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RINGS DEFINED BY $\mathscr{R}$-SETS AND A CHARACTERIZATION OF A CLASS OF SEMIPERFECT RINGS

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Introduction. In this paper we give a characterization of semiperfect rings with projective, essential left socle (Theorems 5.1 and 4.4). This characterization is effected through the use of special $\mathscr{R}$-sets. General $\mathscr{R}$-sets are defined and discussed in §3. Special $\mathscr{R}$-sets are dealt with in §4.

Theorem 5.1 gives necessary and sufficient conditions for a ring with the above properties to be indecomposable, left (or right) perfect, semiprimary, left (or right) artinian, and for the left socle of the ring to be finitely generated. Thus we have, in particular, given a solution to a problem of Goldie [2, p. 268]: "One very interesting problem is the determination of artinian rings with zero singular ideal." In this connection, see also Gordon [4, Theorem 3.1].

Theorem 5.2 is a special case of Theorem 5.1—the determination of semiperfect rings with projective, essential left socle and a unique isomorphism class of minimal left ideals. The simplest natural instance of Theorem 5.2 is exploited in Theorem 5.3. Here we determine those semiperfect rings with projective, essential left socle in which principal indecomposable left ideals have unique simple submodules. This is a generalization of a theorem of Zaks' who handled the semiprimary case [9, Theorem 1.4, p. 67].

In Proposition 5.5 we give necessary and sufficient conditions for a semiperfect ring R with projective, essential left socle to have projective, essential right socle. This proposition implies that the right socle is typically not even a projective submodule of R_R.

Our main result in §2 is the following lemma (Lemma 2.2): If R is a ring in which the identity is a sum of orthogonal idempotents e_i, then the radical of R is left T-nilpotent if and only if the radical of e_iRe_i is left T-nilpotent for every i. Also in §2, we prove what might be a new lemma about reflexive, transitive relations on a finite set (see Lemma 2.7). This lemma allows us to give a normal form for rings of the type characterized in Theorem 5.3. An example (see Remarks 5.4) shows that this normal form simply need not occur in more general cases.

We remark that this paper seems to inherently give rise to some apparently hard problems at various levels of abstraction. One glaringly obvious such instance:

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What are, and what is the structure of, local rings which imbed in the ring of row-finite square matrices (arbitrary cardinality) over a division ring?

We reserve possibly for a later paper the task of giving more specific and special results of our main characterization theorem, Theorem 5.1. To have done so in this paper would, we feel, have added only confusion and acted against the hopefully general tone of the paper.

1. **Preliminaries.** For conventions and definitions utilized in this paper, we refer the reader to our earlier paper [4].

(1.1) Throughout this paper, we use the concepts of Morita equivalence and the reduced ring to simplify our work. We call a ring R' *reduced* if $R'=R'e_1 \oplus R'e_2 \oplus \cdots \oplus R'e_n$ where the e_i are primitive idempotents and $R'e_i \simeq R'e_j$ implies $i=j$. Then, given integers $m_1, \dots, m_n$, there exists a ring

$$R = \bigoplus \sum_{1 \leq i \leq n;\, 1 \leq j \leq m_i} Re_{ij}$$

where the e_{ij} are primitive idempotents and $Re_{ij} \simeq Re_{i'j'}$ if and only if $i=i'$, such that $R'=1'R1'$ where $1'=\sum_{i=1}^{n} e_{i1}$ and $e_{i1}=e_i$. In fact, R may be taken to be the ring of all $n \times n$ blocked matrices

$$\begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & & \vdots \\ B_{n1} & \cdots & B_{nn} \end{bmatrix}$$

where B_{ij} is the $e_iR'e_i - e_jR'e_j$ bimodule of $m_i \times m_j$ matrices with entries in $e_iR'e_j$ (see, for example [3, p. 338]). We call R' a *reduced ring of* R. R' is uniquely determined by the decomposition of R (not the ring R) up to inner automorphism (Osima [8, Theorem 3]).

If the Krull-Schmidt Theorem holds in R, then it holds in R', and R' is the unique (up to inner automorphism) reduced ring of R. In this case R is uniquely determined by its reduced ring R' and the integers $m_1, \dots, m_n$. This is still true if only R' satisfies the Krull-Schmidt Theorem. We do not know if the Krull-Schmidt Theorem must then hold in R. Notice, however, that if R' is semiperfect, then R is also semiperfect and thus trivially Krull-Schmidt (for instance, by [4, Lemma 2.7]).

(1.2) PROPOSITION. *Let R be a ring in which the identity is a sum of orthogonal primitive idempotents and let (in the notation of* 1.1) $R'=1'R1'$ *be a reduced ring of R. Then the categories ${}_R\mathcal{M}$ and ${}_{R'}\mathcal{M}$*(1) *are equivalent. In fact, the functors defined by $M \in {}_R\mathcal{M} \to 1'M \in {}_{R'}\mathcal{M}$ and $M' \in {}_{R'}\mathcal{M} \to R1' \otimes_{R'} M' \in {}_R\mathcal{M}$ are inverse equivalences.*

Proof. Evidently, $R1'R=R$ (for example, [3, Lemma (0.2)]). (So $1'R$ is a progenerator in $\mathcal{M}_R$.) Without using the concept of progenerator, the "Dual Basis

(1) ${}_U\mathcal{M}$ merely denotes the category of left U-modules.

