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EMBEDDING OF ABELIAN SUBGROUPS IN p-GROUPS

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Abstract. Research concerning the embedding of abelian subgroups in p-groups generally has proceeded in two directions; either considering abelian subgroups of small index (cf. J. L. Alperin, *Large abelian subrgoups of p-groups*, Trans. Amer. Math. Soc. **117** (1965), 10–20) or considering elementary abelian subgroups of small order (cf. B. Huppert, *Endliche Gruppen*. I, Springer-Verlag, Berlin, 1967, p. 303). The following new theorems extend these results:

THEOREM A. *Let G be a p-group and M a normal subgroup of G.* (a) *If M contains an abelian subgroup of index p, then M contains an abelian subgroup of index p which is normal in G.* (b) *If $p \neq 2$ and M contains an abelian subgroup of index p^2, then M contains an abelian subgroup of index p^2 which is normal in G.*

THEOREM B. *Let G be a p-group, $p \neq 2$, M a normal subgroup of G, and let k be* 2, 3, 4, *or* 5. *If M contains an elementary abelian subgroup of order p^k, then M contains an elementary abelian subgroup of order p^k which is normal in G.*

Introduction. In the study of finite p-groups, it is quite often useful to know whether the existence of an abelian subgroup of a given order implies the existence of a normal abelian subgroup of that order. Research concerning this question generally has proceeded in two directions; on the one hand, considering abelian subgroups of small index, and on the other hand, considering elementary abelian subgroups of small order. In this paper we extend several results concerning the embedding of abelian subgroups of both small order and small index. We prove the following theorems:

THEOREM A. *Let G be a p-group and M a normal subgroup of G.*

(a) *If M contains an abelian subgroup of index p, then M contains an abelian subgroup of index p which is normal in G.*

(b) *If $p \neq 2$ and M contains an abelian subgroup of index p^2, then M contains an abelian subgroup of index p^2 which is normal in G.*

THEOREM B. *Let G be a p-group and M a normal subgroup of G. Let $p \neq 2$ and k be* 2, 3, 4, *or* 5. *If M contains an elementary abelian subgroup of order p^k, then M contains an elementary abelian subgroup of order p^k which is normal in G.*

Theorem A extends a result of J. Alperin [1, Theorem 4], and Theorem B extends

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results of B. Huppert [4, Hilfssatz III 7.5] and of W. Feit and J. G. Thompson [2, Lemma 8.4].

The restriction that $p\neq 2$ in Theorem A(b) is necessary as is shown by an example in [1, p. 11].

It is interesting to note that the proof of Theorem B in the cases $k=4$ and 5 relies quite heavily on Theorem A. We also remark that Theorem B may be proved directly, and from it we may deduce Theorem A. The critical result upon which each of these proofs is based is Lemma 1.

NOTATION. We use the following notations:

$A\leqq B$, A is a subgroup of B,

$A<B$, A is a proper subgroup of B,

$c_A(B)$, the centralizer of B in A,

$\langle a_1, a_2, \ldots, a_n\rangle$, the group generated by $a_1, a_2, \ldots, a_n$,

$[a, b]$, the commutator $a^{-1}b^{-1}ab$,

G', the commutator subgroup of G,

$Z(G)$, the center of G,

$Z_i(G)$, the ith center of G,

$\Phi(G)$, the Frattini subgroup of G,

$\Omega_1(G)$, the subgroup of G generated by all elements of G of order p,

$|G|$, the order of G.

The proof of Theorem A. Theorem A is a unified statement of two theorems. We will prove these theorems separately as Theorems 1 and 2.

THEOREM 1. *Let G be a p-group and $M\lhd G$ such that M contains an abelian subgroup A of index p. Then M contains an abelian subgroup of index p which is normal in G.*

Proof. If A is the only abelian maximal subgroup of M, then A is characteristic in M, hence, normal in G.

Therefore, we may assume M contains an abelian maximal subgroup B different from A. But then, $M=AB$, $Z(M)\geqq A\cap B$, and $|M:A\cap B|=p^2$. Thus, either M is abelian or $Z(M)=A\cap B$. So either every maximal subgroup of M is abelian or every maximal subgroup of M containing $Z(M)$ is abelian. Thus, some abelian maximal subgroup of M is normal in G. Q.E.D.

The following computational lemma is necessary for the proof of Theorem 2.

LEMMA 1. *Let M be a finite p-group, $p\neq 2$, of class two and L a subgroup of M generated by the elements n, a_1, a_2, and t such that L' is elementary abelian, and the elements $[a_1, n]$, $[a_2, n]$, and $[t, n]$ are independent vectors of the vector space L'. Assume that the following relations hold in L:*

(I) $[a_1, a_2]=1$,

(II) $[a_1, t]=[a_1, n]^{\alpha_1}[a_2, n]^{\alpha_2}[t, n]^{\alpha_3}$,

where the α_i are integers modulo p.

