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TERNARY RINGS

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Abstract. We characterize those additive subgroups of rings which are closed under the triple ring product, then discuss their imbeddings in rings, their representation in terms of two types of modules, a radical theory, the structure of those which satisfy a minimum condition for certain ideals, and finally the classification of those which are simple ternary algebras over an algebraically closed or real closed field.

The ternary version of a Lie algebra, called a Lie triple system, arose in several contexts and received independent attention as early as 1949. Some general results on structure and classification can be found in [2]. Less attention has been given to Jordan triple systems, while the associative analog which we study here seems to have been considered only recently and then in the special form of Hestenes ternary rings. References to date can be found in [4]. Apart from whatever intrinsic interest ternary rings may have, the notions of variant and twist [3] provide a connection with ring theoretic questions which we hope to explore in a sequel. The present work is concerned with their characterization, imbedding in enveloping rings, representation, radical structure and finally with semisimple ternary rings satisfying a minimum condition.

In the development which follows, elementary ring properties and constructions which readily translate to ternary rings are used without comment. Any standard results in ring theory used without citation may be found in [1].

1. **Identities and imbeddings.** Suppose that A is a ring and T_0 an additive subgroup closed under the ternary product $tuv=(tu)v$. We observe that $T_0^2=\{\sum t_it_i'\}$, t_i, t_i' in T_0, is a subring of A, that $T_0+T_0^2=A_0$ is the subring of A generated by T_0, and that $T_0 \cap T_0^2$ is an ideal in A_0. In addition the associativity of A gives rise to associativity conditions on the ternary product in T_0. A natural problem is the characterization of such subsystems.

Definition 1. A *ternary ring* (τ-ring) T is an abelian group in which there is given a ternary product tuv which is left, center and right distributive and satisfies

$$(1) \qquad (tuv)xy = t(uvx)y = tu(vxy).$$

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A *right multiplication* ρ in a τ-ring T is an endomorphism in $(T, +)$ for which

$$(tuv)\rho = tu(v\rho). \tag{2}$$

Left multiplications are defined by

$$(tuv)\lambda = (t\lambda)uv. \tag{2}$$

The sets of right and left multiplications in T separately form subrings of the endomorphism ring of T and commute elementwise with one another. Let $\rho_{xy}: t \to txy$ and $\lambda_{xy}: t \to xyt$.

The identities (1) imply that ρ_{xy} is a right and λ_{xy} a left multiplication and that

$$\rho_{uv}\rho_{xy} = \rho_{(uvx)y} = \rho_{u(vxy)}, \qquad \lambda_{uv}\lambda_{xy} = \lambda_{x(yuv)} = \lambda_{(xyu)v}. \tag{3}$$

In particular, the ρ_{xy} generate a ring P of right multiplications $t \to \sum tx_iy_i$ and the λ_{xy} generate the ring Λ of left multiplications $t \to \sum x_iy_it$.

Let Λ' be anti-isomorphic to Λ and let $F = \Lambda' \oplus \mathrm{P}$, the ring-theoretic external direct sum. Call the subring E of F generated by all $(\lambda_{xy}, \rho_{xy})$ the ring of *inner multiplications* of T.

The group $(T_0, +)$ is made a bimodule for E by setting

$$t\cdot(\lambda, \rho) = t\rho, \qquad (\lambda, \rho)\cdot t = t\lambda. \tag{4}$$

A product in T with values in E is given by $t\cdot u = (\lambda_{tu}, \rho_{tu})$. As additive groups let $A = T \oplus E$. Define a product in A by distributing the ring product in E and the products (4). We have

$$(t\cdot u)\cdot v = (\lambda_{tu}, \rho_{tu})\cdot v = tuv.$$

This is the essential observation in verifying that T is *imbedded in A* in the sense that there is an isomorphism of T into A regarded as a ternary system. We denote this structure on a ring A by A_τ. The following properties of the above construction are all readily confirmed.

PROPOSITION 1. *With operations as defined above $A = T \oplus E$ is a ring, E is a subring, T is imbedded identically in A, $E = T^2$ and, for e in E,*

$$\text{if } eT = Te = 0, \text{ then } e = 0. \tag{5}$$

THEOREM 1. *Let $A = T \oplus E$ be the ring of Proposition 1, and let η be an imbedding of T with enveloping ring $B = T\eta + (T\eta)^2$. Then*

(i) *$\sum (t_i\eta)(t_i'\eta) \to \sum t_i\cdot t_i'$ defines a homomorphism of $(T\eta)^2$ onto E and*

(ii) *if $T\eta \cap (T\eta)^2 = 0$ then $\eta^*: t\eta + \sum (t_i\eta)(t_i'\eta) \to t + \sum t_i\cdot t_i'$ defines a homomorphism of B onto A with kernel $K\eta = \{w \mid w \in (T\eta)^2 \text{ and } w(T\eta) = (T\eta)w = 0\}$.*

Proof. If $\sum (t_i\eta)(t_i'\eta) = 0$ then $\sum (t_i\eta)(t_i'\eta)(t\eta) = 0 = \sum (t_it_i't)$ for every t in T. Since $\eta|T$ is an isomorphism $\sum t_it_i't = 0 = (\sum t_i\cdot t_i')\cdot t$ and similarly on the left. By (5), therefore, $\sum t_i\cdot t_i' = 0$. To verify that the map thus defined by (i) is a homomorphism of $(T\eta)^2$ is routine.

The hypothesis of (ii) implies that $t_\eta \to t$ and the map in (i) jointly define on B a map, necessarily a homomorphism. Clearly the kernel of η^* is contained in $(T_\eta)^2$, from which, by (i) and (5), the conclusion follows.

DEFINITION 2. An *imbedding* η of the τ-ring T is

(i) *direct* in case $(T\eta) \cap (T\eta)^2 = 0$ (in direct imbeddings $(T\eta)^2$ is called the *complementary subring* of $T\eta$);

(ii) *standard* in case it is direct and (5) holds, i.e. for every w in $(T\eta)^2$, $w(T\eta) = (T\eta)w = 0$ implies $w = 0$.

COROLLARY. *A standard imbedding exists for every τ-ring T, and any standard imbedding is equivalent to the imbedding of Proposition* 1.

Theorem 1 asserts that among the direct imbeddings the standard complementary subring and the standard enveloping ring are minimal objects. Because τ-rings are defined by an identity, the usual factor construction applies to insure that any τ-ring T has a universal homomorphism into a ring. Since T has a direct imbedding the universal homomorphism is a direct imbedding.

Whenever convenient an imbedded image of T will be identified with T. Where specification is necessary, products in the standard imbedding will be denoted by $x \circ y$. Thus $A = T \oplus T \circ T$ denotes the standard enveloping ring with T regarded as a τ-subring, or equivalently as a subring of A_τ.

A useful description of τ-rings is obtained from the observation that if σ is an automorphism of a ring A, if $\sigma^2 = 1$, and if

$$\sigma_- = \{a \mid a\sigma = -a\}, \qquad \sigma_+ = \{a \mid a\sigma = a\}$$

then σ_- is a subring of A_τ and $\sigma_-^2 \subset \sigma_+$. In order to eliminate purely additive difficulties we will assume throughout that $t \to 2t$ is an automorphism of $(T, +)$ for each τ-ring T and that the same holds for all enveloping rings considered. In particular this makes the connection between τ-rings and ring automorphisms of period two, which we will call *reflections*, more exact. In the case of standard enveloping rings the condition on T suffices, as the following summary asserts.