Lemma" [6, p. 86, Exercise 1] implies immediately that $R1'$ is a finitely generated, projective right R'-module. The proposition is then an obvious consequence of Morita [7, Theorem 3.4].

2. **Some lemmas.** In this section we prove some lemmas which will be helpful to us in the rest of the paper.

(2.1) LEMMA. *Let $R=\bigoplus \sum Re_i$ be a reduced semiperfect ring (the e_i's being orthogonal primitive idempotents). Then $J(R)=\sum_{i\neq j} e_iRe_j+\sum_k J_k$ where $J_k=J(e_kRe_k)$.*

Proof. Suppose that $e_iRe_j \nsubseteq J$. Then $e_iRe_j \nsubseteq Je_j$. Since R is semiperfect, the e_k's are actually local idempotents. We must have $Re_i \simeq Re_j$ (for example, see [4, Proposition 2.3]). So, since R is reduced, $i=j$. Therefore, $e_iRe_j=e_iJe_j$ if $i\neq j$. Since $J=\sum_{i,j} e_iJe_j$ and $J(e_iRe_i)=e_iJe_i$, the lemma is proved.

(2.2) LEMMA. *Let R be a ring in which the identity is a sum of orthogonal idempotents e_i. Then the following two statements hold.*

(1) *If $J(e_iRe_i)$ is nilpotent for every i, then $J(R)$ is nilpotent.*

(2) *If $J(e_iRe_i)$ is left T-nilpotent for every i, then $J(R)$ is left T-nilpotent.*

Proof. The hypothesis of the lemma implies that $J=J(R)=\sum_{i,j} e_iJe_j$.

(1) Set $N=n\rho$ where ρ is chosen so that $(e_iJe_i)^\rho=0$ for $1\leqq i\leqq n$. Then

$$J^N = \sum_{i_0,\dots,i_N} e_{i_0}Je_{i_1}Je_{i_2}\cdots e_{i_{N-2}}Je_{i_{N-1}}Je_{i_N}$$

where the i_k run independently from 1 to n. Fix a sequence $i_0,\dots,i_N$. By the choice of N, there exists an α with $1\leqq\alpha\leqq n$ and a subsequence $j_0,\dots,j_\rho$ of $i_0,\dots,i_N$ such that $j_k=\alpha$ for every k. Therefore,

$$e_{i_0}Je_{i_1}\cdots e_{i_{N-1}}Je_{i_N} \subseteq e_{i_0}Je_{j_0}Je_{j_1}\cdots e_{j_{\rho-1}}Je_{j_\rho}Je_{i_N} = e_{i_0}J(e_\alpha Je_\alpha)^\rho Je_{i_N} = 0$$

(with the obvious adjustments if $i_0=j_0$ or $i_N=j_\rho$). Thus $J^N=0$.

(2) Suppose false. Then there exists a sequence $\{x_i\}$ of elements of J such that $x_1x_2\cdots x_m\neq 0$ for all m. For each i, we may write $x_i=\sum_{p,q} a^i_{pq}$ where $a^i_{pq}\in e_pJe_q$. Thus

$$x_1\cdots x_m = \sum_{i_0,\dots,i_m} a^1_{i_0i_1}a^2_{i_1i_2}\cdots a^{m-1}_{i_{m-2}i_{m-1}}a^m_{i_{m-1}i_m}$$

where the i_k's run independently from 1 to n. We let sequences $i_0, i_1,\dots, i_m$ with $a^1_{i_0i_1}\cdots a^m_{i_{m-1}i_m}\neq 0$ be vertices of a tree in which edges correspond to adjoining a new index. Obviously, each vertex has finite index. Since $x_1\cdots x_m\neq 0$ for all m, it follows that there exist paths of arbitrary length. Thus the König Graph Theorem implies the existence of an infinite path. That is, there exists a sequence $i_0, i_1, \ldots, i_m, \ldots$ such that $a^1_{i_0 i_1} \cdots a^m_{i_{m-1} i_m} \neq 0$ for all m.

Next we take an integer β with $1 \leqq \beta \leqq n$ and a subsequence $\{i_{f(p)}\}$ of $\{i_p\}$ such that $i_{f(p)} = \beta$ for every p. Set

$$
\begin{aligned}
y_1 &= a^{f(0)+1}_{i_{f(0)} i_{f(0)+1}} \cdots a^{f(1)}_{i_{f(1)-1} i_{f(1)}}, \\
y_2 &= a^{f(1)+1}_{i_{f(1)} i_{f(1)+1}} \cdots a^{f(2)}_{i_{f(2)-1} i_{f(2)}}, \\
&\ \vdots \\
y_p &= a^{f(p-1)+1}_{i_{f(p-1)} i_{f(p-1)+1}} \cdots a^{f(p)}_{i_{f(p)-1} i_{f(p)}}, \\
&\ \vdots
\end{aligned}
$$

So, we have $y_1 y_2 \cdots y_p \neq 0$ for all p. But, for all p, $y_p \in e_\beta J \cdots J e_\beta \subseteq e_\beta J e_\beta$. This contradicts the left T-nilpotency of $J(e_\beta R e_\beta)$.

(2.3) Corollary. *If R is semiperfect, then R is left perfect (semiprimary) if and only if the reduced ring of R is left perfect (semiprimary).*

(2.4) Remark. The left perfect part of the corollary also follows immediately from 1.2. The reason is that Bass' original definition [1, p. 466] of left perfect rings is categorical.