(a) *If M possesses an automorphism τ of the form*

$$\begin{aligned}
\tau\colon a_2 &\to a_2 t^{\gamma_4} a_1^{\gamma_5} n^{\gamma_6} && (\text{mod } Z(L)),\\
t &\to t a_1^{\gamma_2} n^{\gamma_3} && (\text{mod } Z(L)),\\
a_1 &\to a_1 n^{\gamma_1} && (\text{mod } Z(L)),\\
n &\to n && (\text{mod } Z(L)),
\end{aligned}$$

where the γ_i are integers modulo p and $\gamma_4 \not\equiv 0$ (mod p), then $\alpha_2 \equiv \alpha_3 \equiv 0$ (mod p). Hence, $[a_1, tn^{-\alpha_1}] = 1$.

(b) *Assume that in addition to* (I) *and* (II) *the following relation holds in L*:

(III) $[a_2, t] = [a_1, n]^{\beta_1}[a_2, n]^{\beta_2}[t, n]^{\beta_3}$, *where the β_i are integers modulo p. If M possesses an automorphism σ of the form*

$$\begin{aligned}
\sigma\colon a_2 &\to a_2 a_1^{\gamma_5} && (\text{mod } Z(L)),\\
t &\to t n^{\gamma_3} && (\text{mod } Z(L)),\\
a_1 &\to a_1 && (\text{mod } Z(L)),\\
n &\to n && (\text{mod } Z(L)),
\end{aligned}$$

where the γ_i are integers modulo p, then $\gamma_3 \equiv 0$ (mod p). Hence, $t^\sigma \equiv t$ (mod $Z(L)$).

(c) *Assume that relations* (I), (II), *and* (III) *hold in L. If M possesses an automorphism ρ of the form*

$$\begin{aligned}
\rho\colon a_2 &\to a_2 t^{\gamma_4} a_1^{\gamma_5} n^{\gamma_6} && (\text{mod } Z(L)),\\
a_1 &\to a_1 t && (\text{mod } Z(L)),\\
t &\to t n^{\gamma_3} && (\text{mod } Z(L)),\\
n &\to n && (\text{mod } Z(L)),
\end{aligned}$$

where the γ_i are integers modulo p, then $\gamma_3 \equiv 0$ (mod p). Hence, $t^\rho \equiv t$ (mod $Z(L)$).

Proof. The proof of this lemma is purely computational. Since τ (also σ and ρ) is an automorphism of L, the relations (I) and (II) (and (III)) remain valid when we replace a_1, a_2, t, and n by their images under τ (or σ or ρ). In these new expressions for the relations (I) and (II) (and (III)), we eliminate all terms involving $[a_1, t]$ and $[a_2, t]$ by applying relation (II) (and (III)). Thus, we obtain expressions involving only the elements $[a_1, n]$, $[a_2, n]$, $[t, n]$. Since $[a_1, n]$, $[a_2, n]$, and $[t, n]$ are independent elements in the three-dimensional vector space L', each of these expressions yields three equations, one involving the terms in $[a_1, n]$, one involving the terms in $[a_2, n]$, and one involving the terms in $[t, n]$. These are the equations from which the results of the lemma are derived. In the calculations which follow, all congruences are to be considered modulo p.

(a) The action of τ on L gives the following equations:

(1) $\alpha_2\gamma_4 - \gamma_1 \equiv 0$ from relation (I) considering the terms in $[a_2, n]$.

(2) $\alpha_3\gamma_4 - \gamma_1\gamma_4 \equiv 0$ from relation (I) considering the terms in $[t, n]$.

(3) $-\gamma_1 \equiv \alpha_2\gamma_4$ from relation (II) considering the terms in $[t, n]$.

Now since $p>2$, from (1) and (3) we have $0\equiv\gamma_1\equiv\alpha_2\gamma_4$. If $\gamma_4\not\equiv 0$, we have $\alpha_2\equiv 0$. Since $\gamma_1\equiv 0$, equation (2) becomes $\alpha_3\gamma_4\equiv 0$, and hence, $\alpha_3\equiv 0$. So $[a_1, t]=[a_1, n]^{\alpha_1}$, and hence, $[a_1, tn^{-\alpha_1}]=1$.

(b) The action of ρ on L gives the following equations:

(4) $\gamma_3\equiv\alpha_2\gamma_5$ from relation (II) considering the terms in $[a_1, n]$.

(5) $0\equiv\alpha_2\gamma_5+\gamma_3$ from relation (III) considering the terms in $[a_2, n]$.

Hence, $\gamma_3\equiv 0$, and so $t^\sigma\equiv t$ (mod $Z(L)$).

(c) The action of ρ on L gives the following equations:

(6) $\gamma_3\equiv\alpha_2\gamma_5$ from relation (II) considering the terms in $[a_1, n]$.

(7) $\gamma_3+\alpha_2\gamma_5\equiv 0$ from relation (III) considering the terms in $[a_2, n]$.

Thus, $\gamma_3\equiv 0$, so $t^\rho\equiv t$ (mod $Z(L)$). Q.E.D.

The following theorem is a generalization of a result of J. Alperin (Theorem 4 of [1]). The first step of the proof follows Alperin's proof and is reproduced here for the convenience of the reader.

THEOREM 2. *Let G be a finite p-group, $p\neq 2$, and M a normal subgroup of G which contains an abelian subgroup of index p^2. Then M contains an abelian subgroup of index p^2 which is normal in G.*

REMARK. Alperin's result states that if G contains an abelian subgroup A of index p^3, then G contains a normal abelian subgroup of index p^3. This follows trivially from Theorem 2 if M is chosen to be a maximal subgroup of G which contains A.