THEOREM 2. *If A is a ring such that $a \to 2a$ is an automorphism of $(A, +)$ and if σ is a reflection of A, then σ_- is a τ-subring T of A, $T^2 \subset \sigma_+$ and $T \oplus T^2$ is an ideal of A which is σ-invariant. Conversely, if T is a τ-ring such that $t \to 2t$ is an automorphism of $(T, +)$ and if $A = T \oplus T \circ T$ (standard) then for t in T, w in $T \circ T$, $\sigma\colon t + w \to (-t) + w$ is the unique reflection in A with $\sigma_- = T$.*

Proof. We prove only the second assertion. The map σ is readily verified to be a reflection with $T \subset \sigma_-$. If $T \neq \sigma_-$ there is a $w = \sum t_i \circ t_i' \neq 0$ in $\sigma_- \cap \sigma_+$.

$$\begin{aligned} 0 = 2\sum t_i \circ t_i' &\to \sum t_i t_i'(2t) = 0, && \text{all } t \text{ in } T, \\ &\to \sum t_i t_i' t = 0, && \text{all } t \text{ in } T, \\ &\to \sum t_i \circ t_i' = 0, && \text{contradiction.} \end{aligned}$$

2. **Homomorphisms and ideals.** The elementary theory of τ-ring homomorphisms duplicates that for rings. In particular, the fundamental isomorphism theorems hold. Kernels of homomorphisms of a τ-ring T are the subgroups U of $(T, +)$ for which $UTT+TUT+TTU\subset U$. There are three varieties of one-sided ideals.

DEFINITION 3. A subgroup U of $(T, +)$ is

(i) a *right ideal* in case $UTT\subset U$,

(ii) a *left ideal* in case $TTU\subset U$,

(iii) a *medial ideal* in case $TUT\subset U$,

(iv) an *ideal* in case U is right, left and medial.

Any ring A becomes on iteration of its product a τ-ring which we denote by A_τ. If A is a ring with a 1 then the right ideals of A_τ are the right ideals of A, while the medial ideals are the ideals of A_τ and these coincide with the ideals of A.

If $A=T+T^2$ is any enveloping ring for T the right ideals in T are the submodules of T as right T^2 modules. An ideal of a given variety in T generates an ideal of a given variety in A. Some useful basic information on this matter is summarized as follows:

PROPOSITION 2. *Let $A=T+T^2$. If U is a right ideal in T then UT is a right ideal in T^2 and $U+UT$ is a right ideal in A. If U is a left and right ideal in T then UT and TU are ideals in T^2. If U is an ideal in T then $UT+TU$ is an ideal in T^2 and $U+TU+UT$ is an ideal in A.*

3. **Modules.** Two sorts of modules for τ-rings suggest themselves. One arises from the representation of a τ-ring T as a τ-ring of endomorphisms of an abelian group. Every enveloping ring for T becomes such a module under right multiplication by elements of T, but $(T, +)$ is not naturally endowed with this structure. The natural operators on $(T, +)$ are the elements of T^2. Translating this into a purely internal form yields a second, more general, type of module.

DEFINITION 4. (i) A (right) *T-module* is an abelian group M and an abelian group homomorphism $m\otimes t\otimes t'\to m\cdot t\cdot t'$ of $M\otimes T\otimes T$ into M such that

$$(m\cdot t_1\cdot t_2)\cdot t_3\cdot t_4 = m\cdot(t_1t_2t_3)\cdot t_4 = m\cdot t_1\cdot(t_2t_3t_4).$$

(ii) A *special* (right) *T-module* is an abelian group M and a homomorphism $m\otimes t\to m\cdot t$ of $M\otimes T$ into M such that $m\cdot t_1\cdot t_2\cdot t_3=m\cdot(t_1t_2t_3)$.

The group $(T, +)$ becomes a T-module under $t_0\cdot t\cdot t'=t_0tt'$, with right ideals as submodules.

A special T-module X is a module for the universal enveloping ring for T where $x\cdot(tt')=x\cdot t\cdot t'$. Iteration of the action of T on X converts X into a T-module denoted by X_τ. If M is a submodule of X_τ then $M\cap M\cdot T$ is a submodule of X and $M+M\cdot T$ is the enveloping special module of M in X. In the sequel we will often write xt or mtt' for module operations.

Every T-module M can be imbedded in a special T-module X in the sense that M is isomorphic to a submodule of X. To see this let M be a T-module, let

$m \circ t: t' \to mtt'$, and let $M \circ T = \{\sum m_i \circ t_i\}$. Since $M \circ T$ is additively a group of endomorphisms of $(T, +)$, we can consider the abelian group $X = M \oplus M \circ T$.

PROPOSITION 3. *Let $X = M \oplus M \circ T$. The definition $(m + \sum m_i \circ t_i) \circ t = m \circ t + \sum m_i t_i t$ converts X into a special T-module in which M and $M \circ T$ are imbedded T-modules. For m' in $M \circ T$,*

$$\text{(6)} \qquad \text{if } m' \circ T = 0, \text{ then } m' = 0.$$

Proof. The action of T on X is well defined, for if $\sum m_i \circ t_i = 0$ then $(\sum m_i \circ t_i) \circ t = \sum t(m_i \circ t_i) = 0 = \sum m_i t_i t$. The validity of the converse proves the final assertion. The special T-module identity derives from

$$\left(m + \sum m_i \circ t_i\right) \circ u_1 \circ u_2 \circ u_3 = (mu_1u_2) \circ u_3 + \left(\sum m_i t_i u_1\right) \circ u_2 \circ u_3$$

and

$$(mu_1u_2) \circ u_3 : t' \to (mu_1u_2)u_3t', \qquad m \circ (u_1u_2u_3) : t' \to m(u_1u_2u_3)t',$$

$$(mtu_1) \circ u_2 \circ u_3 = (mtu_1)u_2u_3 = mt(u_1u_2u_3).$$

Since $m \circ t \circ u = mtu$, we have at once that X is a special T-module and that M is imbedded in X. Clearly $M \circ T$ is a submodule of X_τ. Its relation to M is given by $(m \circ t) \circ t_1 \circ t_2 = (mtt_1) \circ t_2$.

Proposition 3 shows that every T-module has a direct imbedding in a special T-module. Equivalent imbeddings can generally be identified, and for each T-module M there is a universal imbedding of which we make no significant use here. In any direct imbedding MT will be called the *complementary submodule*. Basic properties of the imbedding constructed in Proposition 3 are given by the module-theoretic analog of Theorem 1.

DEFINITION 5. An enveloping special module X (or imbedding) of a T-module M is called *standard* in case $X = M \oplus MT$ and (6) holds; i.e., for m' in MT, $m'T = 0$ implies $m' = 0$.

THEOREM 3. *Every standard enveloping module for M is equivalent to the module $X = M \oplus M \circ T$ of Proposition 3. If $Y = M \oplus MT$ is any direct special enveloping module, then $m + \sum m_i t_i \to m + \sum m_i \circ t_i$ is a homomorphism of Y onto X with kernel $\{m' \mid m' \in MT \text{ and } m'T = 0\}$.*

Proof. If $m + \sum m_i t_i = 0$ then $m = 0 = \sum m_i t_i$, hence $\sum m_i t_i t = 0$ for every t in T, whence $\sum m_i \circ t_i = 0$. Thus a map is defined, and it is clearly a homomorphism. The element $m + \sum m_i t_i$ is in the kernel if and only if $m = 0$ and $\sum m_i t_i = 0$. The latter is equivalent to $\sum m_i t_i t = 0$ for every t in T.

It follows that the direct imbedding of Proposition 3 is a standard imbedding and that any two standard imbeddings are equivalent. In particular we refer to the *standard complement* $M \circ T$ for M. Unless otherwise indicated the action of T in a standard special enveloping module will be denoted by $x \circ t$.

The relation between M and $M \circ T$ is reflexive in favorable circumstances. We call a T-module or special T-module *completely reducible* in case it is complemented and has no null submodules. The argument in the ring case carries over directly to show that a T-module is completely reducible if and only if it is a sum of irreducible (nonnull) submodules.

