by S with extension idempotent i.

The conditions of Theorem 4 are indeed necessary because one can show by counterexamples that not every ideal extension of $F(*)$ by S/U is a union extension of A by S.

In the subsections 5.1 and 5.2 we derive for semigroups S with composition $U^- \cup V^-$ and $U^- \cup X$ necessary and sufficient conditions in a semigroup A for the existence of union extensions of A by S, and also rules for the construction of such extensions. We use Theorem 4 only in case S has the composition $U^- \cup X$.

5.1. *The composition* $U^- \cup V^-$. If S is a semigroup with composition $U^- \cup V^-$, then Property 4(a) entails that $U = U^- \cup \{i\}$ and $V = V^- \cup \{i\}$ are subsemigroups of S both containing the idempotent element i. Therefore, Theorem 1 provides necessary conditions for the existence of union extensions of a semigroup A by S. These conditions also turn out to be sufficient and the construction of the union extensions can easily be characterized.

THEOREM 5. *Let A be a semigroup, let S be a semigroup containing the idempotent element i and having the composition $U^- \cup V^-$, and let $A \cap S^- = \varnothing$.*

(i) *Union extensions of A by S with extension idempotent i exist if and only if there exists a union extension of A by U with extension idempotent i and there exist ideal extensions of A by V.*

(ii) *Each union extension $E(*)$ of A by S with extension idempotent i is obtained from the unique union extension $F(*)$ of A by U with extension idempotent i by taking an ideal extension $F_1(*_1)$ of A by V and extending the operation $(*)$ on F to the set $E = F \cup V^- = A \cup S^-$ as follows:*

$$a * v = a *_1 v, \qquad v * v' = v *_1 v',$$
$$v * a = v *_1 a, \qquad u * v = v * u = u,$$

for all $a \in A$, all $u \in U^-$ and all $v, v' \in V^-$.

Proof. First we prove the necessity of the conditions in (i) and (ii). Assume that $E(*)$ is a union extension of A by S with extension idempotent i. Since U and V are subsemigroups of S both containing i we know from Theorem 1 that union extensions of A by U and of A by V, both with extension idempotent i, exist. Since i is the zero element of V each union extension of A by V with extension idempotent i is an ideal extension of A by V. Hence the conditions of (i) are necessary. Moreover, since U is a monoid with i as identity element Theorem 2 entails that $F(*)$ is the unique union extension of A by U. Also $F_1(*_1)$ where the set $F_1 = A \cup V^-$ and $(*_1)$ is $(*)$ restricted to F_1, is an ideal extension of A by V. Clearly, $a * v = a *_1 v$, $v * a = v *_1 a$ and $v * v' = v *_1 v'$ for all $a \in A$ and all $v, v' \in V^-$. Furthermore, $uv = (ui)v = u(iv) = ui = u$ and likewise $vu = u$ for all $u \in U^-$ and all $v \in V^-$ so that $u * v = v * u = u$. Hence the conditions of (ii) are necessary.

In order to prove the sufficiency of the conditions in (i) and (ii) we now assume that the unique union extension $F(*)$ of A by S with extension idempotent i exists, that $F_1(*_1)$ is an ideal extension of A by V, and that an operation $(*)$ on the set $E = F \cup V^- = A \cup S^-$ satisfies the conditions of (ii). Clearly, the operation $(*)$ is defined for all pairs of elements of E. Notice that if A contains a zero element o, then for all $a \in A$ and all $v \in V^-$ we have $a *_1 (o *_1 v) = (a *_1 o) * v = o *_1 v$, hence $o *_1 v$ is a right zero element of A so that $o *_1 v = o$, and analogously $v *_1 o = o$. With this remark it is easy to verify that $(*)$ is associative, however we do not write it out as it is a laborious task. We conclude that $E(*)$ is a semigroup satisfying the conditions of Property 3 so that it is a union extension of A by S with extension idempotent i. Hence the conditions of (ii) are indeed sufficient and therefore also those of (i).

Theorem 5 ensures that if the union extension $F(*)$ of A by U with extension idempotent i exists, then there is precisely one union extension of A by S with extension idempotent i for each ideal extension of A by V.

5.2. *The composition* $U^- \cup X$. If S is a semigroup with composition $U^- \cup X$, then Property 4(a) ensures that $U = U^- \cup \{i\}$ is a subsemigroup containing i. Hence Theorem 1 is applicable so that we have a necessary condition for the existence of a union extension of A by S. Moreover, this condition is also sufficient as we shall see. For the construction of the union extensions we use the notion of the two-sided annihilator M_A of a semigroup A containing the zero element o, i.e. the set of all elements m in A such that $ma = am = o$ for all $a \in A$.

THEOREM 6. *Let A be a semigroup, let S be a semigroup containing the idempotent element i and having the composition $U^- \cup X$, and let $A \cup S^- = \varnothing$.*

(i) *Union extensions of A by S with extension idempotent i exist if and only if the union extension of A by U with extension idempotent i exists.*

(ii) *All union extensions of A by S with extension idempotent i are obtained from the unique union extension $F(*)$ of A by U with extension idempotent i by extending the operation $(*)$ on F to the set $E = F \cup X = A \cup S^- = A \cup U^- \cup X$ as follows:*

$$a * x = x * a = xi \quad (= ix \in U^-),$$
$$u * x = ux \quad \text{if } ux \neq i, \qquad x * u = xu \quad \text{if } xu \neq i,$$
$$= o \quad \text{if } ux = i, \qquad = o \quad \text{if } xu = i,$$

where o is the zero element of A, and

$$x * x' = xx' \quad \text{if } xx' \neq i,$$
$$\in M_A \quad \text{if } xx' = i,$$

such that if $xx'x'' = i$ then $x * (x'x'') = (xx') * x''$, hold for all $a \in A$, all $u \in U^-$, and all $x, x', x'' \in X$.