Proof of Theorem 2. (By induction on the order of G.) We may assume that M is a proper subgroup of G, for if $M=G$, then we may apply Theorem 1 to obtain the result. Therefore, we can find a maximal subgroup H of G which contains M. By induction we may assume that M contains an abelian subgroup A of index p^2 which is normal in H. We may also assume that A is not normal in G, and hence, if $G=\langle H, x\rangle$ for some $x\in G\backslash H$, we have $A\neq A^x$.

(i) We now use Alperin's techniques to reduce the problem to the case in which $M/Z(M)$ is elementary abelian of order p^4. For i, any integer modulo p, let $A_i=A^{x^i}$. Thus, $A=A_0$, and if $i\neq j$, we have $A_i\neq A_j$ and so

$$|M|/p^2 = |A_i| < |A_iA_j| = |A_i|\,|A_j|/|A_i\cap A_j| \leqq |M|.$$

Thus, we see that $p^3\leqq|M:A_i\cap A_j|\leqq p^4$. Now we have two cases. First we suppose $|M:A_i\cap A_j|=p^3$ for all i and j with $i\neq j$.

In this case, suppose that $A_2\leqq A_0A_1$. Then $A_i\leqq A_0A_1$ for all i. In fact, if $A_{i-1}\leqq A_0A_1$, then

$$A_i = A_{i-1}^x < (A_0A_1)^x = A_1A_2 \leqq A_1A_0A_1 = A_0A_1.$$

Thus, $A^G\leqq A_0A_1$, so that $A^G=A_0A_1$. Thus, A_0A_1 is normal in G and contains an abelian maximal subgroup, so the result follows from Theorem 1.

Thus, we may assume that $A_2 \nleqq A_0A_1$. Now $|M:A_0A_1|=p$, so $A_2 \nleqq A_0A_1$ implies that $M=A_0A_1A_2$. Consequently,

$$p^2 = |M:A| = |(A_0A_1)A_2:A_2| = |A_0A_1:A_0A_1 \cap A_2|.$$

Also, by assumption, $|A_0A_1:A_0|=|A_0:A_0 \cap A_2|=p$, so that $|A_0A_1:A_0 \cap A_2|=p^2$. However, we have $A_0 \cap A_2 \leqq A_0A_1 \cap A_2$, and these subgroups both have index p^2 in A_0A_1. Thus, $A_0 \cap A_2=A_0A_1 \cap A_2$. Similarly, $A_0A_1 \cap A_2=A_1 \cap A_2$. Therefore, $A_0 \cap A_2=A_0 \cap A_1 \cap A_2$. Hence, since $A_0 \cap A_1 \cap A_2 \leqq Z(A_0A_1A_2) = Z(M)$, $|M/Z(M)| \leqq p^3$. Let K be a normal subgroup of G of index p^2 in M and containing $A_0 \cap A_1 \cap A_2$. Then K is abelian. This completes the proof in this case.

We may now assume that $|A_i:A_j \cap A_i|=p^2$ for some i and j. Conjugating the equation by x^{-i}, we may assume that $i=0$, so $|A:A \cap A_j|=p^2$. Therefore, $M=AA_j$ and $Z(M) \geqq A \cap A_j$. If $Z(M)>A \cap A_j$, then $|M:Z(M)| \leqq p^3$, so if K is a normal subgroup of G of index p^2 in M containing $Z(M)$, then K is abelian. Hence, we may assume $Z(M)=A \cap A_j$.

If $A/Z(M)$ is cyclic, then $A_j/Z(M)$ is cyclic since $Z(M)$ is normal in G and $A_j = x^{-j}Ax^j$. In this case, let $A=\langle a, Z(M)\rangle$ and $A_j=\langle b, Z(M)\rangle$. Since $M=AA_j$, M is of class two; so $a^{p^2} \in Z(M)$ implies $[a^{p^2}, b]=1$, so $[a^p, b^p]=[a, b]^{p^2}=[a^{p^2}, b]=1$, and $L=\langle a^p, b^p, Z(M)\rangle$ is abelian. But $M/Z(M)$ is the direct product of two cyclic groups of order p^2 so that $L/Z(M)$ is the subgroup of $M/Z(M)$ of all elements of order p. Therefore, L is normal in G and of index p^2 in M.

So, we may now suppose that $A/Z(M)$ is elementary abelian of order p^2. Thus, $A_j/Z(M)$ is also elementary abelian and $M/Z(M)$ is elementary abelian of order p^4.

(ii) We now construct a basis of $M/Z(M)$. Let N be a normal subgroup of G which is of index p^3 in M and contains $Z(M)$. If $c_M(N)>N$, then, since $c_M(N) \lhd G$, we can find a normal subgroup K of G contained in $c_M(N)$ which contains N as a subgroup of index p. Then K is abelian, completing the proof in this case. Therefore, we may assume that N is self-centralizing in M.

