FUNCTIONAL ANALYTIC PROPERTIES OF TOPOLOGICAL SEMIGROUPS AND N-EXTREME AMENABILITY(¹)

BY

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Abstract. Let S be a topological semigroup, LUC (S) be the space of left uniformly continuous functions on S, and $\Delta(S)$ be the set of multiplicative means on LUC (S). If (*) LUC (S) has a left invariant mean in the convex hull of $\Delta(S)$, we associate with S a *unique* finite group G such that for any maximal proper closed left translation invariant ideal I in LUC (S), there exists a linear isometry mapping LUC $(G)/I$ one-one onto the set of bounded real functions on G. We also generalise some recent results of T. Mitchell and E. Granirer. In particular, we show that S satisfies (*) iff whenever S is a jointly continuous action on a compact hausdorff space X, there exists a nonempty finite subset F of X such that $sF = F$ for all $s \in S$. Furthermore, a discrete semigroup S satisfies (*) iff whenever $\{T_s; s \in S\}$ is an antirepresentation of S as linear maps from a norm linear space X into X with $\|T_s\| \leq 1$ for all $s \in S$, there exists a finite subset $\sigma \subseteq S$ such that the distance (induced by the norm) of x from $K_x =$ linear span of $\{x - T_s x; x \in X, s \in S\}$ in X coincides with distance of $O(\sigma, x) = \{(1/|\sigma|) \sum_{a \in \sigma} T_{at}(x); t \in S\}$ from 0 for all $x \in X$.

1. **Preliminaries and some notations.** Let S be a *topological semigroup* (i.e. a set with an associative multiplication and a hausdorff topology such that for each $a \in S$, the mappings $s \to a \cdot s$ and $s \to s \cdot a$, $s \in S$, are continuous from S into S) and X a hausdorff topological space. An *action* of S on X is a separately continuous mapping $S \times X \to X$ (i.e. continuous in each one of the variables when the other one of the variables is kept fixed) denoted by $(s, x) \to s \cdot x$, such that $s \cdot (t \cdot x) = (s \cdot t) \cdot x$ for all $s, t \in S$ and $x \in X$. An action on X is *jointly continuous* if the mapping $S \times X \to X$ is continuous when $S \times X$ has the product topology.

Let S be a topological semigroup which acts on a hausdorff topological space X, f be a bounded real function on X, $_s f(x) = f(sx)$ for all $s \in S$, $x \in X$ and $\|f\| = \sup_{x \in X} |f(x)|$; f is called *S-uniformly continuous* if f is continuous, and whenever $s_\alpha \to s_0$, $s_\alpha, s_0 \in S$, then $\lim_\alpha \|_{s_\alpha} f - _{s_0} f\| = 0$. We shall denote by

$m(X) =$ the space of bounded real functions on X,

$C(X)$ = the space of bounded continuous real functions on X,

LUC (S, X) = the space of S-uniformly continuous real functions on X.

When S is discrete, then $C(X)$ = LUC (S, X).

A subspace A of $m(X)$ is called *S-translation invariant* if $_sf \in A$ whenever $f \in A$ and $s \in S$. $C(X)$ [LUC (S, X)] is an S-translation invariant Banach algebra of $m(X)$ [$C(X)$] containing constants.

If T is a subset of X, then $1_T \in m(X)$ such that

$$1_T(t) = 1 \quad \text{if } t \in T,$$
$$= 0 \quad \text{if } t \notin T.$$

Let A be a (norm) closed S-translation invariant subalgebra of $m(X)$ containing constants. For any $a \in S$, define $l_a : A \to A$ by $(l_a f)(x) = _a f(x)$ for all $x \in X$ and $f \in A$ and $L_a : A^* \to A^*$ by $(L_a \phi)(f) = \phi(_a f)$ for all $f \in A$ and $\phi \in A^*$ (where A^* is the conjugate space of A). An element $\phi \in A^*$ is a *mean* if $\phi(f) \geqq 0$ for all $f \geqq 0$ and $\phi(1_X) = 1$; ϕ is *multiplicative* if $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in A$; and ϕ is *S-invariant* if $L_s \phi = \phi$ for all $s \in S$. The set of means on A is w^*- (i.e. $\sigma(A^*, A)$) compact.