(2.5) Lemma. *Let R be a semiperfect ring and M be a completely reducible left R-module. Then the following are equivalent.*

(1) *${}_RM$ is artinian.*

(2) *eM is a finite dimensional left vector space over eRe/eJe for every primitive idempotent e in R.*

Proof. Without loss of generality, R is a reduced ring. Given a primitive idempotent $e \in R$, define a map $\varphi: R \to eRe/eJe$ by $\varphi(r) = ere + eJe$. Then, since R/J is a direct sum of division rings, it follows that eM is the homogeneous component of the completely reducible module ${}_RM$ of isomorphism type Re/Je and that φ is a ring-epimorphism. Also, the diagram

$$
\begin{array}{ccccc}
R & \times & eM & \longrightarrow & eM \\
{\scriptstyle\varphi}\downarrow & & \| & & \| \\
eRe/eJe & \times & eM & \longrightarrow & eM
\end{array}
$$

commutes. Since it does, ker φ is contained in the kernel of the action of R on eM. So eM is the same whether regarded as a left R-module or as a left eRe/eJe-module. But M has the form $M = \bigoplus \sum e_i M$ where the e_i are primitive idempotents of R. The lemma follows.

(2.6) Corollary. *If R is a semiprimary ring, the following are equivalent.*

(1) *$e \cdot J^i M / J^{i+1} M$ is a finite dimensional left vector space over eRe/eJe for every primitive idempotent e and every finitely generated ${}_RM$.*

(2) *$(eJe)^iM/(eJe)^{i+1}M$ is a finite dimensional left vector space over eRe/eJe for every primitive idempotent e and every finitely generated ${}_RM$.*

(3) (1) (*or* (2)) *holds for every indecomposable direct summand of ${}_RR$.*

(4) *R is left artinian.*

(2.7) Lemma. *Suppose that ρ is a reflexive, transitive relation on $\{1, \dots, n\}$. Then there exists a relation ρ^* on $\{1, \dots, n\}$ such that*

(a) *ρ^* is isomorphic to ρ and*

(b) *$i\,\rho^*\,j$ for $i<j$ implies $j\,\rho^*\,i$.*

Proof. Our hypothesis implies the existence of an $m \in \{1, \dots, n\}$ such that $i\,\rho\,m$ always implies $m\,\rho\,i$. There is no generality lost in assuming $m=n$.

Let ρ_1 be the restriction of ρ to $\{1, \dots, n-1\} \times \{1, \dots, n-1\}$. ρ_1 is clearly a reflexive, transitive relation on $\{1, \dots, n-1\}$. By induction, there exist a permutation π of $1, \dots, n-1$ and a relation γ on $\{1, \dots, n-1\}$ such that $i\,\gamma\,j$ and $i<j$ imply $j\,\gamma\,i$; γ being defined by $i\,\gamma\,j$ if and only if $\pi(i)\;\rho_1\;\pi(j)$, $1 \leqq i, j < n$.

Define a permutation π^* of $1, \dots, n$ by

$$\begin{aligned} \pi^*(i) &= \pi(i) \quad \text{if } 1 \leqq i < n, \\ &= n \qquad \text{if } i = n. \end{aligned}$$

Then define the relation ρ^* on $\{1, \dots, n\}$ by $i\,\rho^*\,j$ if $\pi^*(i)\;\rho\;\pi^*(j)$. The reader will easily verify that ρ^* works.

3. The construction.

(3.1) Definition. An $\mathscr{R}$-set is an ordered set

$$(n, L_{ij}, t_{ikj}, \Omega, R^\alpha, f_{ij}^\alpha, \chi_i^\alpha)$$

satisfying

(1) L_{ij} is an abelian group for $1 \leqq i, j \leqq n$ (n is a positive integer).

(2) $\{R^\alpha\}_{\alpha \in \Omega}$ (Ω is an arbitrary nonempty set) is a family of rings and $\{\chi_i^\alpha\}_{\alpha \in \Omega; 1 \leqq i \leqq n}$ s a family of cardinal numbers.

(3) For each triple i, k, j with $1 \leqq i, k, j \leqq n$ and for every $\alpha \in \Omega$,

$$\begin{tikzcd} L_{ik} \otimes L_{kj} \arrow[r, "t_{ikj}"] \arrow[d, "f_{ik}^\alpha \otimes f_{kj}^\alpha"'] & L_{ij} \arrow[d, "f_{ij}^\alpha"] \\ {[R^\alpha]_{ik} \otimes [R^\alpha]_{kj}} \arrow[r, "\text{nat}"] & {[R^\alpha]_{ij}} \end{tikzcd}$$

is a commutative diagram of abelian groups where $[R^\alpha]_{ij}$ is the $R^\alpha - R^\alpha$ bimodule of $\chi_i^\alpha \times \chi_j^\alpha$ row-finite matrices over R^α([2]).

([2]) All tensor products in the remainder of the paper are taken with respect to the ring of integers.

(4) There exist $\varepsilon_i \in L_{ii}$ and $\varepsilon_j \in L_{jj}$ such that $f_{ii}^{\alpha}(\varepsilon_i) = 1_{[R^{\alpha}]_{ii}}$ and $f_{jj}^{\alpha}(\varepsilon_j) = 1_{[R^{\alpha}]_{jj}}$, for all $\alpha \in \Omega$ with $f_{ij}^{\alpha} \neq 0$(3).

(5) For each pair i, j, $\bigcap_{\alpha \in \Omega} \ker f_{ij}^{\alpha} = 0$.