Since N is self-centralizing and has index p^3 in M, we see that N is not contained in any abelian subgroup of index p^2 of M. In particular, $N \nleqq A$, so $A \cap N=Z(M)$ and $|M:AN|=p$. Since the subgroup $[M, N]$ is normal in G, and $AN/[M, N]$ is an abelian maximal subgroup of $M/[M, N]$, we may apply Theorem 1 to $G/[M, N]$ to obtain a subgroup R of index p in M which is normal in G and such that $R/[M, N]$ is abelian. Since N is contained in the center of $M/[M, N]$, we may assume that $N \leqq R$. Now if R contains any abelian subgroup of index p, then we may apply Theorem 1. Thus, we may assume that R contains no abelian maximal subgroup. In particular, $A \nleqq R$. So $M=AR$ and $|R:AN \cap R|=p$.

We choose a basis for $M/Z(M)$ as follows. Let $N=\langle n, Z(M)\rangle$, $R=\langle a_1, t, N\rangle$ where $a_1 \in A$, and $M=\langle a_2, R\rangle$ where $a_2 \in A$.

Now since M has class two and $M/Z(M)$ is elementary abelian, M' is elementary abelian. Since N is self-centralizing and $[a_1^i a_2^j t^k, n]=[a_1, n]^i[a_2, n]^j[t, n]^k$, the only integers i, j, and k for which the equation $1 \equiv [a_1, n]^i[a_2, n]^j[t, n]^k$ is satisfied are

$i \equiv j \equiv k \equiv 0 \pmod p$). Thus $[a_1, n]$, $[a_2, n]$, and $[t, n]$ are linearly independent vectors in the vector space M'.

We note that since $R/[M, N]$ is abelian, $1 \equiv [a_1, t] \pmod{[M, N]}$. Since $[a_1, a_2] = 1$, we see that the center of $M/[M, N]$ contains $\langle a_1, N\rangle/[M, N]$. Since $|M:\langle a_1, N\rangle| = p^2$, we see that either $M/[M, N]$ is abelian or $\langle a_1, N\rangle$ is the preimage of the center of $M/[M, N]$. In the latter case, we note that $\langle a_1, N\rangle$ is normal in G. We now have two cases. Either $\langle a_1, N\rangle \lhd G$, or $\langle a_1, N\rangle \ntriangleleft G$ and $M/[M, N]$ is abelian.

(iii) Assume $\langle a_1, N\rangle \lhd G$. We have the following relations among the generators a_1, a_2, n, and t:

(1) $1 = [a_1, a_2]$.

(2) $[a_1, t] = [a_1, n]^{\alpha_1}[a_2, n]^{\alpha_2}[t, n]^{\alpha_3}$, where the α_i's are integers modulo p.

The action of x on the generators a_1, a_2, n, and t is of the form

$$\begin{aligned} a_2^x &= a_2 t^{\gamma_4} a_1^{\gamma_5} n^{\gamma_6} && (\operatorname{mod} Z(M)), \\ t^x &= t a_1^{\gamma_2} n^{\gamma_3} && (\operatorname{mod} Z(M)), \\ a_1^x &= a_1 n^{\gamma_1} && (\operatorname{mod} Z(M)), \\ n^x &= n && (\operatorname{mod} Z(M)), \end{aligned}$$

where the γ_i are integers modulo p and $\gamma_4 \not\equiv 0 \pmod p$, since AN is not normal in G.

We see from Lemma 1 that these relations give $1 = [a_1, tn^{-\alpha_1}]$. Therefore, R contains the abelian maximal subgroup $\langle a_1, tn^{-\alpha_1}, Z(M)\rangle$. However, this is a contradiction since we assumed R contained no abelian subgroup of index p.

(iv) We now assume that $M/[M, N]$ is abelian and that $\langle a_1, N\rangle \ntriangleleft G$. Since $\langle a_1, N\rangle$ is not normal in G, we see that G/N does not centralize R/N. Let C be the preimage in G of the centralizer of R/N in G/N. Then C is a maximal subgroup of G which contains M. Let y be an element of G which is not contained in C. Then $G = \langle y, C\rangle$ and $R = \langle a_1, [a_1, y], N\rangle$. Thus, without loss of generality, we may assume $t = [a_1, y]$, and by (iii) we may assume that $A \lhd C$. Under these assumptions the action of y on the generators a_1, a_2, t, and n has the form

$$\begin{aligned} a_2^y &= a_2 t^{\gamma_4} a_1^{\gamma_5} n^{\gamma_6} && (\operatorname{mod} Z(M)), \\ a_1^y &= a_1 t && (\operatorname{mod} Z(M)), \\ t^y &= t n^{\gamma_3} && (\operatorname{mod} Z(M)), \\ n^y &= n && (\operatorname{mod} Z(M)), \end{aligned}$$

for γ_i integers modulo p.

Since $M/[M, N]$ is abelian, we have $M' \leqq [M, N]$ so the following relations hold in M:

(3) $1 = [a_1, a_2]$,

(4) $[a_1, t] = [a_1, n]^{\alpha_1}[a_2, n]^{\alpha_2}[t, n]^{\alpha_3}$,

(5) $[a_2, t] = [a_1, n]^{\beta_1}[a_2, n]^{\beta_2}[t, n]^{\beta_3}$.

Using Lemma 1 we see that $t^y \equiv t \pmod{Z(M)}$.