For $a \in X$, let $p_a \in m(X)^*$ be the *point measure* at a, i.e. $p_a(f) = f(a)$ for all $f \in m(X)$. $\phi \in A^*$ is a point measure iff ϕ is the restriction to A of some point measure on $m(X)$. The set of point measure on A is w^*-dense in the set of multiplicative means on A [4, p. 275, proof of Corollary 19]. Furthermore, the set of multiplicative means on $m(X)$ is βX, the Stone-Čech compactification of X.

For any set A, $|A|$ will denote the *cardinality* of A. If A is a subset of a linear space, then Co A will denote the *convex hull* of A.

A topological semigroup S can be considered as an action on S defined by the mapping $(s, t) \to s \cdot t$, $s, t \in S$. In this case, the space of *left uniformly continuous* (i.e. S-uniform continuous) *functions* on S is denoted by LUC (S) (see Mitchell [16] and Namioka [17]).

For any topological semigroup S, and a positive integer n, LUC (S) is *left amenable* (LA) if LUC (S) has a *left invariant mean* (LIM), i.e. an S-invariant mean; LUC (S) is *n-extremely left amenable* (n-ELA), if there exists a subset H_0 of $\Delta(S)$, $|H_0| = n$, which is minimal with respect to the property $L_a H_0 = H_0$ for all $a \in S$, where $\Delta(S)$ denotes the set of multiplicative means on LUC (S). In this case, S/H_0 will denote the factor semigroup of S determined by the equivalence relation E: for any $a, b \in E$, $a \, E \, b$ iff $L_a \phi = L_b \phi$ for all $\phi \in H_0$. Note that if H_1 is another nonempty finite subset of $\Delta(S)$ which is minimal with respect to the property $L_a H_1 = H_1$ for all $a \in S$, then $|H_1| = n$. Furthermore, for any topological semigroup S, LUC (S) has a LIM in Co $\Delta(S)$ iff LUC (S) is n-ELA for some n [12, Remark 3.1 (a), (b)]. Namioka in [17] studies topological semigroups S for which LUC (S) is LA. Topological semigroups S for which LUC (S) is n-ELA have been studied by Mitchell in [16] (for $n=1$) and the author in [12].

A semigroup S is LA [n-ELA] if S when considered as a discrete topological semigroup, LUC $(S) = m(S)$ is LA [n-ELA]. A recent survey of the theory of LA

semigroups can be found in Day [3]. The class of ELA (i.e. 1-ELA) semigroups have been studied by Mitchell [14] and Granirer [6], [7] and [8]. Sorenson in [18] and the author in [12] consider the class of n-ELA semigroups for $n \geq 1$.

The class of ELA semigroups is immense (see for example, Granirer [6], [7], [8]) and n-ELA semigroups include product semigroups $S \times G$ where S is ELA and G is any group of order n. Furthermore, it has been shown by the author in [12] that for each n there exists a huge class of topological semigroups S for which LUC (S) is n-ELA and yet S is not even LA. However, if S is a subsemigroup of a locally compact group, then LUC (S) is n-ELA iff S is a finite group of order n (Granirer and Lau [9, Theorem 3]).

REMARK 1.1. Let S be a topological semigroup. The following are known and will be useful for our purpose:

(a) If LUC (S) has a LIM of the form $(1/n) \sum_1^n \phi_i$, where $\phi_i \in \Delta(S)$ (not necessarily distinct), then LUC (S) is m-ELA for some $m \leq n$, m divides n [12, Lemma 4.7].

