

CONDITIONS ON AN OPERATOR IMPLYING $\operatorname{Re} \sigma(T) = \sigma(\operatorname{Re} T)$

BY
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Abstract. It is shown that the equation of the title is valid for certain classes of not necessarily normal operators (including Toeplitz operators, and operators whose spectrum is a spectral set), and a new proof is given of C. R. Putnam's theorem that it is valid for seminormal operators.

If T is any operator (bounded linear, in Hilbert space) then $f(\sigma(T)) = \sigma(f(T))$ for every rational function $f(\lambda)$ without poles in $\sigma(T)$. On the other hand, if T is normal then for every polynomial $f(\lambda, \lambda^*)$ one has $\sigma(f(T)) = f(\sigma(T)) = \{f(\lambda, \lambda^*) : \lambda \in \sigma(T)\}$ (see [16] for a recent elegant proof). In the present paper, we are concerned with the polynomial $f(\lambda, \lambda^*) = \frac{1}{2}(\lambda + \lambda^*) = \operatorname{Re} \lambda$ for not necessarily normal operators T , i.e., we consider conditions on an operator T such that

$$(*) \quad \operatorname{Re} \sigma(T) = \sigma(\operatorname{Re} T).$$

This is a very humble "spectral mapping theorem", but it is a considerable achievement for a nonnormal operator. For example, the fact that $(*)$ holds for T seminormal, which is due to C. R. Putnam [10], plays a role in his recent proof that a seminormal operator whose spectrum has zero area is normal [13]. In Theorem 1 we give a simple new proof of Putnam's theorem, and condition $(*)$ is verified for some classes of not necessarily seminormal operators in Theorems 2–4. Specifically, we prove

THEOREM 1 (PUTNAM). *If T is a seminormal operator, then $(*)$ holds.*

THEOREM 2. *If T is a Toeplitz operator, then $(*)$ holds.*

THEOREM 3. *If T is an operator such that $\sigma(T)$ is a spectral set for T , then $(*)$ holds.*

THEOREM 4. *If T satisfies the growth condition (G_1) and $\sigma(T)$ is connected, then $(*)$ holds.*

All operators contemplated in Theorems 1–4 are convexoid, i.e., $\operatorname{conv} \sigma(T) = \operatorname{Cl} W(T)$. (Here conv denotes convex hull, Cl denotes closure, and $W(T) = \{(Tx|x) : \|x\| = 1\}$ is the numerical range of T .) Lemmas 4–6 give results weaker than $(*)$ for certain convexoid operators.

Received by the editors March 18, 1970.

AMS 1969 subject classifications. Primary 4730, 4740.

Key words and phrases. Hilbert space, spectrum, spectral mapping theorem, seminormal operator, Toeplitz operator, numerical range.

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LEMMA 1. *If T is hyponormal then $\text{Re } \pi(T) \subset \sigma(\text{Re } T)$.*

Proof. Let $\lambda = \alpha + i\beta \in \pi(T)$, the approximate point spectrum of T , and let x_n be a sequence of unit vectors such that $Tx_n - \lambda x_n \rightarrow 0$. By hyponormality, $\|(T - \lambda I)^* x_n\| \leq \|(T - \lambda I)x_n\|$, therefore also $T^* x_n - \lambda^* x_n \rightarrow 0$. Then $(\text{Re } T)x_n - \alpha x_n = \frac{1}{2}(T + T^*)x_n - \frac{1}{2}(\lambda + \lambda^*)x_n \rightarrow 0$, thus $\alpha \in \sigma(\text{Re } T)$. ■

LEMMA 2. *If T is seminormal then $\text{Re } \sigma(T) \subset \sigma(\text{Re } T)$.*

Proof. One can suppose T hyponormal. Let $\lambda_0 \in \sigma(T)$. The vertical line $\text{Re } \lambda = \text{Re } \lambda_0$ exits the spectrum at a boundary point μ of $\sigma(T)$. Since $\partial\sigma(T) \subset \pi(T)$, one has $\text{Re } \lambda_0 = \text{Re } \mu \in \sigma(\text{Re } T)$ by Lemma 1. ■