Now let $c \in C$. We consider the action of c on the basis a_1, a_2, t, and n. Since c normalizes A and centralizes R/N, c acts on M as follows:

$$\begin{aligned} a_2^c &= a_2 a_1^{\gamma_5} && (\operatorname{mod} Z(M)),\\ a_1^c &= a_1 && (\operatorname{mod} Z(M)),\\ t^c &= tn^{\gamma_3} && (\operatorname{mod} Z(M)),\\ n^c &= n && (\operatorname{mod} Z(M)). \end{aligned}$$

Applying Lemma 1 we see that $t^c \equiv t \pmod{Z(M)}$. Hence, $N^* = \langle t, Z(M)\rangle$ is a normal subgroup of G. Let R^* be a normal subgroup of G such that $\langle a_1, N^*\rangle < R^* < M$. We note that $\langle a_1, N^*\rangle \lhd G$. Hence, if we replace N and R by N^* and R^*, we have the same situation as (iii), which completes the proof.

The Proof of Theorem B. As we did in the proof of Theorem A, we will prove the various cases of Theorem B as separate theorems. The case $k=2$ is a theorem of B. Huppert which will be useful in the proofs which follow.

THEOREM 3 [4, Hilfssatz III 7.5]. *If G is a finite p-group, $p \neq 2$, and M a non-cyclic normal subgroup of G, then M contains an elementary abelian subgroup of order p^2 which is normal in G.*

In the proofs of Theorems 4, 5, and 6, the following lemma is used.

LEMMA 2. *Let G be a finite p-group, $p \neq 2$. Let M be a normal subgroup of G which contains an elementary abelian subgroup N of order p^k which is normal in G. If M contains a subgroup X of exponent p such that $N < X \leqq c_M(N)$, then M contains an elementary abelian subgroup of order p^{k+1} which is normal in G.*

Proof. (By induction on $|G|$.) Since $c_X(N) > N$, we can find a subgroup X^* of $c_X(N)$ which contains N and has order p^{k+1}. Then, since X has exponent p, X^* is elementary abelian.

If $|G| = p^{k+1}$, then $G = X^*$ and the result is obvious. So assume that $|G| > p^{k+1}$ and that the lemma is true for all groups of order less than $|G|$. Thus, X^* is contained in a proper normal subgroup of G. Hence, we may assume that $M < G$. Applying the induction hypothesis to M yields an elementary abelian subgroup U which is normal in M, contains N, and has order p^{k+1}.

Now U is normal in the centralizer of N in M, and since $|U/N| = p$, we have $U \leqq Z_2(c_M(N))$. Since U has exponent p, $N < U \leqq \Omega_1(Z_2(c_M(N))) = T$. We note that $T \lhd G$ and so we can find a subgroup V which is normal in G such that $N < V \leqq T$ and $|V:N| = p$. Now since the class of T does not exceed two and $p \neq 2$, T has exponent p, hence, V has exponent p. Since V centralizes N, V is elementary abelian of order p^{k+1}, and $N < V \lhd G$. Q.E.D.

THEOREM 4. *Let G be a finite p-group, $p \neq 2$, and M a normal subgroup of G which contains an elementary abelian subgroup of order p^3. Then M contains an elementary abelian subgroup of order p^3 which is normal in G.*

Proof. (By induction on the order of G.) If G has order p^3 or p^4, then all subgroups of G of order p^3 are normal in G. So we may assume $|G| \geqq p^5$ and that the theorem is true for all groups of order less than $|G|$.

Since $|G| \geqq p^5$, every subgroup of G of order p^3 is contained in a proper normal subgroup of G. Thus, it will suffice to prove the theorem for M proper in G.

Now since M is not cyclic, we apply Theorem 3 to obtain an elementary abelian subgroup W of M of order p^2 which is normal in G. Since $|M| < |G|$, M contains a normal elementary abelian subgroup S of order p^3. We consider the subgroup SW. If $SW \neq S$, then $|SW| \geqq p^4$; and, since $c_{SW}(W)$ has index at most p in SW, we see that $|c_{SW}(W)| \geqq p^3$. Since $W \triangleleft M$, we see that $c_{SW}(W) \triangleleft M$. Thus, M contains a normal subgroup S^*, which is elementary abelian of order p^3 and contains W.

We apply Lemma 2, letting $S^* = X$, to complete the proof.

THEOREM 5. *Let G be a finite p-group, $p \neq 2$, and M a normal subgroup of G. If M contains an elementary abelian subgroup of order p^4, then M contains an elementary abelian subgroup of order p^4 which is normal in G.*

Proof. (By induction of $|G|$.) If G has order p^4 or p^5, then all subgroups of G of order p^4 are normal in G. So we may assume $|G| \geqq p^6$ and that the theorem is true for all groups of order less than $|G|$. Since $|G| \geqq p^6$, every subgroup of G of order p^4 is contained in a proper normal subgroup of G. Thus, it will suffice to prove the theorem for M proper in G.

We apply the induction hypothesis to M to obtain an elementary abelian subgroup S of order p^4 which is normal in M. Applying Theorem 4 to G and M we can find an elementary abelian subgroup N of M of order p^3 which is normal in G.