(3.2) Definition. The *ring defined by the $\mathscr{R}$-set* in 3.1 is the ring R of all $n \times n$ matrices whose i, jth entry is an arbitrary element of L_{ij}. Addition in R is defined coordinatewise. Multiplication in R is defined by $(a_{ij})(b_{ij}) = (c_{ij})$ where

$$c_{ij} = \sum_k t_{ikj}(a_{ik} \otimes b_{kj}).$$

To see that R is actually a ring, we need the following propositions.

(3.3) Proposition.

$$\begin{tikzcd}
(L_{ip} \otimes L_{pk}) \otimes L_{kj} \arrow[r, "t_{ipk} \otimes 1"] \arrow[d, "\mathrm{nat}"'] & L_{ik} \otimes L_{kj} \arrow[r, "t_{ikj}"] & L_{ij} \arrow[d, equal] \\
L_{ip} \otimes (L_{pk} \otimes L_{kj}) \arrow[r, "1 \otimes t_{pkj}"] & L_{ip} \otimes L_{pj} \arrow[r, "t_{ipj}"] & L_{ij}
\end{tikzcd}$$

are commutative diagrams.

Proof. Let $x \in L_{ip}$, $y \in L_{pk}$, $z \in L_{kj}$ and set

$$A = t_{ikj}(t_{ipk}(x \otimes y) \otimes z) \quad \text{and} \quad B = t_{ipj}(x \otimes t_{pkj}(y \otimes z)).$$

Then,

$$f_{ij}^{\alpha}(A) = f_{ik}^{\alpha}(t_{ipk}(x \otimes y))f_{kj}^{\alpha}(z) = (f_{ip}^{\alpha}(x)f_{pk}^{\alpha}(y))f_{kj}^{\alpha}(z)$$

and, similarly, $f_{ij}^{\alpha}(B) = f_{ip}^{\alpha}(x)(f_{pk}^{\alpha}(y)f_{kj}^{\alpha}(z))$. So $f_{ij}^{\alpha}(A) = f_{ij}^{\alpha}(B)$ for all $\alpha \in \Omega$. By 3.1(5), $A = B$.

(3.4) Proposition. *For $\varepsilon_i \in L_{ii}$ and $\varepsilon_j \in L_{jj}$ satisfying the condition in* 3.1(4), *the diagrams*

$$\begin{tikzcd}
 & \varepsilon_i \otimes L_{ij} \arrow[dr, "t_{iij}"] & \\
L_{ij} \arrow[ur] \arrow[rr, equal] \arrow[dr] & & L_{ij} \\
 & L_{ij} \otimes \varepsilon_j \arrow[ur, "t_{ijj}"'] &
\end{tikzcd}$$

commute.

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(3) A slight alteration here is due to E. C. Dade.

Proof. Let $x \in L_{ij}$. Then

$$f_{ij}^{\alpha}(t_{iij}(\varepsilon_i \otimes x)) = f_{ii}^{\alpha}(\varepsilon_i)f_{ij}^{\alpha}(x) = f_{ij}^{\alpha}(x) = f_{ij}^{\alpha}(x)f_{jj}^{\alpha}(\varepsilon_j) = f_{ij}^{\alpha}(t_{ijj}(x \otimes \varepsilon_j))$$

for every α. So $t_{iij}(\varepsilon_i \otimes x) = x = t_{ijj}(x \otimes \varepsilon_j)$ by 3.1(5).

(3.5) Remark. Propositions 3.3 and 3.4 show that t_{iii} induces a ring structure (associative with identity) on L_{ii}. With respect to this ring structure, every f_{ii}^{α} is a ring-homomorphism. Furthermore, L_{ij} becomes, in the natural way, a (unitary) $L_{ii}-L_{jj}$ bimodule.

Propositions 3.3 and 3.4 also show that the ring R defined by the $\mathscr{R}$-set 3.1 is really a ring. By 3.4, the identity of R is the matrix

$$\begin{bmatrix} 1_{L_{11}} & & 0 \\ & \ddots & \\ 0 & & L_{L_{nn}} \end{bmatrix}.$$

The only thing left which causes any trouble is the associative law. This follows from 3.3.

(3.6) Proposition. *Let φ_{ij} be the map which sends $x \in L_{ij}$ to the $n \times n$ matrix with x in the (i, j)-position and 0's elsewhere and set $e_i = \varphi_{ii}(1_{LL_{ii}})$. Then the following hold.*

(1) *The e_i are orthogonal idempotents whose sum is the identity of R.*

(2)
$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
L_{ik} \otimes L_{kj} \arrow[r, "t_{ikj}"] \arrow[d, "\varphi_{ik} \otimes \varphi_{kj}"'] & L_{ij} \arrow[d, "\varphi_{ij}"'] \\
e_iRe_k \otimes e_kRe_j \arrow[r, "\nu_{ikj}"] \arrow[d] & e_iRe_j \arrow[d] \\
0 & 0
\end{tikzcd}$$

are exact, commutative diagrams of abelian groups where the ν_{ikj} are the natural maps induced by multiplication in R.

(3) *Every φ_{ii} is a ring-isomorphism.*

(4) *The ring defined by the $\mathscr{R}$-set $(n, e_iRe_j, \nu_{ikj}, \Omega, R^{\alpha}, f_{ij}^{\alpha} \circ \varphi_{ij}^{-1}, \chi_i^{\alpha})$ is isomorphic to R by the map which sends $(a_{ij}) \in R$ to $(\varphi_{ij}(a_{ij}))$.*

Proof. (1) follows from 3.4. (2) is immediate from the definitions. (3) comes from 3.5 and the fact that the diagrams in (2) commute when $i=j=k$. (4) is obvious.