(b) If LUC (S) is n-ELA, then there exists a collection $\mathscr{F}$ of n distinct open and closed subsets of S, with union S, such that $1_A \in$ LUC (S) for all $A \in \mathscr{F}$ and $\mathscr{F}$ is the decomposition of S by cosets of S/H for any finite subset $H \subseteq \Delta(S)$ satisfying $L_a H = H$ for all $a \in S$ [12, Theorem 4.1]. Consequently, S/H is a group of order n, and is a continuous homomorphic image of S. Furthermore, for any $\psi \in H$ and σ coset representative of S/H, $(1/n) \sum_{a \in \sigma} L_a \psi$ is a LIM on LUC (S).

2. Actions of a topological semigroup on compacta.

The basis to our work lies on the generalisation of a fixed point theorem of T. Michell in [16, Theorem 1]. We first prove the following useful lemma:

LEMMA 2.1. *For any jointly continuous action of a topological semigroup S on a hausdorff topological space X, the mapping $S \to C(X)$ defined by $s \to {}_s f$, $s \in S$ and $f \in C(X)$ is continuous when $C(X)$ has the topology of uniform convergence on compacta. In particular, if X is compact, then LUC $(S, X) = C(X)$.*

Proof. Let $f \in C(X)$, $s_0 \in S$ and $\varepsilon > 0$. For any $x_0 \in X$, there exist neighbourhoods V_{x_0} of s_0 and U_{x_0} of x_0 such that

$$V_{x_0} \times U_{x_0} \subseteq \{(s, x); |f(s \cdot x) - f(s_0 \cdot x_0)| < \varepsilon\}.$$

If K is a compact subset of X, there exists $\{x_1, \ldots, x_n\} \subseteq K$ such that $\bigcup_{i=1}^n U_{x_i} \supseteq K$. Consequently, if $V = \bigcap_{i=1}^n V_{x_i}$, then

$$\sup_{x \in K} |{}_s f(x) - {}_{s_0} f(x)| < \varepsilon \quad \text{for all } s \in V.$$

For any topological semigroup S, by "S acts on $\Delta(S)$" we shall mean the jointly continuous mapping of $S \times \Delta(S) \to \Delta(S)$ defined by $(s, \phi) \to L_s \phi$ for all $\phi \in \Delta(S)$ and $s \in S$ (see Mitchell [16, proof of Theorem 1]).

THEOREM 2.2. *For any topological semigroup S and fixed n*:

(a) *If* LUC (S) *is n-ELA, then for any jointly continuous action of S on a compact hausdorff space X,*

(*) *there exists a nonempty subset $F \subseteq X$ such that $|F| \leq n$, $|F|$ divides n and $aF = F$ for all $a \in S$.*

(b) *If S satisfies* (*) *when S acts on $\Delta(S)$, then* LUC (S) *is m-ELA for some $m \leq n$, m divides n.*

Proof. (a) Let $H_0 = \{\phi_1, \ldots, \phi_n\} \subseteq \Delta(S)$ be such that $L_a H_0 = H_0$ for all $a \in S$, and $z_0 \in X$ be fixed. Define $\psi_i \in$ LUC $(S, X)^*$ by $\psi_i(f) = \phi_i(T_{z_0} f)$ where $(T_{z_0} f)(s) = f(sz_0)$, $f \in$ LUC (S, X), $s \in S$ and $1 \leq i \leq n$. Then $\{\psi_i, 1 \leq i \leq n\}$ are multiplicative means on LUC (S, X) and $(1/n) \sum_1^n \psi_i$ is S-invariant on LUC $(S, X) = C(X)$ (Lemma 2.1). Since X is compact, each ψ_i becomes a point measure p_{x_i} on $C(X)$, $1 \leq i \leq n$, $x_i \in X$ [4, Lemma 2.5, p. 278]. Hence $\sum_1^n f(sx_i) = \sum_1^n f(x_i)$ for all $s \in S$, $f \in C(X)$. Since $C(X)$ separates closed sets, we have $sK = K$ for all $s \in S$, where $K = \{x_1, \cdots, x_n\}$. Furthermore, if $a, b \in S$ are such that $L_a \phi = L_b \phi$ for all $\phi \in H_0$, then $f(ax_i) = \phi_i(T_{z_0}(af)) = \phi_i(T_{z_0}(bf)) = f(bx_i)$ for all $f \in C(X)$, $1 \leq i \leq n$. Since $C(X)$ separates points, we have $ax = bx$ for all $x \in K$. Consequently, we may consider the finite group $G = S/H_0$ of order n (Remark 1.1 (d)) as a group of transformation from K into K defined by $\bar{a}(x) = ax$ for all $x \in K$, $a \in S$, where $\bar{a}$ is the homomorphic image of a in S/H_0. Let $x_0 \in K$ be fixed and $F = \{gx_0; g \in G\}$. Then $gF = F$ for all $g \in G$. Define on G the equivalence relation E: for any $a, b \in G$, $a E b$ iff $ax_0 = bx_0$. Let $\{g_1, \ldots, g_k\}$ be representatives from the equivalence classes $J_1, \ldots, J_k$ of G with respect to E, where $g_i \in J_i$ and g_1 is the identity of G. Since the mappings from H_1 onto H_i defined by $g \to g_i g$ for all $g \in H_1$ is one-one, $|J_1| = |J_i|$ for all $1 \leq i \leq k$. Consequently $|G| = k|J_1|$, i.e. $k = |F|$ divides $|G| = n$ and $sF = F$ for all $s \in S$.