THEOREM 4. *If $M'=M \circ T$ and $M''=M' * T$, the standard complement of M', there is a homomorphism of MTT onto M'' with kernel $\{m \mid m \in MTT$ and $mTT=0\}$.*

Proof. If $M'T$ is any complementary module for M' then the homomorphism of Theorem 3 induces a homomorphism φ of $M'T$ onto $M' * T$. But $MTT=(M \circ T) \circ T$ is such a complement and according to Theorem 3 the kernel is

$$\{m \mid m \in MTT \text{ and } mT = m \circ T = 0\},$$

which is equivalent to the assertion.

COROLLARY. *If M is completely reducible then M is a standard complement for $M \circ T$ and $X=M \oplus M \circ T$ is a standard special enveloping module for both M and $M \circ T$.*

Proof. The submodule MTT has a complement in M which must be null. Thus φ is a homomorphism of M onto M'' whose kernel is a null submodule.

For subsets, Y, Z of a special T-module we use $(Y{:}Z)=\{t \mid Zt \subset Y\}$.

DEFINITION 6. The *kernel of a special T-module X* is $(0{:}X)$. The *kernel of a T-module M* is $K(M)=\{t \mid MtT=0\}$. A special kernel, as the kernel of a homomorphism, is an ideal. The kernel of a module is a left-right ideal.

PROPOSITION 4. *If $X=M \oplus M \circ T$ then* $(0{:}X)=K(M) \cap K(M \circ T)$.

Proof. For $X \circ t=0$ it is necessary and sufficient that $M \circ t=0$ and $M \circ T \circ t =0$. The former is equivalent to $MtT=0$ and for the latter we have $M \circ T \circ t =MTt$.

It will be convenient to make use of an easily verified analog of Theorem 2. By a *semireflection* in a special T-module X we mean a semi-endomorphism Σ of X with $\Sigma^2=1$, $\Sigma \neq 1$, and with $t \to -t$ as associated automorphism of T. Thus $(xt)\Sigma =-(x\Sigma)t$.

THEOREM 5. *If X is a special T-module for which $x \to 2x$ is an automorphism of $(X, +)$ and if Σ is a semireflection of X, then Σ_- is a T-module M and $M \oplus MT$ is Σ-invariant. Conversely, if M is a T-module for which $m \to 2m$ is an automorphism of $(M, +)$, if $X=M \oplus M \circ T$, and if $\Sigma\colon m+m' \to (-m)+m'$ for m in M, m' in $M \circ T$, then Σ is the unique semireflection in X for which $M=\Sigma_-$.*

As for τ-rings we limit our consideration to modules and special modules in which $x \to 2x$ is an additive automorphism. Theorem 5 shows that $M \oplus M \circ T$ inherits this property from M.

THEOREM 6. *A special enveloping module of an irreducible T-module M is completely reducible if and only if $MT \cong M \circ T$. In this case the standard complement $M \circ T$ is irreducible and $X = M \oplus M \circ T$ is reducible if and only if $M = Y_\tau$ for some irreducible special T-module Y, in which case $X = Y_1 \oplus Y_2$, $Y_i \cong Y$ and $M \cong M \circ T = Y_\tau$.*

Proof. If $Z = M + MT$ is any enveloping module, either $M \cap MT = 0$ or $M \subset MT$, in which case $MT \subset MTT \subset M$ so $M = MT$ and $M \cong Y_\tau$ for some special irreducible Y. But then the homomorphism determined by $mt \to m \circ t$ (Theorem 3) is an isomorphism of M onto $M \circ T$ since M is irreducible. Should $Z = M \oplus MT$, let N be the kernel of the natural homomorphism of MT onto $M \circ T$. Because $NT = 0$, Z is completely reducible only if $N = 0$.

Let $N \subset M \circ T$ be a submodule. If $N \neq 0$, $N \circ T \neq 0$ and $N \circ T$ is a submodule of M, hence $N \circ T = M$ and $N \supset N \circ T \circ T = M \circ T$ which is therefore irreducible. Now let Y be a proper submodule of X and let Σ be the semireflection of X with $\Sigma_- = M$. If $Y\Sigma = Y$ then $Y = (Y \cap M) \oplus (Y \cap M \circ T)$. In case $Y \cap M \neq 0$, $Y \supset M$ and $Y = X$ follows. By the corollary to Theorem 4 this argument also applies if $Y \cap M \circ T \neq 0$. We conclude that $Y\Sigma \neq Y$. But now $Y + Y\Sigma$ and $Y \cap Y\Sigma$ are Σ-invariant submodules of X, from which it follows that $X = Y \oplus Y\Sigma$. The projection of X onto Y along $Y\Sigma$ is an endomorphism of X which maps M isomorphically since $Y\Sigma$ has no nontrivial Σ-invariant subsets. Moreover, the image M' of M in Y must have $M' \oplus M'\Sigma = X$, and we therefore conclude that $Y_\tau \cong M$. By the corollary to Theorem 4 and the fact that $M \circ T$ is irreducible, $M \circ T \cong Y_\tau$.

Conversely, if $M \cong Y_\tau$ let $\varphi\colon \sum y_i t_i \to \sum y_i \circ t_i$ be the natural isomorphism of Y_τ onto $Y_\tau \circ T$. Then $\{y + y\varphi\}$ is a proper special submodule of $Y_\tau \oplus Y_\tau \circ T$, for $(yt + y \circ t) \circ t' = (yt) \circ t' + ytt' = y' \circ t' + y't'$.

The situation detailed above can be described as follows: if M is irreducible $M \oplus M \circ T$ is Σ-irreducible i.e. has no Σ-invariant submodules. A Σ-irreducible module is either irreducible or the direct sum of isomorphic, irreducible submodules.

There is no difficulty in establishing the T-module structure of an irreducible special T-module.

THEOREM 7. *If X is an irreducible special T-module then either X_τ is irreducible or $X = M \oplus M \circ T$ for some irreducible T-module M and $M \neq Y_\tau$ for any special T-module Y.*

4. **The radical.** Definition 6 provides a natural basis for the specification of a Jacobson radical. In this section we show that with the appropriate definition of *modular*, the theory can be developed along expected lines.

DEFINITION 7. The *radical* $R(T)$ of a τ-ring T is the intersection of the kernels of its irreducible modules. The τ-ring T is *semisimple* in case $R(T) = 0$ and *primitive* in case it has a faithful irreducible special module.

From Proposition 4 it follows that T primitive implies T semisimple. Furthermore, $R(T)=\bigcap K(M)=\bigcap(K(M)\cap K(M\circ T))=\bigcap(0:M\oplus M\circ T)$, the intersection being taken over all irreducible M. But if $X=M\oplus M\circ T$, M irreducible, then by Theorem 6 either X is irreducible or $K(M)=K(M\circ T)=(0:X)$ and $K(M)=K(Y)$ where $Y_\tau\cong M$ and Y_τ, hence Y, is irreducible. This proves the essential part of the following, the remainder being either immediate consequences or else the direct translation of applicable ring-theoretic arguments.

Proposition 5. *For any τ-ring T*

(i) *$R(T)$ is the intersection of the kernels of its faithful irreducible special modules,*

(ii) *$R(T)$ is the intersection of the ideals W of T such that T/W is primitive,*

(iii) *$R(T)$ is an ideal and $T/R(T)$ is semisimple.*

Definition 8. A right ideal U of T is *modular* provided there exist t_0t_1 in T such that $\{t-t_0t_1t\}=(1-t_0t_1)T\subset U$.

Theorem 8. *A right ideal U of T is maximal and modular if and only if there exists a T-module M and a nonzero element e of M such that $U=\{t \mid etT=0\}$. A T-module M is irreducible if and only if $M\cong T-U$ for some maximal and modular right ideal U of T.*

Proof. Suppose U is maximal and modular. Then $T^3\not\subset U$, where $T^3=\{\sum t_it_i't_i''\}$, so $M'=T-U$ is irreducible. Let $M=M'\circ T$, choose t_0, t_1 so that $(1-t_0t_1)T\subset U$, and let $\varphi: t\to t+U$. Now $(t_0\varphi)\circ t_1\circ t=(t_0t_1t)\varphi=t\varphi$ for every t in T, and as a consequence if $w_0=(t_0\varphi)\circ t_1$, $w_0TT=M$, i.e. w_0 is a generator of M. By Theorem 6, M is irreducible. By Theorem 4, if $w_0\circ t\circ T=0$ then $w_0\circ t=0$ whence $U=\{t \mid w_0\circ t\circ T=0\}$.