(3.7) Definition. Let $\mathscr{L} = (n, L_{ij}, t_{ikj}, \Omega, R^{\alpha}, f_{ij}^{\alpha}, \chi_i^{\alpha})$ and

$$\mathscr{M} = (n', M_{ij}, s_{ikj}, \Lambda, U^{\lambda}, g_{ij}^{\lambda}, \psi_i^{\lambda})$$

be $\mathscr{R}$-sets. We say that $\mathscr{L}$ and $\mathscr{M}$ are *equivalent* if $n=n'$ and there exist a permutation π of $\{1,\ldots,n\}$ and maps $\varphi_{ij}\colon L_{ij}\to M_{\pi(i)\pi(j)}$ such that

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ L_{ik}\otimes L_{kj} \arrow[r, "t_{ikj}"] \arrow[d, "\varphi_{ik}\otimes\varphi_{kj}"'] & L_{ij} \arrow[d, "\varphi_{ij}"'] \\ M_{\pi(i)\pi(k)}\otimes M_{\pi(k)\pi(j)} \arrow[r, "s_{ikj}"] \arrow[d] & M_{\pi(i)\pi(j)} \arrow[d] \\ 0 & 0 \end{tikzcd}$$

are exact, commutative diagrams of abelian groups.

This clearly defines an equivalence relation on the class of all $\mathscr{R}$-sets.

(3.8) LEMMA. *Suppose that L and M are the rings defined by, respectively, the $\mathscr{R}$-sets $\mathscr{L}$ and $\mathscr{M}$ in* 3.7. *Then $\mathscr{L}$ equivalent to $\mathscr{M}$ implies L isomorphic to M. Conversely, if L and M are isomorphic, the L_{ii} and M_{jj} are completely primary rings*([4]) *and if the Krull-Schmidt Theorem holds in* (*say*) *M, then $\mathscr{L}$ and $\mathscr{M}$ are equivalent.*

Proof. Suppose $\mathscr{L}$ is equivalent to $\mathscr{M}$. The map which sends

$$(m_{ij})^{1\leqq i,j\leqq n}\in M\to(m_{\pi(i)\pi(j)})^{1\leqq i,j\leqq n}$$

is clearly a ring-automorphism of M. So the map $\varphi\colon L\to M$ defined by $\varphi(l_{ij})=(\varphi_{ij}(l_{ij}))$, the φ_{ij} being the same maps as in 3.7, is at least a group-isomorphism. That φ is actually a ring-isomorphism follows from the diagrams in 3.7.

Suppose next that there exists a ring-isomorphism $\theta\colon L\to M$ and that the rest of the hypotheses in the converse of the lemma are fulfilled. We have decompositions $L=Le_1\oplus\cdots\oplus Le_n$ and $M=Mf_1\oplus\cdots\oplus Mf_{n'}$ where $\{e_i\}$ and $\{f_j\}$ are sets of orthogonal idempotents defined as in 3.6. Since the L_{ii} and M_{jj} are completely primary, the e_i and f_j are primitive. But $M=M\theta(e_1)\oplus\cdots\oplus M\theta(e_n)$. Since M is Krull-Schmidt, $n=n'$ and there exist a permutation π on $\{1,\ldots,n\}$ and an inner automorphism τ of M such that $\tau(\theta(e_i))=f_{\pi(i)}$ for $1\leqq i\leqq n$ (see, for example [6, Proposition 3, p. 77]). Therefore, the diagrams

$$\begin{tikzcd} e_iLe_k\otimes e_kLe_j \arrow[r, "\text{nat}"] \arrow[d, "\theta\otimes\theta"'] & e_iLe_j \arrow[d, "\theta"'] \\ \theta(e_i)M\theta(e_k)\otimes\theta(e_k)M\theta(e_j) \arrow[r, "\text{nat}"] \arrow[d, "\tau\otimes\tau"'] & \theta(e_i)M\theta(e_j) \arrow[d, "\tau"'] \\ f_{\pi(i)}Mf_{\pi(k)}\otimes f_{\pi(k)}Mf_{\pi(j)} \arrow[r, "\text{nat}"] & f_{\pi(i)}Mf_{\pi(j)} \end{tikzcd}$$

([4]) By a completely primary ring we mean a (not necessarily local) ring in which the identity is a primitive idempotent.

are commutative; the vertical maps being isomorphisms. So $\mathscr{L}$ equivalent to $\mathscr{M}$ follows from 3.6.

(3.9) PROPOSITION. *Suppose that R is the ring defined by the $\mathscr{R}$-set* 3.1 *and that the following conditions hold.*

(a) *Every L_{ii} is a completely primary ring.*

(b) *For each pair i, j with $i \neq j$, either t_{iji} or t_{jij} is nonepic.*

Then R is a reduced ring. If, in addition, each L_{ii} is local, then R is semiperfect.

Proof. Using the same notation as in 3.6, we write $R = \bigoplus \sum Re_i$ where the e_i are orthogonal idempotents. (a) implies the e_i are primitive. If R is not reduced, then there exist $i \neq j$ such that $Re_i \simeq Re_j$. According to Jacobson [5, Proposition 4, p. 51], $e_iRe_jRe_i = e_iRe_i$ and $e_jRe_iRe_j = e_jRe_j$. This contradicts (b).