Proof. (i) The necessity follows immediately from Theorem 4, the sufficiency follows from the sufficiency of the conditions of (ii).

(ii) Assume that the unique union extension $F(*)$ of A by S with extension idempotent i exists and that $E(*)$ is obtained as described in (ii). Clearly, $E = A \cup S^-$ and we have to verify that $(*)$ is associative and hence, according to Property 3, $E(*)$ is indeed a union extension of A by S with extension idempotent i. We shall not write out the verification because it is rather lengthy though straightforward. For this verification the following remarks are important. From Property 4(h) it follows that $ix, xi \in U^-$ for all $x \in X$. Therefore $ix = ixi = xi$ and $ixx' = xix' = xx'i$ for all $x, x' \in X$. Furthermore, if there are elements $x, x' \in X$ such that $xx' = i$, then $i = i(xx')i = (ix)(x'i)$. Hence $i \in U^-U^-$ so that, according to Theorem 2, A contains a zero element, say o, and thus $M_A \neq \varnothing$. We conclude that the conditions of (ii) are sufficient.

Conversely, assume that $E(*)$ is a union extension of A by S with extension idempotent i. Theorem 4 entails that the unique union extension $F(*)$ of A by U with extension idempotent i exists. Moreover, since $E(*)$ is an ideal extension of $F(*)$ by S/U clearly the set $E=F\cup(S\backslash U)=F\cup X=A\cup U^-\cup X=A\cup S^-$, and $E(*)$ is obtained as an extension of the operation $(*)$ on F to the set E. As remarked above, $ix=xi\neq i$ so that Theorem 4 entails $a*x=x*a=ix=xi\in U^-$ for all $a\in A$ and all $x\in X$. Since $E(*)$ is a union extension Property 3 entails that $u*x=ux$ if $ux\neq i$, that $x*u=xu$ if $xu\neq i$, and that $x*x'=xx'$ if $xx'\neq i$ for all $u\in U^-$ and all $x, x'\in X$. Now suppose that for $u\in U^-$ and $x\in X$ holds $ux=i$. Then Theorem 4 entails that $u*x\in A$ so that $a*(u*x)=a(u*x)$, while Theorem 2 entails that $(a*u)*x=u*x$ for all $a\in A$. Hence $u*x$ is a right zero element of A. Moreover, $i=(ux)i=u(xi)$ so that $i\in U^-U^-$ and A contains a zero element o according to Theorem 2. But then $u*x=o$. Analogously, if $u\in U^-$ and $x\in X$ are such that $xu=i$, then $x*u=o$. Now, let us suppose $x, x'\in X$ such that $xx'=i$. Then Property 3 entails that $x*x'\in A$, and for all $a\in A$ we have $a*(x*x')=a(x*x')$ and $(x*x')*a=(x*x')a$. However, $(a*x)*x'=(ix)*x'=o$ because $ix\in U^-$, and $x*(x'*a)=x*(x'i)=o$ because $x'i\in U^-$. Hence $x*x'\in M_A$. The last condition of (ii) is obvious from the associativity of $(*)$. Hence the conditions of (ii) are necessary.

Notice that all freedom in the construction of A by S resides in the choice of the element $x*x'\in M_A$ if $xx'=i$. Even this choice is limited by the last condition of (ii) which ensures the associativity of $(*)$ on E.

In the case where the annihilator of A consists only of the zero element of A the following corollary of Theorem 6 is easily proved.

COROLLARY 6.1. *Let A be a semigroup containing the zero element o and let $M_A=\{o\}$. Let S be a semigroup containing the idempotent element i and having the composition $U^-\cup X$, and let $A\cap S^-=\varnothing$. Then there is a unique union extension $E(*)$ of A by S with extension idempotent i. This extension is obtained by defining the operation $(*)$ on the set $E=A\cup S^-=A\cup U^-\cup X$ as follows:*

$$\begin{aligned} a*b &= ab, & & \\ a*u &= u*a = u, & s*t &= st \quad \text{if } st\neq i,\\ a*x &= x*a = xi \quad (=ix\in U^-), & &= o \quad \text{if } st = i,\end{aligned}$$

hold for all $a, b\in A$, all $u\in U^-$, all $x\in A$ and all $s, t\in S^-$.

In the case where the semigroup S is such that $i\notin U^-U^-$ Theorem 6 has the following simple consequence because then $xx'\neq i$ for all $x, x'\in X$.

COROLLARY 6.2. *Let A be a semigroup, let S be a semigroup containing the idempotent element i and having the composition $U^-\cup X$, and let $A\cap S^-=\varnothing$. If $i\notin U^-U^-$ then there is a unique union extension $E(*)$ of A by S with extension* *idempotent i. This extension is obtained by defining the operation* $(*)$ *on the set* $E=A\cup S^{-}=A\cup U^{-}\cup X$ *as follows*:

$$a*b=ab,\qquad s*t=st,\qquad a*u=u*a=u,$$

and

$$a*x=x*a=xi\quad(=ix\in U^{-}),$$

hold for all $a,b\in A$, *all* $u\in U^{-}$, *all* $x\in X$ *and all* $s,t\in S^{-}$.

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Delft University of Technology,
Delft, The Netherlands