Conversely, if $U=\{t \mid etT=0\}$ where $e\neq 0$ is an element of an irreducible M, then $eTT=M$ and so there exists t_0 with $et_0T\neq 0$, which implies $et_0T=M$. In particular, $et_0t_1=e$ for some t_1. Now $e\circ(t_0t_1t-t)=(et_0t_1)\circ t-e\circ t=0=e(t_0t_1t-t)T$ for every t in T. The final relation asserts $(1-t_0t_1)T\subset U$.

To prove the second assertion it remains to consider an irreducible M and nonzero e in M. The module $M\circ T$ is irreducible and has a generator $e\circ t_0$. The map $t\to et_0t$ is a homomorphism of T onto M with kernel $U=\{t \mid et_0t=0\}$. But $M\circ T$ is irreducible and M is a standard complement for $M\circ T$. Thus $et_0t=0$ if and only if $(e\circ t_0)\circ t\circ T=0$, hence U is maximal and modular by the first assertion of this theorem.

Lemma. *If U is maximal and modular then $(U:T^2)=\{t \mid T^2t\subset U\}$ is the kernel of an irreducible module and $(U:T^2)\subset U$.*

Proof. Let M be an irreducible module and e a generator for M for which $U=\{t \mid etT=0\}$. Then $MtT=(eTT)tT=e(T^2t)T$, so that $MtT=0$ if and only if $T^2t\subset U$. Because $MtT=0$ implies $etT=0$, $(U:T^2)\subset U$.

THEOREM. 9. *For any τ-ring T, $R(T)=\bigcap U$, U maximal and modular.*

Proof. By the lemma $\bigcap U \supset \bigcap (U:T^2) \supset R(T)$. If M is an irreducible T-module $K(M)=\bigcap \{t \mid etT=0\}$ over all nonzero e in M where, according to Theorem 8, these are maximal and modular. But $R(T)=\bigcap K(M)$, M irreducible, so $R(T)\supset \bigcap U$, U maximal and modular.

DEFINITION 9. A right ideal V of a τ-ring T is (right) *quasiregular* whenever v is in $(1-vt)T$ for every v in V and t in T.

The quasiregular condition is equivalent to $(1-vt)T=T$ for all v in V, t in T. We simply note that if v is in $\{t'-vtt'\}$ so is vtt' and therefore also t'.

THEOREM 10. *The radical $R(T)$ is quasiregular and contains every q.r. right ideal.*

Proof. Should $R(T)$ fail to be q.r. then $U_0=(1-r_0t_0)T$ is a proper right ideal for some r_0 in $R(T)$ and t_0 in T. By definition U_0 is modular. No proper right ideal containing U_0 has r_0 as a member, and by a Zorn argument the set of right ideals containing U_0 and excluding r_0 has a maximal element which must therefore be a maximal and is necessarily a modular ideal U. But $U\supset R(T)$ and the hypothesis is therefore contradicted.

Now suppose the V is q.r. that M is an irreducible T-module but that $V\not\subset K(M)$. Choose e in M and v_0 in V so that $ev_0T\neq 0$. Because M is irreducible $ev_0T=M$ and a t_0 exists for which $ev_0t_0=e$. This and the fact that V is q.r. imply the contradicting conclusion $e(v_0t_0t-t)T=0=eTT$.

The preceding theorems provide a basis for the straightforward derivation of τ-ring analogs of many results concerning the radical of a ring. Some are immediate, for example the right-left symmetry of the radical. Others, however, fail fundamentally. A right ideal which is nilpotent in the obvious sense need not be in the radical as we shall see from an example. An efficient way to read off certain properties of the τ-ring T is to relate $R(T)$ to the radical $R(T+T^2)$ of enveloping rings, a matter of interest in its own right. The main theorem is separated into three lemmas:

LEMMA A. *If $A=T+T^2$ is any enveloping ring for T then $R(T)\subset R(A)$.*

Proof. Let X be an irreducible A-module with kernel $K(X)$. By restriction X is a special T-module which is irreducible since T generates A. If r is in $R(T)$ then r is in the kernel of this module by Proposition 5. Thus r is in $\bigcap K(X)$, over all kernels of irreducible A-modules, and this is $R(A)$.

LEMMA B. *If $A=T\oplus T^2$ then $R(A)\cap T\subset R(T)$.*

Proof. If r is in $R(A)\cap T$ then $rT\subset R(A)$ so $(1-rt)A=A$ for every t in T. But since $(1-rt)T\subset T$ and $(1-rt)T^2\subset T^2$, $(1-rt)T=T$ and r is in $R(T)$.

LEMMA C. *If $A=T\oplus T^2$ then $R(A)\cap T^2=R(T^2)$.*

Proof. If r is in $R(A)\cap T^2$ then $(1-rtt')A=A$ for every t and t' in T, which implies $(1-rtt')T^2=T^2$, i.e. rtt' is in $R(T^2)$. But $\{tt'\}$ generates T^2 and therefore $rT^2\subset R(T^2)$. This last implies [1, p. 9] that r is in $R(T^2)$.

If w is in $R(T^2)$ then $(1-wtt')T^2=T^2$ for every t and t' in T. Letting $r=wt$ we conclude that $U=(1-rt)T$ is a modular right ideal in T. But then $UT=T^2$ and $UTT=T^3\subset U$, hence $t'=u+rtt'$ is in U for every t' in T, and therefore $U=T=(1-rt)T$. Now each of the following implies its successor: $rT\subset R(T)\subset R(A)$, $rA\subset R(A)$, r in $R(A)$; $wT\subset R(A)$, $wA\subset R(A)$, w in $R(A)$.

THEOREM 11. *If* $A=T\oplus T^2$ *then* $R(A)=R(T)\oplus R(T^2)$.

Proof. Because the reflection σ in A with $\sigma_-=T$ preserves quasiregularity $R(A)$ is σ-invariant and $R(A)=R(A)\cap T\oplus R(A)\cap T^2$. The conclusion now follows from the lemmas.

Theorem 11 has the consequences listed below. The first two are immediate. To obtain the third suppose that $A=T\oplus T\circ T$ and $B=U\oplus(UT+TU)$. Using the property $R(B)=R(A)\cap B$ [1, p. 10], and noting that $U\oplus U^2$ is an ideal in B, we infer

$$\begin{aligned} R(U) &= R(U\oplus U^2)\cap U = R(B)\cap(U\oplus U^2)\cap U \\ &= R(B)\cap U = R(A)\cap B\cap U \\ &= R(A)\cap U = (R(T)\oplus R(T^2))\cap U = R(T)\cap U. \end{aligned}$$

COROLLARIES. (1) *If* $A=T\oplus T^2$ *and* T *is semisimple then* $R(A)\subset T^2$ *and* $R(A)=\{a\mid aT=0\}$.

(2) *If* T *is semisimple then* $A=T\oplus T^2$ *is semisimple if and only if* $A=T\oplus T\circ T$ *is standard.*

(3) *If* U *is an ideal in* T *then* $R(T)\cap U=R(U)$.