LEMMA 3. *If T is normal then (*) holds.*

Proof. This is immediate from the continuous functional calculus; see also the remarks at the beginning of the paper. Alternatively, see Theorem 3. ■

Proof of Theorem 1. One can suppose T hyponormal. In view of Lemma 2, it suffices to show that $\sigma(\text{Re } T) \subset \text{Re } \sigma(T)$. Write $T = H + iJ$, H and J selfadjoint, and let $D = T^*T - TT^*$; $D \geq 0$ by hypothesis, and one has

$$(1) \quad HJ - JH = -\frac{1}{2}iD.$$

Changing Hilbert space [2], one can suppose $\sigma(H) = \pi_0(H)$ (the point spectrum of H).

Assuming $\alpha \in \sigma(H)$, it is to be shown that $\alpha \in \text{Re } \sigma(T)$. Let $\mathcal{M} = N(H - \alpha I)$, the null space of $H - \alpha I$; since α is an eigenvalue, $\mathcal{M} \neq \{0\}$. We show that \mathcal{M} is invariant under J . Let $x \in \mathcal{M}$, i.e., $(H - \alpha I)x = 0$. By (1),

$$(H - \alpha I)J - J(H - \alpha I) = -\frac{1}{2}iD,$$

therefore

$$-\frac{1}{2}i(Dx|x) = (Jx|(H - \alpha I)x) - (J(H - \alpha I)x|x) = 0,$$

thus $0 = (Dx|x) = \|D^{1/2}x\|^2$, $D^{1/2}x = 0$, $Dx = 0$. Then (1) yields

$$0 = H(Jx) - J(Hx) = H(Jx) - \alpha(Jx),$$

thus $Jx \in \mathcal{M}$.

Let $J_1 = J|_{\mathcal{M}}$. Of course \mathcal{M} is also invariant under H , with $H|_{\mathcal{M}} = \alpha I$. Therefore \mathcal{M} is invariant under $T = H + iJ$, and $T|_{\mathcal{M}} = \alpha I + iJ_1$; since J_1 is selfadjoint, clearly $T|_{\mathcal{M}}$ is normal; since T is hyponormal, it follows that \mathcal{M} reduces T [1, p. 160, Example 9]. Writing $T_1 = T|_{\mathcal{M}}$ and $T_2 = T|_{\mathcal{M}^\perp}$, we have $T = T_1 \oplus T_2$, $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, therefore

$$(2) \quad \text{Re } \sigma(T) = \text{Re } \sigma(T_1) \cup \text{Re } \sigma(T_2).$$

Since T_1 is normal, by Lemma 3 we have

$$\text{Re } \sigma(T_1) = \sigma(\text{Re } T_1) = \sigma(\alpha I) = \{\alpha\},$$

therefore $\alpha \in \text{Re } \sigma(T)$ by (2). ■

REMARK. The proof of Theorem 1 (and of Lemma 8 below) uses the fact that the representation $T \rightarrow T^0$ of [2] preserves spectrum; this fact is not proved explicitly in [2], but it follows at once from the observation that T is invertible iff T^*T and TT^* are invertible iff $T^*T \geq cI$ and $TT^* \geq cI$ for some $c > 0$. {The underlying general principle is that if $b \in B \subset A$, where A is a C^* -algebra with unity and B is a closed $*$ -subalgebra containing the unity, then $\sigma_B(b) = \sigma_A(b)$ [6, p. 8].}

LEMMA 4. *Let T be any convexoid operator. If $[\alpha_0, \beta_0]$ is the smallest interval containing $\operatorname{Re} \sigma(T)$, then $\alpha_0, \beta_0 \in \sigma(\operatorname{Re} T)$.*