(b) If $F \subseteq \Delta(S)$ is such that $L_a F = F$ for all $a \in S$, and $|F| \leq n$, $|F|$ divides n, then $(1/|F|) \sum_{\phi \in F} \phi$ is a LIM on LUC (S). Our assertion now follows from Remark 1.1 (a).

REMARK 2.3. (a) When $n = 1$, Theorem 2.2 reduces to Mitchell's fixed point theorem [16, Theorem 1].

(b) Note that in Theorem 2.2 (a), $F \subseteq X$ chosen in our proof satisfies the additional property that if F_0 is a nonempty subset of F such that $aF_0 \subseteq F_0$ for all $a \in S$, then $F_0 = F$. For otherwise, let $g_0 x_0 \in F_0$, $g_0 \in G$ (x_0 and G are as in the proof), then $F = \{gg_0 x_0; g \in G\} \subseteq F_0 \subseteq F$, i.e. $F_0 = F$.

(c) In general $|F|$ in Theorem 2.2 (a) need not be n (e.g. consider the action of S on the discrete space $X = \{x_0\}$ defined by $sx_0 = x_0$ for all $s \in S$). However, if n is prime, then F is either n or 1 (in this case, F becomes a fixed point).

A subsemigroup G of a semigroup S is a *left ideal group* if G is a group and a left ideal of S.

COROLLARY. *Let S be a compact topological semigroup with jointly continuous multiplication (i.e. the mapping from $S \times S \to S$ defined by $(s, t) \to s \cdot t$, $s, t \in S$ is*

continuous when $S \times S$ has the product topology). Then LUC (S) *is n-ELA iff S has a left ideal group of n elements.*

Proof. If LUC (S) is n-ELA, then by applying Theorem 2.2 and Remark 2.3 (b) to the jointly continuous action of S on S defined by $(s, t) \to s \cdot t, s, t \in S$, we obtain a minimal left ideal F of S, $|F| \leq n$. Necessarily F is a group. Since $H_0 = \{p_s; s \in F\} \subseteq \Delta(S)$ satisfies $L_a H_0 = H_0$ for all $a \in S$, $|F| = n$. The converse is trivial.

3. Maximal translation invariant ideals. In this section we associate with each topological semigroup S for which LUC (S) is n-ELA with a *unique* finite group G such that if I is a maximal proper closed left translation invariant (i.e. if $f \in I$ then $_s f \in I$ for all $s \in I$) ideal in LUC (S), then there exists a linear isometry mapping LUC $(S)/I$ one-one into $m(G)$.