5. **Examples.** (a) Let X_1, X_2 be vector spaces over a division ring Δ and let $T=T(X_1,X_2)=\mathrm{Hom}_\Delta(X_1,X_2)\oplus\mathrm{Hom}_\Delta(X_2,X_1)$. Set $X=X_1\oplus X_2$ and regard T as a subset of $A=\mathrm{Hom}_\Delta(X)$. With the operations induced from A, T is a τ-ring and $T\oplus T^2=T\oplus T\circ T=A$. Let $U_{12}=\mathrm{Hom}(X_1,X_2)$ and $U_{21}=\mathrm{Hom}(X_2,X_1)$. Then U_{12} and U_{21} are left-right ideals in T with $U_{12}^2=U_{21}^2=0$. Since A is simple and semisimple, T is simple ($T^3\neq 0$ and T has no proper ideals) and semisimple. If X_1 and X_2 have finite dimensions n_1 and n_2 then T can be identified with the τ-ring of all n_1+n_2 by n_1+n_2 matrices with zeros in disjoint n_1 by n_1 and n_2 by n_2 square submatrices on the diagonal.

In the natural way X_1 is a T-module and $X=X_1\oplus X_1\circ T$. Clearly, X is a faithful special T-module, in fact, $K(X_1)=X_2$ and $K(X_2)=X_1$.

(b) In example (a) let X_0 be a subspace of X_1 and take

$$T_0 = \{t\mid t\in T \text{ and } X_0t = 0\}.$$

Then $T_0=U_0\oplus U_{21}$ where $U_0=\{t\mid t\in U_{12}\text{ and } X_0t=0\}$. We find that

$$R(T_0) = \{t\mid t\in U_{21}\text{ and } X_2t\subset X_0\}.$$

In this case we still have $T_0\oplus T_0^2=T_0\oplus T_0\circ T_0$ so that $R(T_0\oplus T_0^2)=R(T_0)\oplus(T_0R(T_0)+R(T_0)T_0)$. However, in terms of modules we find $X=X_2\oplus X_2T_0$

but $X_1 = X_2T_0$ is not a standard complement for X_2 if $X_0 \neq 0$. For every x_2 in X_2 and t in $R(T_0)$, $x_2t_0T \subset X_0T_0 = 0$. But if $x_2 \neq 0$ there is a t_0 for which $x_2t_0 \neq 0$.

(c) Let Φ be a field and let $A_0 = \Phi^*[u, v]$ be the ring of polynomials with zero constant term. The automorphism σ of A_0 which fixes Φ and for which $u\sigma = -u$, $v\sigma = -v$ determines the τ-subring $T_0 = \Phi^*_-[u, v]$ consisting of odd polynomials. Clearly $T_0^2 = \Phi^*_+[u, v]$, the ring of even polynomials. Let $B_0 = uvA_0$, observe that B_0 is σ-invariant and set

$$A = A_0/B_0 = (T_0/uvT_0) \oplus (T_0^2/uvT_0^2) = T \oplus T^2$$

where $T = T_0/uvT_0$. Evidently, if x and y denote the cosets of u and v respectively, then T has basis $\{x^{2k-1}, y^{2k-1}\}$, T^2 has basis $\{x^{2k}, y^{2k}, xy\}$ and the multiplicative relations derive from $x^2y = y^2x = 0$. This means that T has ideals $\Phi^*_-[x]$, $\Phi^*_-[y]$ and $T = \Phi^*_-[x] \oplus \Phi^*_-[y]$, so that, as the sum of semisimple ideals, T is semisimple. On the other hand $W = \{\alpha xy\}$, α in Φ, is $R(T^2)$.

Theorem 11 does not settle the relation between $R(T)$ and $R(T+T^2)$ in case $T+T^2$ is not direct. In certain cases the crucial Lemma B holds and this implies Lemma C and Theorem 11. For example, if $T = T^2 = A_\tau$ for some ring A or if $R(T+T^2)$ is nil then $R(A) \cap T \subset R(T)$. We have no example to show that the general form of Lemma B fails.

6. **Special types of τ-rings.** A τ-ring cannot contain an identity but certain τ-rings generate identities in the sense defined below. This determines the ternary version of inverse. The ternary form of commutativity is clear.

DEFINITION 10. (i) A τ-ring is *commutative* in case it satisfies $tuv = utv = uvt$.

(ii) A τ-ring T *admits an identity* provided there exist elements $\{e_i, e_i'\}$ such that $\sum e_ie_i't = \sum te_ie_i' = t$ for every t in T.

With the above usages as a basis the definitions of *inverse*, *τ-division ring*, and *τ-field* are obvious. If T admits an identity specified by $\{e_i, e_i'\}$ then a module M for T is *unital* provided $\sum me_ie_i' = m$ identically in M. A summary of elementary properties follows.

PROPOSITION 6. (i) *If T admits an identity, then every enveloping ring for T has an identity, and if some $A = T \oplus T^2$ has an identity then T admits an identity.*

(ii) *If T admits an identity, its standard enveloping ring is universal and $T^2 \cong T \circ T$ in every enveloping ring.*

(iii) *If T admits an identity, the standard enveloping module of every unital T-module M is universal and $MT \cong M \circ T$ in every enveloping special T-module.*

(iv) *If T is commutative, so is its standard enveloping ring, and T is commutative if it has a commutative enveloping ring.*

(v) *If T is a τ-division ring, then T^2 is a division ring in any enveloping ring.*

(vi) *If the standard enveloping ring of T is a division ring, then T is a τ-division ring.*

Proof. The first assertion in (i) is immediate. Suppose then that $e = e_+ + e_-$ is an identity in $A = T \oplus T^2$. For t in T we have $et = e_+t + e_-t = t$, hence $e_+t = t$ and

$e_-t=0$. This implies $e_-A=0$ which with $e_-e=e_-$ implies $e_-=0$ and $e=e_+ = \sum e_ie_i'$ for suitable $\{e_i, e_i'\}$ in T. As for (ii) note that if $A=T+T^2$ has an identity then it has no left or right annihilators. In particular by Proposition 1 and part (i) of Theorem 1 the kernel of the canonical homomorphism of T^2 onto $T \circ T$ is zero. By part (ii) of Theorem 1 the kernel of the canonical homomorphism of any $T \oplus T^2$ onto $T \oplus T \circ T$ is also zero.

To verify (ii) suppose that M is unital and $Y=M+MT$. Then MT is unital for

$$mt\left(\sum e_ie_i\right) = \sum mt(e_ie_i') = m\left(\sum te_ie_i'\right) = mt.$$

As a consequence Y is (special) unital and the result now follows by consideration of the canonical homomorphism of Theorem 3 and the canonical homomorphism $\sum m_it_i \to \sum m_i \circ t_i$ of any MT onto $M \circ T$.

To establish the first part of (iv) it is sufficient to prove that every element of T commutes with every element of T^2, which is obvious, and that the elements of T commute. But $(uv-vu)T=T(uv-vu)=0$ identically in T, which in the standard imbedding implies $uv=vu$.

For (v), if T is a τ-division ring it has no proper one-sided ideals. Moreover if $W\neq 0$ is a right ideal in the nonnull ring T^2, then WT is a right ideal in T, hence $WT=T$ and $WT^2=T^2=W$. Thus T^2 is a division ring. The final property (vi) requires only the observation that the inverse of an element of T must also be in T.

The limitations in (iv) and (v) above are necessary. In particular a τ-division ring may have an enveloping ring which is not a division ring. The following pair of examples are instructive. Let $1, i, j, k$ be a standard basis for the division algebra of real quaternions. The space spanned by j, k is a τ-division ring with the complete ring of quaternions as enveloping ring. On the other hand, using matrix unit notation, consider the subspaces of the algebra of 2 by 2 real matrices spanned by $e_{12}-e_{21}$ and $e_{11}-e_{22}$. This is a τ-division ring with the full ring of matrices as enveloping ring. In both examples T^2 is the complex field.

The radical theory of commutative τ-rings appears to lack novelty. We give one example of an expected result, using the appropriate notion of subdirect sum [1, p. 13].