Now consider the subgroup $S \cap N$. If $|S \cap N| = p^3$, then $S \geqq N$, so we may apply Lemma 2 using $S = X$ to complete the proof in this case. If $|S \cap N| = 1$, then, since S and N are both normal in M, S and N commute, so apply Lemma 2 using $X = SN$ to complete the proof in this case. If $|S \cap N| = p$, then $SN/c_{SN}(N)$ acts on N as a subgroup of the group of all matrices of the form

$$\begin{bmatrix} 1 & 0 & \alpha_{13} \\ 0 & 1 & \alpha_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where } \alpha_{ij} \text{ are integers modulo } p.$$

Thus, $|SN/c_{SN}(N)| \leqq p^2$. Since $|SN/N| = p^3$, we see that $c_{SN}(N) > N$, so we may apply Lemma 2 using $X = c_{SN}(N)$ to complete the proof in this case.

The case $|S \cap N| = p^2$ is not quite so trivial. Since $N \triangleleft G$ and N is not cyclic, using Theorem 3 we see that N contains an elementary abelian subgroup W of order p^2 which is normal in G. If $S \ngeqq W$, then $|SW| \ngeqq p^5$. Since $c_{SW}(W)$ has index at most p in SW and is elementary abelian and normal in M, $c_{SW}(W)$ contains a subgroup S^* which is normal in M, elementary abelian of order p^4, and contains W. In this case replace S by S^*.

Thus, we may assume that $S \cap N = W \triangleleft G$. Let D be the centralizer of W in M. Then S and N are contained in D, and $D \triangleleft G$.

We now examine the embedding of S and N in D. Since $D/c_D(N)$ is isomorphic to a subgroup of abelian group $\mathscr{A}$ given by all matrices of the form

$$\begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where } \alpha_{ij} \text{ are integers modulo } p,$$

we see that $D' \leqq c_D(N)$. Thus, $[S, D] \leqq c_D(N)$. Recall that $S \lhd D$. Thus, we have $[S, D] \leqq c_D(N) \cap S = c_S(N)$. Now if $c_S(N) > W$, then $c_{SN}(N) > N$ and we may apply Lemma 2 letting $X = c_{SN}(N)$ to complete the proof. If $c_S(N) = W$, then $[S, D] \leqq W \leqq Z_1(D)$.

Thus, $S \leqq \Omega_1(Z_2(D)) = F$. We note that since $p \neq 2$, F has exponent p.

Now if $c_F(N) > N$, we may apply Lemma 2 letting $X = c_F(N)$, so we may assume $c_F(N) = N$. Thus,

$$|F| = |F : c_F(N)| \, |N| \leqq |\mathscr{A}| \, |N| = p^5.$$

Since $SN \leqq F$, $p^5 = |SN| \leqq F$. Thus, $|F| = p^5$ and F contains an abelian maximal subgroup. So we may apply Theorem 1 to complete the proof.

THEOREM 6. *Let G be a finite p-group, $p \neq 2$, and M a normal subgroup of G which contains an elementary abelian subgroup of order p^5. Then M contains an elementary abelian subgroup of order p^5 which is normal in G.*

Proof. (By induction on the order of G.) If G has order p^5 or p^6, then all subgroups of G of order p^5 are normal. So we may assume $|G| \geqq p^7$ and that the theorem is true for all groups of order less than $|G|$. Since $|G| \geqq p^7$, every subgroup of order p^5 of G is contained in a proper normal subgroup of G. Thus, it will suffice to prove the theorem for M proper in G.

We apply the induction hypothesis to M to obtain an elementary abelian subgroup S of order p^5 which is normal in M. Applying Theorem 5 to G and M, we can find an elementary abelian subgroup N of M of order p^4 which is normal in G.

(i) We now consider the subgroup $S \cap N$. If the order of $S \cap N$ is 1, p, or p^4, then we will show that $c_{SN}(N) > N$. Now since S and N are normal in M, $[S, N] \leqq S \cap N \leqq Z_1(SN)$, so SN has class 2. Hence, since $p \neq 2$, SN has exponent p. Thus, for the cases in which $c_{SN}(N) > N$, we may apply Lemma 2, letting $X = c_{SN}(N)$, to complete the proof.

If $|S \cap N| = 1$, then $[S, N] \leqq S \cap N = 1$ so $SN = c_{SN}(N)$. If $|S \cap N| = p^4$, then $S > N$, so $S = c_{SN}(N) > N$. If $|S \cap N| = p$, then since $[S, N] \leqq S \cap N$, $SN/c_{SN}(N)$ is isomorphic to a subgroup of the group $\mathscr{A}$ of all matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & \alpha_{14} \\ 0 & 1 & 0 & \alpha_{24} \\ 0 & 0 & 1 & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{where } \alpha_{ij} \text{ are integers modulo } p.$$

Thus, $|SN/c_{SN}(N)| \leqq p^3$. Since $|SN/N| = p^4$, we see that $c_{SN}(N) > N$.

(ii) If $|S \cap N| = p^2$ or p^3, the proof is more difficult since $c_{SN}(N)$ may equal N in these cases. The remainder of the proof deals with these two cases.

Since $N \lhd G$, N contains a subgroup W of order p^2 which is normal in G. If $W \nleqq S$, then the group SW has order at least p^6. Since W has order p^2, $|SW : c_{SW}(W)| \leqq p$, and since $SW \lhd M$, $c_{SW}(W) \lhd M$. Thus, SW contains a subgroup S^* which is elementary abelian of order p^5, contains W and is normal in M. In this case replace S by S^*. Thus, without loss of generality, we may assume that $W \leqq S \cap N$.