THEOREM 3.1. *For any topological semigroup S,* LUC (S) *is n-ELA iff there exists a unique (up to isomorphism) finite group G which is a continuous homomorphic image of S such that (*) G has order n and for any maximal proper left translation invariant norm closed ideal $I \subseteq$ LUC (S), there exists a multiplicative linear isometry T mapping* LUC $(S)/I$ *one-one into $m(G)$, and T satisfies:*

    (a) $T(\bar{1}_S) = 1_G$,
    (b) $T(\bar{f}) \geq 0$ *if* $f \geq 0, f \in$ LUC (S),
    (c) $T(_s \bar{f}) = {}_s T(\bar{f})$ *for all* $s \in S, f \in$ LUC (S).
(*Note: $\bar{s}$ denotes the homomorphic image of S in G and $\bar{f}$ the equivalence class of $f \in$ LUC (S) in LUC $(S)/I$.*)

Proof. If LUC (S) is n-ELA, let G be the finite group (of order n) S/H_0, where H_0 is a finite subset of $\Delta(S)$ such that $L_a H_0 = H_0$ for all $a \in S$. Then G is a continuous image of S (Remark 1.1 (b)). For any maximal proper left translation invariant closed ideal $I \subseteq$ LUC (S) (which exists by Zorn's lemma), let $\phi_0 \in \Delta(S)$ be such that $\phi_0(f) = 0$ for all $f \in I$ and $K = w^*$-closure of $\{L_s \phi_0; s \in S\}$. Applying Theorem 2.2 (a) to the jointly continuous action of S on K defined by $(s, \phi) \to L_s \phi$ for $s \in S$, $\phi \in K$, we obtain a finite subset $H_1 \subseteq \Delta(S)$ such that $L_s H_1 = H_1$ for all $s \in S$. Let $\psi_0 \in H_1$ be fixed. If $a, b \in S$ and $f, g \in$ LUC (S) are such that $L_a \phi = L_b \phi$ for all $\phi \in H_0$ and $f - g \in I$, then $L_a \psi_0 = L_b \psi_0$ (Remark 1.1 (b)) and hence $\psi_0(_a f - {}_b g) = \psi_0(_a(f - g)) = 0$. Consequently we may define a linear transformation T: LUC $(S)/I \to m(G)$ by $T(\bar{f})(\bar{s}) = \psi_0(_s f)$. It is simple to check that T is multiplicative and satisfies (a), (b) and (c). Furthermore, T is one-one since if $\bar{f} \neq \bar{g}$, then $f - g \notin I$ and hence $\psi_0(_a(f - g)) \neq 0$ for some $a \in S$, which implies $T(\bar{f})(\bar{a}) \neq T(\bar{g})(\bar{a})$.

To see that T is onto, we let $a_1, \ldots, a_n$ be representative from the coset decomposition of S by $G = S/H_0$, $S_1, \ldots, S_n$ respectively. Then for each $1 \leq i \leq n$, $1_{S_i} \in$ LUC (S) and $\mu = (1/n) \sum_1^n L_{a_i} \psi_0$ is a LIM on LUC (S) (Remark 1.1 (b)). Since $\mu(1_{S_1} + \cdots + 1_{S_n}) = 1$, $S_1, \ldots, S_n$ are disjoint and $L_{a_i} \psi_0(1_{S_j})$ takes only values 0

or 1 for each i, j, it follows that for each i, $1 \leq i \leq n$, there exists exactly one element from $S_1, \ldots, S_n$, say S_i for convenience, such that $(L_{a_i}\psi_0)(1_{S_i}) = 1$. Now for any $\pi \in m(G)$, define $f = \sum_{i=1}^n \pi(\bar{a}_i)1_{S_i}$, then $T(\bar{f}) = \pi$.

To show that T is an isometry, it suffices to show that for each $f \in \mathrm{LUC}\,(S)$, $\inf_{g \in I} \|f+g\| = \sup_{\phi \in K} |\phi(f)|$. In one direction $\sup_{\phi \in K} |\phi(f)| = \sup_{\phi \in K} |\phi(f+g)| \leq \|f+g\|$ for all $g \in I$. In the other direction, we shall produce $g_0 \in I$ such that $\|f+g_0\| = \sup_{\phi \in K} |\phi(f)|$. We first observe that $\{f \in \mathrm{LUC}\,(S); \phi(f) = 0$ for all $\phi \in K\}$ is a closed left translation invariant proper ideal in $\mathrm{LUC}\,(S)$ containing I and hence equals to I. Let h be the continuous extension of f when restricted to K without increasing in norm, and put $g_0 = h - f$. Then $g_0 \in I$ and $\|f+g_0\| = \|h\| = \sup_{\phi \in K} |\phi(f)|$.