THEOREM 12. *Every semisimple commutative τ-ring is a subdirect sum of τ-fields.*

Proof. Let T be such a τ-ring and $A=T \oplus T \circ T$. Because A is commutative and semisimple (corollary of Theorem 11), A is a subdirect sum of fields F_α, with each of which there is given an epimorphism α: $A \to F_\alpha$. Clearly these maps imbed T in the complete direct sum U of the τ-rings $T_\alpha = T\alpha$. Since $F_\alpha = T_\alpha + T_\alpha^2$ and F_α is simple, either $T_\alpha = T_\alpha^2$ and T_α is the τ-field of a field or $F_\alpha = T_\alpha \oplus T_\alpha^2$ and T_α is a τ-field by (vi) of Proposition 6.

If M is a T-module, Definition 4(i) implies that the maps $m \to \sum mt_it_t$ form a ring of endomorphisms of $(M, +)$. It is natural to take the centralizer of this subring as the *centralizer* $H(M)$ of M. Of course $H(M)$ is a ring and if M is irreducible $H(M)$ is a division ring. In order to recover the τ-ring structure it is necessary to imbed M in a special τ-module and to use the natural definition of the centralizer $H(X)$ of a special T-module X.

PROPOSITION 7. *If $X=M\oplus MT$ and if σ is the semireflection of X determining M then*

(i) *$\sigma^*: \gamma\to\sigma\gamma\sigma^{-1}$ is a semireflection in the ring $H(X)$,*

(ii) *$\sigma^*_- = \{\gamma \mid M\gamma\subset MT$ and $(MT)\gamma\subset M\}$,*

(iii) *$\sigma^*_- = \{\gamma \mid M\gamma\subset M$ and $(MT)\gamma\subset MT\}$, and*

(iv) *if $MT\cong M\circ T$ then $\sigma^*_+\cong H(M)$.*

Proof. Only (iv) requires more than a simple check. Suppose then that γ_0 is in $H(M)$. Clearly if γ_0 extends to an element γ of $H(X)$ that extension is unique. But if $MT\cong M\circ T$ then the definition $\gamma: \sum m_i\circ t_i\to\sum (m_i\gamma_0)\circ t_i$ of γ on $M\circ T$ is unambiguous and together with $\gamma_0=\gamma$ on M defines an element of $H(X)$.

Even if (iv) holds it may occur that $\sigma^*_-\oplus(\sigma^*_-)^2\neq H(X)$. If, however, M is irreducible then the nonzero elements of σ^*_- are isomorphisms of M and $M\circ T$ so that either $\sigma^*_-=0$ (in example (a) of §5 take $M=X_1$ and $\dim X_1\neq\dim X_2$) or σ^*_- is a τ-division ring and its standard complement is $\sigma^*_+=(\sigma^*_-)^2\cong H(M)$.

7. **Nonstandard imbeddings and primitive τ-rings.** According to Theorem 3 a nonstandard complement MT for a T-module M is distinguished by the fact that $\sum m_it_i\to\sum m_i\circ t_i$ is a proper homomorphism. Application of this to the regular representation of T in any enveloping ring provides a connection with complementary subrings. In general it is easy to produce examples of either (see examples (b) and (c) in §5). The ready examples do not, however, exhibit a nonstandard complement for an irreducible submodule of a faithful irreducible special module. From Theorem 7 it follows that if $X=M+MT$ is faithful and special-irreducible then $MT\cong M\circ T$. The question is the following: Can M have other imbeddings? If T admits an identity the answer is negative (Proposition 6 (iii)) and in §8 we will show this applies to primitive τ-rings satisfying a chain condition. It can also be shown that the answer is negative if T is commutative. Thus an example for which the answer is affirmative not only sheds light on the structure of $H(X)$ (Proposition 7) but has independent interest as well.

Let V be a vector space over a field Φ of characteristic 0 with a countable basis $e_0, e_1, \ldots$. Let u be the linear transformation defined by $e_0u=0$, $e_iu=ie_{i+1}$, $i=1, 2, \ldots$; and let v be given by $e_0v=0$, $e_iv=e_{i-1}$, $i=1, 2, \ldots$. Let $A=\Phi^*[u, v]$ be the algebra strictly generated by u, v. In the space $V-[e_0]=\bar{V}$, u and v induce maps $\bar{u}$ and $\bar{v}$ which generate the homomorph $\bar{A}$ of A. We assert that if T is the τ-ring generated by u, v then $A=T\oplus T^2$. Moreover the subspaces $V_1=[e_1, e_3, e_5, \ldots]$ and $V_2=[e_0, e_2, e_4, \ldots]$ are T-modules with $V_2=V_1T$. Since $e_0T=0$, V_1T is not standard, yet V_1 is irreducible and $\bar{V}\cong V_1\oplus\bar{V}_2$ is a faithful special T-module.

What requires proof is that V_1 is irreducible and that $\bar{V}$ is faithful. The key properties are:

(i) The set $\{\bar{v}^k\bar{u}^l\}$, $k\geqq 0$, $l\geqq 0$ ($\bar{v}^0\bar{u}^0=1$), is a basis for $\bar{A}$ and $\{v^k u^l\}$, $k\geqq 0$, $l\geqq 0$, is a basis for A, provided we take $v^0u^0=w$, where $e_0w=0$, $e_iw=e_i$, $i=1, 2, \ldots$. Thus the canonical homomorphism of A onto $\bar{A}$ is an isomorphism and $\bar{V}$ is therefore a faithful A-module.

(ii) The formulas

$$\bar{e}_i[(k+1)\bar{v}^{k-1}\bar{u}^k-\bar{v}^k\bar{u}^{k+1}] = \alpha_{ik}\bar{e}_{k+1}$$

and

$$\bar{e}_i[(k-1)\bar{v}^{k-1}\bar{u}^{k-2}-\bar{v}^k\bar{u}^{k-1}] = \beta_{ik}\bar{e}_{k-1}$$

hold, where $\alpha_{ik}=(k+1)!\delta_{ik}$ and $\beta_{ik}=(k-1)!\delta_{ik}$.

This implies that $\bar{A}$ contains all linear transformations whose matrices have finitely many nonzero entries and thus that $\bar{A}$ is dense.

8. **Semisimple τ-rings with minimum condition.** As for rings, the condition that every set of right ideals in a τ-ring has a minimal element is equivalent to the nonexistence of infinite, properly descending chains of right ideals. We call these rings and τ-rings *Artinian.* In general, however, an enveloping ring of an Artinian τ-ring need not be Artinian. For semisimple τ-rings the situation is otherwise and the structural implications are strong.

THEOREM 13. *A τ-ring T is semisimple and Artinian if and only if its standard enveloping ring A is semisimple and Artinian.*

Proof. Let $A=T\oplus T\circ T$. By Theorem 11 if A is semisimple so is T, and any properly descending chain of right ideals of T generates (in any direct imbedding) a properly descending chain of right ideals in A.

Suppose then that T is semisimple and Artinian. By the corollary to Theorem 11 A is semisimple. The Artinian condition implies that any intersection of right ideals of T is a finite intersection. Let $U_1, \ldots, U_n$ be maximal modular right ideals of T for which $0=\bigcap U_i$. Each U_i generates the modular right ideal U_i+U_iT of A, and every proper modular ideal of A is contained in a maximal modular ideal B_i of A [1, p. 6]. From $B_i\cap T=U_i$ it follows that $(\bigcap B_i)\cap T=0$. Let σ be the reflection in A with $\sigma_-=T$. We write B^σ for $B\sigma$.

$$[(\bigcap B_i)\cap T]^\sigma = (\bigcap B_i^\sigma)\cap T = 0 \quad \text{and} \quad B_i^\sigma \supset U_i.$$

As a consequence if $B=\bigcap B_i$ then $B^\sigma=\bigcap B_i^\sigma$ and $B\cap B^\sigma\cap T=0$. Since $B\cap B^\sigma$ is σ-invariant $B\cap B^\sigma\subset T^2$, consequently $(B\cap B^\sigma)T=0$, and by the semisimplicity of A, $B\cap B^\sigma=0$.