Proof. Changing Hilbert space, one can suppose that $W(T)$ is closed and that $\pi(T) = \pi_0(T)$ [5]. Choose $\lambda_0 \in \sigma(T)$ with $\operatorname{Re} \lambda_0 = \alpha_0$. Clearly λ_0 is a boundary point of $\sigma(T)$ (by the extremality of α_0), therefore $\lambda_0 \in \partial\sigma(T) \subset \pi(T) = \pi_0(T)$, hence also $\lambda_0 \in W(T)$. On the other hand, since T is convexoid, $W(T)$ is closed and $\operatorname{Re} \sigma(T) \geq \alpha_0$, we have

$$\operatorname{Re} W(T) = \operatorname{Re} \operatorname{conv} \sigma(T) \geq \alpha_0;$$

thus λ_0 is clearly a boundary point of $W(T)$, therefore it is a normal eigenvalue of T , i.e., $N(T - \lambda_0 I) = N(T^* - \lambda_0^* I)$ [8, Satz 2]. Choose $u \neq 0$ with $Tu = \lambda_0 u$; then also $T^*u = \lambda_0^* u$, therefore $(\operatorname{Re} T)u = \alpha_0 u$, thus $\alpha_0 \in \sigma(\operatorname{Re} T)$. Similarly $\beta_0 \in \sigma(\operatorname{Re} T)$. ■

LEMMA 5. *Suppose T is convexoid and $\sigma(T)$ is connected. If $[\alpha, \beta]$ is the smallest interval containing $\sigma(\operatorname{Re} T)$, then $\sigma(\operatorname{Re} T) \subset [\alpha, \beta] \subset \operatorname{Re} \sigma(T)$.*

Proof. Since $\sigma(T)$ is connected, $\operatorname{Re} \sigma(T)$ is an interval, thus it will suffice to show that $\alpha, \beta \in \operatorname{Re} \sigma(T)$. Assume to the contrary, e.g., that $\alpha \notin \operatorname{Re} \sigma(T)$. Thus, if L is the vertical line $\operatorname{Re} \lambda = \alpha$, then L is disjoint from $\sigma(T)$. Since $\sigma(T)$ is connected, it must lie strictly to one side of L . Suppose, e.g., that it lies to the right. Then there exists $\varepsilon > 0$ such that $\operatorname{Re} \sigma(T) \geq \alpha + \varepsilon$. Since T is convexoid, it follows that $\operatorname{Re} W(T) \geq \alpha + \varepsilon$ thus $W(\operatorname{Re} T) \geq \alpha + \varepsilon$, i.e., $\operatorname{Re} T \geq (\alpha + \varepsilon)I$, hence $\sigma(\operatorname{Re} T) \geq \alpha + \varepsilon$; in particular, $\alpha \geq \alpha + \varepsilon$, contrary to $\varepsilon > 0$. The proof with "left" in place of "right" is similar, with $\alpha + \varepsilon$ replaced by $\alpha - \varepsilon$ and the inequalities reversed. ■

LEMMA 6. *If T is a convexoid operator such that both $\sigma(T)$ and $\sigma(\operatorname{Re} T)$ are connected, then (*) holds.*

Proof. Say $\sigma(\operatorname{Re} T) = [\alpha, \beta]$ and $\operatorname{Re} \sigma(T) = [\alpha_0, \beta_0]$. By Lemma 5, $[\alpha, \beta] \subset [\alpha_0, \beta_0]$; by Lemma 4, $\alpha_0, \beta_0 \in [\alpha, \beta]$ and so $[\alpha_0, \beta_0] \subset [\alpha, \beta]$. ■

Proof of Theorem 2. If T is a Toeplitz operator, then T is convexoid [7, Problem 196, Corollary 4] and $\sigma(T)$ is connected by a theorem of H. Widom [17]; since $\operatorname{Re} T$ is also a Toeplitz operator, $\sigma(\operatorname{Re} T)$ is connected too (cf. [7, Problem 199]), thus Lemma 6 is applicable. ■

LEMMA 7. *If T is a spectral set for T , then $\sigma(\operatorname{Re} T) \subset \operatorname{Re} \sigma(T)$.*