G is unique for otherwise let $G' = \{t'_1, \ldots, t'_m\}$ be another finite group of order m which is a continuous homomorphic image of S satisfying (*), where $\bar{s}$, s' denote the homomorphic images of S in G, G' respectively. Let I be a maximal closed left translation invariant proper ideal in $\mathrm{LUC}\,(S)$; by assumption, there exists a multiplicative isometry T mapping $\mathrm{LUC}\,(S)/I$ one-one onto $m(G')$ which satisfies conditions (a), (b) and (c). Define $\psi_i \in \Delta(S)$ by $\psi_i(f) = T(\bar{f})(t'_i)$ for each $1 \leq i \leq m$, $f \in \mathrm{LUC}\,(S)$. Let $H_2 = \{\psi_1, \ldots, \psi_m\}$, then $L_s H_2 = H_2$ for all $s \in S$. For any $a, b \in S$, if $\bar{a} = \bar{b}$, then $L_a \psi = L_b \psi$ for all $\psi \in H_2$ (Remark 1.1 (b)) and hence $T(\bar{f})(a't'_i) = T(\bar{f})(b't'_i)$ for all $f \in \mathrm{LUC}\,(S)$, $1 \leq i \leq m$. Since T is onto and $m(G')$ separates points, we have $a't'_i = b't'_i$ for all $1 \leq i \leq m$, which implies $a' = b'$. Conversely if $a' = b'$, then $\psi_i(_a f) = T(_a\bar{f})(t'_i) = T(\bar{f})(a't'_i) = T(\bar{f})(b't'_i) = \psi_i(_b f)$ for all $1 \leq i \leq m$ and $f \in \mathrm{LUC}\,(S)$; hence $\bar{a} = \bar{b}$. Consequently, the mapping $G \to G_0$ defined by $\bar{a} \to a'$ is an isomorphism onto and hence $m = n$.

Conversely, if there exists a finite group G which satisfies (*), let T be a multiplicative linear isometry from $\mathrm{LUC}\,(S)/I$ one-one onto $m(G)$ satisfying (a), (b) and (c), where I is a maximal closed left translation invariant proper ideal in $\mathrm{LUC}\,(S)$, and $G = \{\bar{s}_1, \ldots, \bar{s}_n\}$. Define $\phi_i \in \Delta(S)$ by $\phi_i(f) = T(\bar{f})(\bar{s}_i)$. Then $(1/n)\sum_1^n \varphi_i$ is a LIM on $\mathrm{LUC}\,(S)$. Consequently, $\mathrm{LUC}\,(S)$ is m-ELA for some $m \leq n$ (Remark 1.1 (a)). By uniqueness of G and what we have proved, $m = n$.

4. n-ELA semigroups.

Let n be fixed, X be a norm linear space, $\{T_s; s \in S\}$ be an antirepresentation of S as linear transformations from X into X such that $\|T_s\| \leq 1$ for all $s \in S$, and $\sigma = \{a_1, \ldots, a_n\} \subseteq S$. We may define a linear transformation T_σ from X into X by $T_\sigma(x) = (1/n)\sum_1^n T_{a_i}(x)$ for all $x \in S$. Furthermore, if $x \in X$, denote by

$O(x) = \{T_s(x); s \in S\}$,

$O(\sigma, x) = \{T_{\sigma \circ t}(x); t \in S\}$,

$K_X = $ linear span of $\{x - T_s x; s \in S, x \in X\}$ in X.