The last statement of the proposition comes from 3.6 and [4, Lemma 2.7].

(3.10) PROPOSITION. *If L_{ii} is a local ring, then the following statements are equivalent.*

(1) *t_{iji} is nonepic.*

(2) *$f_{ij}^{\alpha}(L_{ij})f_{ji}^{\alpha}(L_{ji}) \subset f_{ii}^{\alpha}(L_{ii})$ for some nonzero f_{ii}^{α}.*

(3) *$f_{ij}^{\alpha}(L_{ij})f_{ji}^{\alpha}(L_{ji}) \subset f_{ii}^{\alpha}(L_{ii})$ for every nonzero f_{ii}^{α}.*

Proof. Consider the commutative diagrams

$$\begin{array}{ccc} L_{ij} \otimes L_{ji} & \xrightarrow{t_{iji}} & L_{ii} \\ {\scriptstyle f_{ij}^{\alpha} \otimes f_{ji}^{\alpha}} \big\downarrow & & \big\downarrow {\scriptstyle f_{ii}^{\alpha}} \\ [R^{\alpha}]_{ij} \otimes [R^{\alpha}]_{ji} & \xrightarrow{\text{nat}} & [R^{\alpha}]_{ii} \end{array}$$

(2) $\Rightarrow$ (1) is immediate. To show (1) $\Rightarrow$ (3), assume t_{iji} nonepic. Then $H = \operatorname{im} t_{iji}$ is a proper ideal in L_{ii}. Since L_{ii} is local, $H \subseteq J(L_{ii})$. So $f_{ii}^{\alpha}(H) \subseteq f_{ii}^{\alpha}(J(L_{ii})) \subseteq J(f_{ii}^{\alpha}(L_{ii}))$. Therefore $f_{ii}^{\alpha}(H) \subset f_{ii}^{\alpha}(L_{ii})$ whenever $f_{ii}^{\alpha} \neq 0$. But $f_{ii}^{\alpha}(H) = f_{ij}^{\alpha}(L_{ij})f_{ji}^{\alpha}(L_{ji})$. Since (3) $\Rightarrow$ (2) is trivial, the proof is completed.

(3.11) LEMMA. *Let R be the ring defined by the $\mathscr{R}$-set* 3.1. *Then the following are true.*

(1) *R has nilpotent radical if and only if every L_{ii} has nilpotent radical.*

(2) *R has left T-nilpotent radical if and only if every L_{ii} has left T-nilpotent radical.*

(3) *R is left artinian if and only if every L_{ii} is a semiprimary local ring with the property that $J_i^{\sigma}L_{ij}/J_i^{\sigma+1}L_{ij}$ is a finite dimensional left vector space over L_{ii}/J_i for every j and σ where J_i is the radical of L_{ii}.*

Proof. By 3.6, the identity of R is a sum of orthogonal idempotents. So (1) and (2) are immediate by 2.2. (3) follows from (1) and 2.6.

(3.12) Set $N_{\alpha} = \{i \mid f_{ii}^{\alpha} \neq 0 \text{ and } 1 \leq i \leq n\}$ and $n_{\alpha} = |N_{\alpha}|$. Note that n_{α} may be zero for a given α in Ω. But $n_{\alpha} = 0$ for all $\alpha \in \Omega$ is impossible. This comes from 3.1(5).

Note further that as a consequence of 3.1(4) we have $f_{ij}^{\alpha}\neq 0$ implies $f_{ii}^{\alpha}\neq 0$ and $f_{jj}^{\alpha}\neq 0$.

For $i, k, j \in N_\alpha$, let ν_{ikj}^{α} be the natural map $f_{ik}^{\alpha}(L_{ik}) \otimes f_{kj}^{\alpha}(L_{kj}) \to f_{ij}^{\alpha}(L_{ij})$ induced by matrix multiplication and let ι^{α} be the injection $f_{ij}^{\alpha}(L_{ij}) \to [R^{\alpha}]_{ij}$. Finally let τ_α be the bijection $\{1,\ldots,n_\alpha\}\to N_\alpha$ which preserves the natural $\leqq$ order and denote $\tau_\alpha(i)$ by i'. The next proposition is clear.

(3.13) PROPOSITION. *Except in the trivial case where $n_\alpha=0$, $(n_\alpha, f_{i'j'}^{\alpha}(L_{i'j'}), \nu_{i'k'j'}^{\alpha}, \{\alpha\}, R^{\alpha}\iota_{i'j'}^{\alpha}, \chi_{i'}^{\alpha})$ is an $\mathscr{R}$-set.*

(3.14) LEMMA. *The ring R defined by the $\mathscr{R}$-set in* 3.1 *is a subdirect sum of the rings R_α defined by the $\mathscr{R}$-sets in* 3.13.

Proof. Let U_α be the ring of all $n\times n$ matrices of the form (u_{ij}^{α}), $u_{ij}^{\alpha}\in f_{ij}^{\alpha}(L_{ij})$. Define $f^{\alpha}\colon R\to U_\alpha$ by $f^{\alpha}((l_{ij}))=(f_{ij}^{\alpha}(l_{ij}))$. f^{α} is clearly a group-epimorphism. The diagrams in 3.1(3) imply that the f^{α}'s are ring-epimorphisms. Furthermore, if $(l_{ij})\in \ker f^{\alpha}$ for every α, then $l_{ij}\in\bigcap_\alpha \ker f_{ij}^{\alpha}$ for each pair i,j with $1\leqq i, j\leqq n$. So, by 3.1(5), $\bigcap_\alpha \ker f^{\alpha}=0$. To finish, it is enough then to show that U_α is ring-isomorphic to R_α. This is easy: Map $(u_{ij}^{\alpha})\in U_\alpha$ to the $n_\alpha\times n_\alpha$ matrix in R_α whose (i,j) entry for $(i',j')\in N_\alpha\times N_\alpha$ is $u_{i'j'}^{\alpha}$ where $i'=\tau_\alpha(i)$, the τ_α's being as defined in 3.12. This works.