Proof. By hypothesis, $f(T)$ is normaloid for all rational functions f without poles in $\sigma(T)$ [3]. In particular, $T - \lambda I$ is normaloid for all complex λ , therefore T is

convexoid [14], [8, Satz 9, (i)]. {Better yet, $(T - \lambda I)^{-1}$ is normaloid for all $\lambda \notin \sigma(T)$, i.e., T satisfies condition (G_1) .}

Let $\alpha \in \sigma(\operatorname{Re} T)$ and assume to the contrary that $\alpha \notin \operatorname{Re} \sigma(T)$. Then, if L is the vertical line $\operatorname{Re} \lambda = \alpha$, L is disjoint from $\sigma(T)$.

Suppose first that $\sigma(T)$ lies to one side—say the left—of L . Then there exists $\varepsilon > 0$ such that $\operatorname{Re} \sigma(T) \leq \alpha - \varepsilon$; since T is convexoid, the argument in Lemma 5 yields the contradiction $\alpha \leq \alpha - \varepsilon$. Similarly if $\sigma(T)$ lies to the right of L .

Suppose finally that $\sigma(T)$ is disconnected by L . Write $\sigma(T) = X_1 \cup X_2$, where X_1 lies to the left, and X_2 to the right, of L . Since $\sigma(T)$ is a spectral set for T , by a theorem of J. P. Williams [18, Theorem 4] there is an orthogonal decomposition $T = T_1 \oplus T_2$ with $\sigma(T_k) = X_k$ a spectral set for T_k . Then $\operatorname{Re} T = \operatorname{Re} T_1 \oplus \operatorname{Re} T_2$, thus $\alpha \in \sigma(\operatorname{Re} T) = \sigma(\operatorname{Re} T_1) \cup \sigma(\operatorname{Re} T_2)$. Say $\alpha \in \sigma(\operatorname{Re} T_k)$; then the application of the preceding paragraph to T_k yields a contradiction. ■

The following lemma is a special case of a theorem of Putnam [12, Theorem 4], with a considerably simpler proof:

LEMMA 8. *If T satisfies condition (G_1) , then $\operatorname{Re} \sigma(T) \subset \sigma(\operatorname{Re} T)$.*

Proof. We can suppose $\pi(T) = \pi_0(T)$ [2], and therefore $\partial\sigma(T) \subset \pi(T) = \pi_0(T)$.

Suppose $\alpha_0 \in \operatorname{Re} \sigma(T)$. Thus, if L is the vertical line $\operatorname{Re} \lambda = \alpha_0$, the assumption is that L intersects $\sigma(T)$. Let λ_0 be a point where L exits the spectrum; then $\lambda_0 \in \partial\sigma(T)$ and $\operatorname{Re} \lambda_0 = \alpha_0$.

It will suffice to construct a sequence of unit vectors x_n such that $(T - \lambda_0 I)x_n \rightarrow 0$ and $(T^* - \lambda_0^* I)x_n \rightarrow 0$, for this will imply $\alpha_0 \in \sigma(\operatorname{Re} T)$ as in the proof of Lemma 1. To this end, we construct a sequence λ_n of normal eigenvalues of T such that $\lambda_n \rightarrow \lambda_0$, as follows.