Our next result, which partially generalises Theorem 5 I(b), II(b) of Graniner [8] asserts that if S is n-ELA, then $d(K_X, x) = d(O(\sigma, x), 0)$ for some finite subset $\sigma \subseteq S$, $|\sigma| \leq n$ and all $x \in X$ and d is the metric induced by the norm on X. We first prove the following lemma:

LEMMA 4.1. *Let S be a semigroup with f.i.p.r.i. and $\sigma_0 \subseteq S$, $|\sigma_0| = n$. If for each $a \in S$, and $f \in m(S)$, there exists a net $\{t_\alpha\}$ in S, depending on a and f such that $\lim \|l_{t_\alpha}(l_{\sigma_0}(f - l_a))\| = 0$, then S is m-ELA for some $m \leq n$, m divides n.*

Proof. For each $a \in S$ and $f \in m(S)$, let

$$K_{a,f} = \left\{ \phi \in \beta S; \frac{1}{n} \sum_{s \in \sigma_0} L_s \phi(_a f) = \frac{1}{n} \sum_{s \in \sigma_0} L_s \phi(f) \right\}.$$

Then $K_{a,f}$ are nonempty, w^*-closed subsets of βS. Furthermore, the family $\{K_{a,f}; a \in S, f \in m(S)\}$ has finite intersection property. In fact if $\alpha > 0$, $\{a_1, \ldots, a_k\} \subseteq S$ and $\{f_1, \ldots, f_k\} \subseteq m(S)$, let $t_1^\alpha, \ldots, t_k^\alpha$ be such that $\|l_{t_i^\alpha}(l_{\sigma_0}(f_i - _{a_i} f_i))\| < \alpha$. Let $c_\alpha \in \bigcap_{i=1}^k t_i S$, then $c_\alpha = t_i^\alpha s_i^\alpha$ for some $s_i^\alpha \in S$, and $\|p_{c_\alpha}(l_{\sigma_0}(f_i - _{a_i} f_i))\| \leq \|l_{t_i^\alpha}(l_{\sigma_0}(f_i - _{a_i} f_i))\| < \alpha$ for $1 \leq i \leq k$. If ϕ is a cluster point of the net $\{p_{c_\alpha}; \alpha > 0\}$, then $\phi \in \bigcap_{i=1}^k K_{a_i, f_i}$. By w^*-compactness of βS, there exists $\phi_0 \in \beta S$ such that

$$\frac{1}{n} \sum_{s \in \sigma_0} L_s \phi_0(_a f) = \frac{1}{n} \sum_{a \in \sigma_0} L_s \phi_0(f)$$

for all $f \in m(S)$, $a \in S$. By Remark 1.1 (a), S is m-ELA for some $m \leq n$, m divides n.

Let S be a (discrete) semigroup with f.i.p.r.i. (i.e. finite intersection property for right ideals), define as in Granirer [5] on S the equivalence relation (r): $a(r)b$ iff $ac = bc$ for some $c \in S$. Then (r) is two-sided stable (i.e. $a(r)b$ implies $ac(r)bc$ and $ca(r)cb$ for all $a, b, c \in S$) and the factor semigroup induced by (r) is denoted by $S/(r)$.

REMARK 4.2. We will need in what follows the following known results:

(a) A semigroup S is n-ELA iff S has f.i.p.r.i. and $S/(r)$ is a group of order n (see Sorenson [18, Theorem 3.3.6] and Lau [12, Theorem 5.2]).

(b) If S is an n-ELA semigroup, and F_0 is a coset representative of $S/(r)$, then for each finite subset $\sigma \subseteq S$, there exists $t_\sigma \in S$, depending on σ, such that $aF_0 t_\sigma = F_0 t_\sigma$ for all $a \in \sigma$ (see Granirer [6, Theorem 1] for $n = 1$, and Lau [12, Theorem 5.3]).