(iii) Now we consider the case in which $|S \cap N| = p^2$. Let D be the centralizer of W in M. Then $D \lhd G$ and $SN \leqq D$. Let H be the subgroup of D such that H/W is the centralizer of N/W in D/W. Then $H \lhd G$ and since $[S, N] \leqq W$, S and N are both in H.

Now the action of $H/c_H(N)$ on N is given by a subgroup of the group of matrices $\mathscr{M}$ given by

$$\begin{bmatrix} 1 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{where } \alpha_{ij} \text{ are integers modulo } p.$$

We note that $\mathscr{M}$ is abelian. Thus, $H' \leqq c_H(N)$. In particular, we note that

$$[H, S] \leqq H' \cap S \leqq c_H(N) \cap S = c_S(N).$$

Now we may assume $c_S(N) = S \cap N$ for otherwise $c_{SN}(N) > N$ and the result follows from Lemma 2. Thus, $[H, S] \leqq S \cap N = W \leqq Z_1(H)$, so $S \leqq Z_2(H)$. Since $N \leqq Z_2(H)$ (by the definition of H), we have $SN \leqq \Omega_1(Z_2(H)) = F$. Now since F has class 2, is generated by elements of order p, and $p \neq 2$, we see that F has exponent p.

We may assume $c_F(N) = N$ for otherwise Lemma 2 applies. Since $F/c_F(N)$ is isomorphic to a subgroup of $\mathscr{M}$ we have

$$p^7 = |SN| \leqq |F| = |F : c_F(N)|\,|c_F(N)| \leqq |\mathscr{M}|\,|N| = p^8.$$

If $|F| = p^7$, then since $|F : S| = p^2$ we may apply Theorem 2 to complete the proof in this case.

If $|F| = p^8$, then $F/c_F(N) \simeq \mathscr{M}$. Let Y be a normal subgroup of G such that $W < Y < N$. Then $c_F(Y)$ has index p^2 in F. Since $c_F(Y) \lhd G$, it will suffice to show that $c_F(Y)$ contains an abelian subgroup of order p^5 (index p), for then we may apply Theorem 1 to complete this case. Let $N = \langle n, Y\rangle$ and s_1, s_2 be elements of $c_F(Y)$ such that $\langle [s_1, n], [s_2, n]\rangle = W = S \cap N$. We may choose s_1 and s_2 to commute, for if $1 \neq [s_1, s_2]$, then $[s_1, s_2] = [s_1, n]^\alpha [s_2, n]^\beta$, so replacing s_1 by $s_1^* = s_1 n^\beta$ and s_2 by $s_2^* = s_2 n^{-\alpha}$, we see that $1 = [s_1^*, s_2^*]$. Now the group $S = \langle Y, s_1^*, s_2^*\rangle$ is abelian of order p^5, completing the proof in the case in which $|S \cap N| = p^2$.

(iv) If $|S \cap N| = p^3$, then we note that S has the following properties:

(a) $S \lhd M$.

(b) S is elementary abelian of order p^5.

(c) $W < S \cap N < N$.

Now if S is the only subgroup of M satisfying (a), (b), and (c), then S is characteristic in M, hence, $S \lhd G$. If S is not the only such subgroup of M, then among all subgroups T satisfying (a), (b), and (c) either there is some T such that $S \cap N \neq T \cap N$, or for all subgroups T of M satisfying (a), (b), and (c), $T \cap N = S \cap N$.

(v) If for some T, $S \cap N \neq T \cap N$, then since $S \cap N$ and $T \cap N$ are different subgroups of N of order p^3, $N=(S \cap N)(T \cap N)$. Using condition (c) we see that $S \cap T \cap N = W$. If $S \cap T > W$, then there is an element $t \in ST \backslash N$ which centralizes $S \cap N$ and $T \cap N$, hence, $t \in c_{ST}(N) \backslash N$, and we may apply Lemma 2, using $X = c_{ST}(N)$, to complete this case.

So we may assume $S \cap T = W$. Thus,

$$|ST| = |S|\ |T|/|S \cap T| = p^8.$$

Now since S and T are normal in M, $[S, T] = S \cap T = W$. Let D be the centralizer of W in M. Then D contains S and N, and $D \lhd G$. Let E be the subgroup of D so that E/W is the centralizer of N/W in D/W. Then since $N \leqq ST$ and $(ST)' = [S, T] \leqq W$, we see that $[N, S] \leqq (ST)' \leqq W$. Hence, E contains SN and $E \lhd G$.

We now consider the action of $E/c_E(N)$ on N. Since E centralizes N/W and W, $E/c_E(N)$ is isomorphic to a subgroup of the abelian group $\mathscr{M}$ of all matrices of the form

$$\begin{bmatrix} 1 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{where } \alpha_{ij} \text{ are integers modulo } p.$$

We may assume that $c_{ST}(N) = N$, for otherwise the result would follow from Lemma 2.