REMARKS. (i) If conditions (a) and (b) of 3.9 hold in R, then R is a reduced ring. However, it is ostensibly quite possible that, in general, some (or even all) of the nonzero R_α's fail to be reduced.

(ii) An $\mathscr{R}$-set $(n, L_{ij}, t_{ikj}, \{1\}, R^1, f_{ij}, \chi_i)$ is loosely speaking, equivalent to the existence of $L_{ii}-L_{jj}$ bimodules L_{ij} such that the diagrams

$$\begin{array}{ccccccc}
L_{ii} & \times & L_{ij} & \times & L_{jj} & \xrightarrow{\text{nat}} & L_{ij}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
[R^1]_{ii} & \times & [R^1]_{ij} & \times & [R^1]_{jj} & \xrightarrow{\text{nat}} & [R^1]_{ij}
\end{array}$$

are commutative where the nonzero vertical maps are monic and

$$\operatorname{im}(L_{ik}\to[R^1]_{ik})\operatorname{im}(L_{kj}\to[R^1]_{kj})\subseteq \operatorname{im}(L_{ij}\to[R^1]_{ij}).$$

The following alternate way of looking at rings defined by $\mathscr{R}$-sets was pointed out to the author by E. C. Dade.

(3.15) THEOREM. *R is the ring defined by an $\mathscr{R}$-set if and only if there exist a family $\{R^{\alpha}\}_{\alpha\in\Omega}$ of rings, a family $\{M^{\alpha}\}_{\alpha\in\Omega}$ of $R^{\alpha}-R$ bimodules and a decomposition $1=e_1+\cdots+e_n$ of the identity of R into a sum of orthogonal idempotents, such that*

(a) *each $M^{\alpha}e_i$, $i=1,\ldots,n$; $\alpha\in\Omega$, is a free R^{α}-module;*

(b) *if $r\in R$ and $M^{\alpha}r=0$ for all $\alpha\in\Omega$, then $r=0$.*

Proof. In the notation of Lemma 3.14, consider the ring-epimorphism $g^{\alpha}\colon R\to R_\alpha$ defined in the proof of the lemma. g^{α} is a representation of R as $(\chi_{1'}+\cdots+\chi_{n'_\alpha})$

$\times(\chi_1'+\cdots+\chi_{n'_\alpha})$ matrices over R^α. Let M^α be the corresponding representation module. Inspection of the representations g^α verifies condition (a). Since

$$\bigcap_{\alpha\in\Omega} \ker g^\alpha = 0,$$

(b) also holds.

The straightforward verification of the converse of the theorem is left to the reader.

We remark that if one takes the R^α to be local rings, then (a) can be replaced by the cleaner looking condition:

(a′) M^α is a projective R^α-module, for all $\alpha\in\Omega$.

The reason for this is Kaplansky's well-known result that projective modules over local rings are free.

4. **Special $\mathscr{R}$-sets.** There are some rather obvious and interesting ways to extend our definition of $\mathscr{R}$-set in 3.1. We have opted to resist doing this. First of all, we do not really know what the rings defined by $\mathscr{R}$-sets 3.1 are even in the reduced, semiperfect case. Secondly, the present definition is convenient for the main task of this paper—to characterize semiperfect rings with projective essential left socle. It will be clear to the reader that, at least in some cases, we could handle rings more general than semiperfect ones. (In particular, rings which are artinian modulo their radical.) We resist this temptation for the sake of simplicity and, hopefully, clarity.

(4.1) Definition. Let ρ be a reflexive relation on a set $\mathscr{S}$. We say that $v\in\mathscr{S}$ is *ρ-maximal* if $u\in\mathscr{S}$ and $u\,\rho\,v$ always implies $u=v$. If $\{v_\alpha\}_{\alpha\in\Omega}$ is the family of ρ-maximal elements of $\mathscr{S}$ and $V_\alpha=\{s\in\mathscr{S}\mid v_\alpha\,\rho\,s\}$, then the restriction ρ_α of ρ to $V_\alpha\times V_\alpha$ is a reflexive relation on V_α with unique ρ_α-maximal element v_α. We call ρ a *special relation* if

(a) ρ has maximal elements and

(b) $\rho=\bigcup_{\alpha\in\Omega}\rho_\alpha$.

(4.2) Definition. Suppose ρ is a special relation on $\{1,\ldots,n\}$ with ρ-maximal elements $v_1,\ldots,v_\kappa$ and let ρ_α be the restriction of ρ to $V_\alpha\times V_\alpha$ where

$$V_\alpha = \{i \mid v_\alpha\rho i \text{ and } 1\leqq i\leqq n\}.$$

We say an $\mathscr{R}$-set $(n, L_{ij}, t_{ikj}, \Omega, R^\alpha, f_{ij}^\alpha, \chi_i^\alpha)$ is a *special $\mathscr{R}$-set* if the following conditions hold.