For $n = 1, 2, 3, \dots$, let $D_n = \{\lambda : |\lambda - \lambda_0| \leq 1/n\}$. Since $\lambda_0 \in \partial\sigma(T)$, D_n contains a point μ_n of the resolvent set of T such that $|\mu_n - \lambda_0| < 1/2n$. Clearly $\operatorname{dist}(\mu_n, \sigma(T))$ is assumed in D_n ; say $\lambda_n \in \sigma(T)$ with $\operatorname{dist}(\mu_n, \sigma(T)) = |\mu_n - \lambda_n|$. Thus $\lambda_n \in \sigma(T)$ lies on the circumference of a closed disc (centered at μ_n) whose interior contains no point of $\sigma(T)$; since T satisfies (G_1) , it follows that $N(T - \lambda_n I) = N(T^* - \lambda_n^* I)$ [4, Lemma 2]. {The definition of “semibare point” in [4] is unnecessarily restrictive; there is no harm in other points of $\sigma(T)$ lying on the circumference of the disc (in any case, a smaller disc will shake them off).} Obviously $\lambda_n \in \partial\sigma(T)$, so λ_n is an eigenvalue; thus λ_n is a normal eigenvalue. Let x_n be a unit vector with $(T - \lambda_n I)x_n = 0$, $(T^* - \lambda_n^* I)x_n = 0$. Then

$$(T - \lambda_0 I)x_n = (T - \lambda_n I)x_n + (\lambda_n - \lambda_0)x_n = (\lambda_n - \lambda_0)x_n,$$

therefore $\|(T - \lambda_0 I)x_n\| = |\lambda_n - \lambda_0| \leq 1/n$; thus $(T - \lambda_0 I)x_n \rightarrow 0$, and similarly $(T^* - \lambda_0^* I)x_n \rightarrow 0$. ■

Proof of Theorem 3. If $\sigma(T)$ is a spectral set for T , then $\sigma(\operatorname{Re} T) \subset \operatorname{Re} \sigma(T)$ by Lemma 7; as observed in the proof, T satisfies (G_1) , therefore $\operatorname{Re} \sigma(T) \subset \sigma(\operatorname{Re} T)$ by Lemma 8. ■

Proof of Theorem 4. Assume T satisfies (G_1) and $\sigma(T)$ is connected. By Lemma 8, $\text{Re } \sigma(T) \subset \sigma(\text{Re } T)$. Let $[\alpha, \beta]$ be the smallest interval containing $\sigma(\text{Re } T)$; since T is convexoid [9], [15], it follows from Lemma 5 that $[\alpha, \beta] \subset \text{Re } \sigma(T)$. Thus

$$\sigma(\text{Re } T) \subset [\alpha, \beta] \subset \text{Re } \sigma(T) \subset \sigma(\text{Re } T). \quad \blacksquare$$

If T satisfies the hypotheses of one of the above theorems, then so does $\lambda T + \mu I$ for all complex numbers λ and μ . We have remarked that the operators in Theorems 1–4 are all convexoid. In a sense, this is not an accident: If T is an operator such that $\lambda T + \mu I$ satisfies (*) for all complex λ, μ (equivalently, λT satisfies (*) for all $|\lambda| = 1$), then T is convexoid; the details of the proof are similar to those in Lemmas 5 and 7.

On the other hand, a convexoid operator need not satisfy (*). For example, consider the 5×5 matrix $T = A \oplus B$, where

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $B = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ is a 3×3 diagonal matrix whose eigenvalues lie off the imaginary axis and are the vertices of a triangle that contains the disc $D_{1/2} = \{\lambda : |\lambda| \leq 1/2\}$. Since $W(A) = D_{1/2}$ (cf. [7, Problem 166]) and $W(B) = \text{conv}\{\lambda_1, \lambda_2, \lambda_3\} \supset D_{1/2}$, it is easy to see that T is convexoid (cf. [7, p. 113]). One has $\sigma(T) = \sigma(A) \cup \sigma(B) = \{0\} \cup \{\lambda_1, \lambda_2, \lambda_3\}$, and in particular $0 \in \text{Re } \sigma(T)$. On the other hand, $\text{Re } T = \text{Re } A \oplus \text{Re } B$, therefore $\sigma(\text{Re } T) = \sigma(\text{Re } A) \cup \sigma(\text{Re } B) = \{-\frac{1}{2}, \frac{1}{2}\} \cup \text{Re } \sigma(B)$ and so $0 \notin \sigma(\text{Re } T)$. The dimension of the example is optimal, since a convexoid operator on a space of dimension less than 5 is necessarily normal [8, Satz 9, (ii)].

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