THEOREM 4.3. (a) *Let X be a norm linear space and $\{T_s; s \in S\}$ be an antirepresentation of a semigroup S as linear transformations from X into X with $\|T_s\| \leq 1$ for all $s \in S$. If S is n-ELA, then for each σ_0, coset representative of $S/(r)$,*

(*) *$d(K_X, x) = d(O(\sigma_0, x), 0)$ for all $x \in X$ where d is the metric induced by the norm on X.*

(b) *If S is a semigroup whose antirepresentation $\{l_s; s \in S\}$ on $m(S)$ (with sup norm) satisfies (*) for some $\sigma_0 \subseteq S$, $|\sigma_0| = n$, then S is m-ELA for some $m \leq n$, m divides n.*

Proof. (a) Let $\|\cdot\|$ denote the norm on X and $x \in X$ be arbitrary but fixed. If $x_0 \in K_X$, then $x_0 = \sum_1^m k_i(x_i - T_{s_i} x_i)$ where k_i are scalars, $x_i \in X$, and $s_i \in S$, $i = 1, \ldots, m$. Choose $t_0 \in S$ such that $s_i \sigma_0 t_0 = \sigma_0 t_0$ for $i = 1, \ldots, m$. Then as readily checked, $T_{\sigma_0 t_0}(x_0) = 0$ and

$$\|x - x_0\| \geq \|T_{\sigma_0 t_0}(x - x_0)\| = \|T_{\sigma_0 t_0}(x)\|,$$

i.e. $d(K_X, x) \geqq d(O(\sigma_0, x), 0)$. Since

$$T_{\sigma_0 t}(x) = (1/n) \sum_{s \in \sigma_0} (T_{st}(x) - x) - x \quad \text{and} \quad (1/n) \sum_{s \in \sigma_0} (T_{st}(x) - x) \in K_X$$

for all $t \in S$, it follows that (*) holds.

(b) Condition (*) implies that $d(K_{m(S)}, 1) \geqq 1$ and hence $K_{m(S)}$ is not uniformly dense in $m(S)$. Consequently, S is left amenable (this follows from Proposition 3.2 of Namioka [17]; see also Day [3, Lemma 2.2]) and so S has f.i.p.r.i. Furthermore, if $f \in m(S)$, then $d(\text{Co } O(f), 0) = (1/n)d(O(\sigma_0, f), 0)$, where Co $O(f)$ is the convex hull of $\{l_s; s \in S\}$. In fact, $d(f + K_{m(S)}, 0) \geqq (1/n)d(O(\alpha_0, f), 0) \geqq d(\text{Co } O(f), 0) \geqq d(f + K_{m(S)}, 0)$. The last inequality follows from Co $O(f) \subseteq f + K_{m(S)}$ as readily checked and shown in [8, p. 59]. If $f \in m(S)$, $a \in S$, $\|(1/n) \sum_{i=1}^{n} (l_{a^i}(f - {}_a f)\| \leqq 2\|f\|/n \to 0$ if n is large; hence $d(O(\sigma_0, f - {}_a f), 0) = d(\text{Co } O(f - {}_a f), 0) = 0$ for all $a \in S$, and $f \in m(S)$. By Lemma 4.1 S is m-ELA for some $m \leqq n$, m divides n.

REMARK. It follows from the proof of Theorem 4.3 that in order for S to be m-ELA for some $m \leqq n$, m divides n, it is sufficient for S to have f.i.p.r.i. and there exists $\sigma_0 \subseteq S$, $|\sigma_0| = n$ such that for each $f \in m(S)$, $d(\text{Co } O(f), 0) \geqq (1/n)d(O(\sigma_0, f), 0)$. For $n = 1$, as shown by Granirer [8, Theorem 5 II(b)], in order for S to be ELA, it is enough that for each $f \in m(S)$, $d(\text{Co } O(f), 0) = d(O(f), 0)$. However, we do not know whether or not in order for S to be m-ELA for some $m \leqq n$, $n \geqq 2$, it is enough to have $\sigma_0 \subseteq S$, $|\sigma_0| = n$ such that $d(\text{Co } O(f), 0) \geqq (1/n)d(O(\sigma_0, f), 0)$ for all $f \in m(S)$ without imposing the condition that S has f.i.p.r.i.

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