Let Y be a normal subgroup of G such that $W < Y < N$. Recall that $c_{ST}(N) = N$ and $|ST| = p^8$. Thus, $c_{ST}(Y)$ has index p^2 in ST. Now, as in case (iii), we see that $c_{ST}(Y)$ contains an abelian subgroup of order p^5, so applying Theorem 1 to M and $c_{ST}(Y)$, we see that $c_{ST}(Y)$ contains an elementary abelian subgroup $\hat{S}$ of order p^5, and $\hat{S} \lhd M$.

We now show that $\hat{S} \leqq Z_2(c_E(Y))$. Since $|c_E(Y)/c_E N| = p^2$, $c_E(Y)/c_E N$ is abelian. Thus, the derived group of $c_E(Y)$ is contained in $c_E N$. In particular, $[\hat{S}, c_E(Y)] \leqq c_E N$. Now since $\hat{S}$ is normal in M, we have

$$[\hat{S}, c_E(Y)] \leqq \hat{S} \cap c_E N = c_{\hat{S}}(N) \leqq Y \leqq Z_1(c_E(Y)),$$

i.e., $\hat{S}$ and N are both contained in $\Omega_1(Z_2(c_E(Y))) = K$. Since the class of K does not exceed two and p is odd, K has exponent p. Since $N = c_K(N)$ and $|K : c_K(N)| \leqq p^2$, we have

$$p^6 = |\hat{S}N| \leqq |K| = |K : c_K(N)|\ |c_K(N)| \leqq p^6.$$

Thus, $K=\hat{S}N$ and has order p^6. So we apply Theorem 1 to K to complete the proof in this case.

(vi) If all T satisfying (a), (b), and (c) are such that $T \cap N = S \cap N$, then $S \cap N = \bigcap_{g\in G} (S^g \cap N) = \bigcap_{g \in G} (S \cap N)^g \lhd G$. Let $C = c_M(S \cap N)$.

Then $C \lhd G$, and S and N are contained in C. Since the group of automorphisms induced by $C/c_C(N)$ on N is isomorphic to a subgroup of the abelian group of all matrices of the form

$$\begin{bmatrix} 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{where } \alpha_{ij} \text{ are integers modulo } p,$$

we have $C' \leqq c_C(N)$. Thus,

$$[S, C] \leqq S \cap c_C(N) \leqq c_S(N) = S \cap N \leqq Z_1(C),$$

and hence, $S \leqq Z_2(C)$. Let $L = \Omega_1(Z_2(C))$. Then $SN \leqq L \lhd G$ and L has exponent p, since $p \neq 2$.

We may assume $c_L(N) = N$ (otherwise apply Lemma 2). So we have

$$p^6 = |SN| \leqq |L| \leqq |L : c_L(N)|\,|c_L(N)| = p^7.$$

Thus, either $L = SN$ and L has order p^6, so we can apply Theorem 1 to complete the proof, or $|L| = p^7$ and we apply Theorem 2 to complete the proof.

Applications. We conclude with serveral applications of Theorem B. The following corollaries are generalizations of Sätze 7.7 and 7.8 of [4].

DEFINITION. Let G be a p-group. Then G is said to have depth k if

$$k = \max \{\text{rank } A \mid A \text{ is an abelian subgroup of } G\}.$$

COROLLARY 1. *Let G be a p-group, $p \neq 2$. If $Z_k(G)$ has depth $k-1$, for $k=2$, 3, 4, or 5, then G has depth $k-1$.*

Proof. Suppose G does not have depth $k-1$. Then G contains an abelian subgroup which requires k generators, hence, G contains an elementary abelian subgroup of order p^k. We apply Theorem B to show that G contains a normal elementary abelian subgroup N of order p^k. But $N \leqq Z_k(G)$ (by [3, Satz III 7.26]), a contradiction.

COROLLARY 2. *Let G be a p-group, $p \neq 2$, and let k take the value 2, 3, 4, or 5.*

(a) *If $Z_k(G) \cap G'$ has depth $k-1$, then G' has depth $k-1$.*

(b) *Let l be an integer greater than or equal to $k/2$. If $Z_l(G')$ has depth $k-1$, then G' has depth $k-1$.*

Proof. (a) Suppose G' does not have depth $k-1$, then G' contains an elementary abelian subgroup of order p^k. Using Theorem B, we see that G' contains an elementary abelian subgroup N of order p^k which is normal in G. Thus, $N \leqq Z_k(G)$, so $N \leqq Z_k(G) \cap G'$, a contradiction to the assumption that $Z_k(G) \cap G'$ has depth $k-1$.

(b) Suppose G' has depth greater than $k-1$. Then using Theorem B we see that G' contains an elementary abelian subgroup N of order p^k which is normal in G. Since $N \leqq Z_k(G)$, we have

$$\underbrace{[N, G', \ldots, G']}_{(l \text{ times})} = 1,$$

i.e., $N \leqq Z_l(G')$, a contradiction.

Conclusion. With regard to the more general question of finding sufficient conditions for a p-group to have an abelian subgroup of maximal order which is normal, the results of Theorems A and B give only a partial answer. An example given in [4, p. 349] shows that, for $p \neq 2$, there is a p-group of order p^{2p^2+1} in which no abelian subgroup of maximal order is normal. In fact, the abelian subgroups of maximal order have order p^{2p^2-2p+2} (index p^{2p-1}) and are elementary abelian. Thus, the generalizations of Theorems A and B to larger values of k will certainly depend on the primes involved.

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