This last relation shows that 0 is a finite intersection of maximal modular right ideals of A. Assuming this intersection irredundant we construct a composition series for right ideals: $A\supset B_1\supset B_1\cap B_2\supset\cdots$. But any operator-group with a composition series is Artinian.

COROLLARIES. *If T is semisimple and Artinian:*

(1) *T admits an identity.*

(2) *A is universal for T and every enveloping ring of T is semisimple and Artinian.*

(3) *In any enveloping ring for T, $T^2 \cong T \circ T$.*

(4) *Every homomorphic image of T is semisimple and Artinian.*

The first assertion follows from the fact that every semisimple Artinian ring has an identity. Proposition 6 (ii) gives the next two, and the last follows from the corresponding property for A.

THEOREM 14. *If T is semisimple and Artinian, then*

(i) *T^2 is semisimple and Artinian,*

(ii) *every unital T-module is completely reducible,*

(iii) *T is the direct sum of a finite number of minimal right ideals, and*

(iv) *T is the direct sum of a finite number of simple ideals.*

Proof. Theorem 11 asserts the semisimplicity of $T \circ T$ and the corollary above extends this to any T^2. The ring T^2 is Artinian since every properly descending chain of right ideals in T^2 generates such a chain in $A = T \oplus T^2$, which is Artinian by Theorem 13. The assertion (ii) follows from the observation that a unital T-module is a unital T^2-module with the same submodules. Application of (ii) to T as a T-module shows that T is the direct sum of minimal right ideals, necessarily finite since T is Artinian.

The final property can be obtained by using the fact that $A = A_1 \oplus \cdots \oplus A_n$, where A is the standard enveloping ring for T and each A_i is a simple ideal. Let σ be the reflection in A with $T = \sigma_-$. If $A_i\sigma = A_i$ let T_i be τ-ring corresponding to $\sigma|A_i$. If A_i is not σ-invariant let $B_i = A_i \oplus A_i\sigma$ and let T_i be the τ-ring corresponding to $\sigma|B_i$. In the first case any proper ideal in T_i generates a proper ideal in A_i; hence T_i is simple. In the second case we need to observe that this proper ideal is σ-invariant and that B_i is σ-simple in the sense that it has no proper σ-invariant ideals. Thus, again, T_i is simple. Clearly the T_i are ideals in T so that T is the direct sum of the distinct T_i.

The argument above gives more than is claimed in Theorem 14. It shows that simple Artinian τ-rings are of two types. These can be characterized very simply in the manner suggested by Theorem 6.

THEOREM 15. *If T is simple and Artinian, then either*

(i) *$T \cong B_\tau$ for some simple, Artinian B and $A = T \oplus T \circ T$ is the direct sum of two simple ideals isomorphic to B, or*

(ii) *$T \not\cong B_\tau$ for any B and A is simple.*

Proof. The proof of Theorem 14 shows that the simplicity of T is equivalent to the σ-simplicity of A. Suppose that A is σ-simple but not simple and B is a proper ideal. Since $B \cap B\sigma$ and $B + B\sigma$ are σ-invariant ideals, $A = B \oplus B\sigma$. If φ is the corresponding projection of A onto B, then $\varphi|T$ is a monomorphism which must be an isomorphism of T and B_τ because T generates A.

Conversely if $T \simeq B_\tau$ for some simple B then T is simple. For if U is a right ideal in B_τ, $U \supset UBB = UB$. Furthermore the simplicity of B assures that an isomorphism ψ of T and B_τ is extended uniquely to an epimorphism from A to B by

$$t + \sum t_i \circ t_i' \to t\psi + \sum (t_i\psi)(t_i'\psi),$$

and the kernel of this map is a proper ideal in A.

One consequence of the preceding results is that up to isomorphism T^2 is independent of the imbedding, and we will therefore refer to *the* ring T^2. Example (a) in §5 shows that for a simple, Artinian T the corresponding T^2 is not necessarily simple. Example (a) exhibits, however, the only other possibility.

THEOREM 16. *If T is simple and Artinian, then either*
(i) *T has no proper right-left ideals and T^2 is simple or*
(ii) *T has precisely two proper right-left ideals K_1, K_2 and K_1K_2, K_2K_1 are distinct simple ideals in T^2 such that $K_1^2 = K_2^2 = 0$ and $T = K_1 \oplus K_2$, $T^2 = K_1K_2 \oplus K_2K_1$.*

Proof. Suppose first that T has a proper right-left ideal K_1. Let $M = T - K_1$ as a right T-module, and set $K_1' = \{t \mid MTt = 0\}$, $K_2 = \{t \mid MtT = 0\}$. (See Definition 6 and Proposition 4 of §3.) Then $K_1' = \{t \mid T^2t \subset K_1\} \supset K_1$ and $K_2 = \{t \mid TtT \subset K_1\}$. Now $K_1' \cap K_2$ is an ideal, and if $K_1' = T$ then $T^3 \subset K_1$, which contradicts $T^3 = T$. We conclude that $K_1' \cap K_2 = 0$. Furthermore K_1' is completely reducible as a left module by Theorem 14 and therefore $K_1' = L \oplus K_1$. But $T^2K_1' \subset K_1$ and thus $T^2L = 0$ which, by the semisimplicity of T, implies $L = 0$.

From $M(TK_1T)T \subset MTK_1 = 0$ it now follows $TK_1T \subset K_2$ and, symmetrically, $TK_2T \subset K_1$. But then $K_1^2T \subset K_1 \cap K_2 = 0$ and so $K_1^2 = K_2^2 = 0$. In T^2, K_1K_2 and K_2K_1 are ideals and $T^2 = (K_1 + K_2)^2 = K_1K_2 + K_2K_1$. The fact that

$$(K_1K_2 \cap K_2K_1)K_1 = (K_1K_2 \cap K_2K_1)K_1 = 0$$

implies $K_1K_2 \cap K_2K_1 = 0$. Finally, the ring K_1K_2 is simple for if W is a proper ideal in K_1K_2, $WT \subseteq K_1$, $TW \subseteq K_2$ and $WT + TW$ is an ideal in T. This argument also shows that if T^2 is not simple T has a proper right-left ideal.

Finally we verify that $K_1K_2T = K_1$ and $K_2K_1T = K_2$. But since the only proper ideals of T^2 are K_1K_2 and K_2K_1, we have for any proper right-left ideal L of T either $K_1K_2T = L$ or $K_2K_1T = L$. Thus $L = K_1$ or $L = K_2$.

Theorem 17 describes the way in which the classification of structure given in Theorem 16 reflects in representations.

THEOREM 17. *If T is simple and Artinian a necessary and sufficient condition that all irreducible T-modules be isomorphic is that T have no proper right-left ideals. If T has proper right-left ideals K_1, K_2 then these are the homogeneous components of T as a right T-module, and consequently T has precisely two nonisomorphic irreducible modules.*

Proof. Let M be an irreducible T-module. Since T is completely reducible as a right T-module, $M \simeq T - U \simeq V$ where U is a maximal and V a minimal right ideal.

Let $K_1=\sum U_\alpha$ over all $U_\alpha\cong V$ as right T-modules. For w_0 in $T\circ T$, the map $t\to w_0t$ is an endomorphism of T as a right T-module and hence maps K_1 into itself. Thus $T^2K_1\subset K_1$ so that K_1 is a right-left ideal. As a consequence all irreducible T-modules are isomorphic if and only if $K_1=T$.

In case $K_1\neq T$, by Theorem 16, $T=K_1\oplus K_2$ where K_1, K_2 are the only proper right-left ideals in T. It follows that K_2 is the homogeneous component of its minimal right ideals and since any minimal right ideal is a direct summand it is isomorphic as a T-module to a submodule either of K_1 or K_2.

Observe that if $T=A_\tau$ then T has no proper right-left ideals, for since $T^2=T$, every right-left ideal is an ideal.

9. **Simple ternary algebras.** To complete the results in the preceding section we sketch the classification of those simple, Artinian ternary rings which are algebras, in the obvious sense, over an algebraically closed field or the real (actually any formally real closed) field.

Suppose then that T is a finite-dimensional simple ternary algebra over P, an algebraically closed field. Place T in class I if it has a nonsimple standard enveloping algebra. By Theorem 15 this class consists of the ternary algebras of n by n matrices $(\mathrm{P}_n)_\tau$. If T is not in class I put it in class II, consisting of these T with simple standard enveloping algebras. Classes I and II are disjoint and exhaustive, again by Theorem 15.

According to Theorem 2 each T in class II is the ternary algebra σ_- for some reflection σ of a simple ternary algebra A. It is easily verified that $\sigma_-\cong\sigma'_-$ if and only if σ and σ' are conjugate in the automorphism group of A. In the case at hand the simple algebras are the P_n and the automorphisms are all inner, i.e. $\sigma: x\to s^{-1}xs$ for some nonsingular n by n matrix s. The condition $\sigma^2=I$ is equivalent to $s^2=\alpha I$, α in P. Choose β in P with $\beta^2\alpha=1$. The automorphism of P_n determined by $s'=\beta s$ is also σ and $(s')^2=I$. The matrix s' is similar to a diagonal matrix $s^{(r)}=\operatorname{diag}(1,\ldots,1,-1,\ldots,-1)$, where r denotes the number of ones. Since the corresponding $\sigma^{(r)}$ is a conjugate of σ, class II consists of the ternary algebras $T^{(r)}$ consisting of all matrices

$$\left[\begin{array}{c|c} 0_r & (\alpha_{ij}) \\ \hline (\alpha'_{ij}) & 0_{n-r} \end{array}\right].$$

By restricting r to $1\leqq r\leqq [n/2]$ we assure that no two of the $T^{(r)}$ are isomorphic.

In order to classify the simple real ternary algebras we need to recall that the simple real algebras fall into three classes.

A. The matrix algebras Φ_n, Φ the real field.

B. P_n, P the complex field.

C. Δ_n, Δ the division ring of real quaternions.

The automorphisms of the algebras in A or C are all inner, but these in B have additional automorphisms of the form $x\to s^{-1}\bar{x}s$.

As before let class I consist of the ternary algebras $(\Phi_n)_\tau$, $(P_n)_\tau$, and $(\Delta_n)_\tau$. Let class IIA designate those with some Φ_n as enveloping algebras. Suppose then σ is a reflection in Φ_n given by $x \to s^{-1}xs$; where $s^2 = \alpha I$. If $\alpha > 0$ we may proceed as before to replace s by an s' with $(s')^2 = 1$. The reflections produce the ternary algebras $T^{(r)}(\Phi)$ of *real* matrices. If $\alpha < 0$ the matrix s may be replaced by an s' with $(s')^2 = -I$. In this case $n = 2m$ and every such matrix is similar to the following matrix.

$$\left[\begin{array}{c|c} 0 & I_m \\ \hline -I_m & 0 \end{array}\right]$$

This implies that σ_- is isomorphic to the ternary algebra U consisting of all matrices.

$$\left[\begin{array}{c|c} x & y \\ \hline y & -x \end{array}\right]$$

To determine the ternary algebras of class IIB, those with some P_n as enveloping algebras, we first observe that the inner automorphisms give rise to the ternary algebras $T^{(r)}(P)$. The automorphisms $x \to s^{-1}\bar{x}s$ can be identified with the conjugate-linear reflections of the algebra of linear transformations in an n-dimensional complex vector space V. If σ is such a reflection then (using capitals for maps on V and small letters for elements of V) σ: $X \to S^{-1}XS$ for some conjugate-linear transformation S and all linear transformations X in V. Again, $S^2 = \lambda I$ for some complex λ. Letting αS: $x \to (\alpha x)S$ and $s\alpha$: $x \to \alpha(xS)$, c-linearity becomes $\bar{\alpha}S = S\alpha$. Thus $(\alpha S)^2 = \bar{\alpha}\alpha S^2$ so that S may be replaced by αS for any $\alpha > 0$. Furthermore $S^3 = S\lambda = \lambda S = \bar{\lambda}S$ so that λ is real. It follows that a c-linear transformation S determining σ may be chosen so that either $S^2 = I$ or $S^2 = -I$.

In case $S^2 = I$ we have $V = V_1 \oplus V_{-1}$ where $V_1 = \{x + xS\}$ and $V_{-1} = \{x - xS\}$; consequently S has a diagonal matrix $s^{(r)}$. Now regard S as a *linear* transformation in V over Φ; and choose a basis $e_1, \ldots, e_n, ie_1, \ldots, ie_n$ for V over Φ such that the matrix of S (in V) relative to $e_1, \ldots, e_n$ is $s^{(r)}$. The matrix of S in V over Φ is determined by $(ie_k)S = -i(e_kS)$, and the *linear* transformations in V have matrices characterized by $(ie_k)X = i(e_kX)$.

From this information it is possible to get an explicit matrix representation of each real, simple ternary algebra which belongs to P_n via an automorphism of the class under consideration. The result is that each is isomorphic to one of the ternary algebras $T_*^{(r)}$ where $T_*^{(r)}$ denotes the set of n by n complex matrices having the form

$$\left[\begin{array}{c|c} iX_1 & X_3 \\ \hline X_4 & iX_2 \end{array}\right]$$

where each X_i is real and X_1 is r by r. Again redundance is elimintated by requiring $0 \leqq r \leqq [n/2]$.

In case $S^2 = -I$ we note that there are no one-dimensional spaces invariant under S since $xS = \lambda x$ implies $xS^2 = (\lambda x)S = \bar{\lambda}\lambda x = -x$. Thus for any $x \neq 0$ in V, $\{x, xS\}$ spans a two-dimensional S-irreducible subspace. It follows that V is the direct sum of such spaces and consequently that $n = 2m$ is even. There is a basis $e_1, \ldots, e_n$ for V such that $e_k S = e_{k+m}$, $e_{k+m}S = -e_k$, $k = 1, \ldots, m$. As before, consider V as a real space, S as a real linear transformation, and extend the given basis to a basis $e_1, \ldots, e_n, ie_1, \ldots, ie_n$ for V over Φ. An isomorph U_* of σ_- can then be exhibited in matrix form. In terms of complex matrices it consists of all

$$\left[\begin{array}{c|c} X & Y \\ \hline \bar{Y} & -\bar{X} \end{array}\right]$$

where X and Y are m by m.

It remains to describe class IIC, the set of simple, real ternary algebras with the real algebras Δ_n as enveloping algebras. As before we may suppose that σ is a reflection determined by a linear transformation S in an n-dimensional vector space V over Δ and that either $S^2 = I$ or $S^2 = -I$. In the first case S has a diagonal matrix and σ_- is therefore isomorphic to the ternary matrix algebra $T^{(r)}(\Delta)$.

In the second case let $1, i, j, k$ be a standard real basis for Δ and let P be the complex subfield spanned by $1, i$. Regard S as a linear transformation in the complex space V_{P}. The decomposition $2x = (x + ix) + (x - ix)$ produces a decomposition into characteristic subspaces: $Y_{\mathrm{P}} = V_i \oplus V_{-i}$. Since the map $x \to jx$ induces a nonsingular semilinear transformation of V_i onto V_{-i}, any basis $e_1, \ldots, e_n$ for V_i produces a basis $e_1, \ldots, e_n, je_1, \ldots, je_n$ for V_{P}. It follows that $e_1, \ldots, e_n$ is a basis for V relative to which S has matrix iI, and that if $T = \sigma_-$ then T is isomorphic to the real ternary algebra $T_*^{(0)}(\Delta)$ of all n by n matrices with entries in the space spanned by j, k.

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