(1) $L_{ij}\neq 0$ if and only if $i\,\rho\,j$ and every L_{ii} is a local ring.

(2) For every pair $i\neq j$, either t_{iji} or t_{jij} is nonepic.

(3) $\Omega=\{v_1,\ldots,v_\kappa\}$, $R^\alpha=L_{v_\alpha v_\alpha}$ and the $L_{v_\alpha v_\alpha}$'s are division rings.

(4) $f_{ij}^\alpha\neq 0$ implies $i\,\rho_\alpha\,j$ and every nonzero $f_{v_\alpha i}^\alpha$ is epic.

(5) $\chi_{v_\alpha}^\alpha=1$ for $1\leqq\alpha\leqq\kappa$.

In the sequel, we denote $L_{v_\alpha v_\alpha}$ by L^α.

(4.3) Proposition. *If $v_\alpha\,\rho\,i$, then $f_{v_\alpha i}^\beta$ is monic if $\alpha=\beta$ and zero otherwise.*

Proof. Assume $f^{\beta}_{v_\alpha i} \neq 0$. By (4), $v_\alpha \, \rho_\beta \, i$. Since ρ is special, this implies in particular that $v_\beta \, \rho \, v_\alpha$. But the v_γ's are ρ-maximal. So $v_\beta = v_\alpha$—i.e. $\alpha = \beta$. Next, by 3.1(5), $0 = \bigcap_\gamma \ker f^{\gamma}_{v_\alpha i} = \ker f^{\alpha}_{v_\alpha i}$. Since $L_{v_\alpha i} \neq 0$ (because $v_\alpha \, \rho \, i$), $f^{\alpha}_{v_\alpha i}$ is monic as required.

(4.4) THEOREM. *Let R be the ring defined by the special $\mathscr{R}$-set in* 4.2. *Then R is a reduced semiperfect ring with projective, essential left socle.*

Proof. That R is reduced semiperfect follows from 4.2(1), 4.2(2) and 3.9. By Proposition 3.6, we may write $R = \bigoplus \sum_{i=1}^{n} Re_i$, the e_i being orthogonal local idempotents. According to the same proposition, given i, k, j, we have for each α a commutative diagram

$$\begin{array}{ccc} e_iRe_k \otimes e_kRe_j & \xrightarrow{\text{nat}} & e_iRe_j \\ \Big\downarrow {\scriptstyle f^{\alpha}_{ik} \otimes f^{\alpha}_{kj}} & & \Big\downarrow {\scriptstyle f^{\alpha}_{ij}} \\ [L^\alpha]_{ik} \otimes [L^\alpha]_{kj} & \xrightarrow{\text{nat}} & [L^\alpha]_{ij} \end{array}$$

of abelian groups.

Let S be the left socle of R and set $T = \sum_{\alpha=1}^{\kappa} e_{v_\alpha} R$. By 4.2(1), we have

$$Re_{v_\alpha}R = \sum_i e_iRe_{v_\alpha}R = \sum_{i;\, i\rho v_\alpha} e_iRe_{v_\alpha}R = e_{v_\alpha}Re_{v_\alpha}R = e_{v_\alpha}R$$

by the ρ-maximality of v_α. So every $e_{v_\alpha}R$ is an ideal. In particular, T is an ideal.

According to 2.1, the radical J of R is given by $J = \sum_{i \neq j} e_iRe_j + \sum_k e_kJe_k$ (since $J(e_kRe_k) = e_kJe_k$). Notice that each $e_{v_\alpha}Re_{v_\alpha}$ is a division ring. This follows from 4.2, 4.3, and the assumption that the v_α's are ρ-maximal. Hence $e_{v_\alpha}Je_{v_\alpha} = 0$ for all α. But then $Je_{v_\alpha}R = \sum_{i \neq v_\alpha} e_iRe_{v_\alpha}R = 0$ by 4.2(1). So $JT = 0$. Therefore, since a semiperfect ring is artinian modulo its radical [1, Theorem 2.1], $T \subseteq S$.

Next suppose $Tx = 0$, $x \in R$, and write $x = \sum_{i,j} x_{ij}$ where $x_{ij} \in e_iRe_j$. Fix p, q with $x_{pq} \neq 0$. Since the right annihilator of T is an ideal, we have $e_{v_\alpha}Rx_{pq} = 0$ for all α. But for each α with $f^{\alpha}_{pq} \neq 0$, $f^{\alpha}_{v_\alpha p} \neq 0$ and

$$\begin{array}{ccccc} 0 & & & & 0 \\ \Big\downarrow & & & & \Big\downarrow \\ e_{v_\alpha}Re_p & \times & e_pRe_q & \longrightarrow & e_{v_\alpha}Re_q \\ \Big\downarrow {\scriptstyle f^{\alpha}_{v_\alpha p}} & & \Big\downarrow {\scriptstyle f^{\alpha}_{pq}} & & \Big\downarrow {\scriptstyle f^{\alpha}_{v_\alpha q}} \\ [L^\alpha]_{v_\alpha p} & \times & [L^\alpha]_{pq} & \longrightarrow & [L^\alpha]_{v_\alpha q} \\ \Big\downarrow & & & & \Big\downarrow \\ 0 & & & & 0 \end{array}$$

is an exact, commutative diagram. Recalling that $\chi^{\alpha}_{v_\alpha} = 1$, it is easy to see that the natural action of $[L^\alpha]_{pq}$ on $[L^\alpha]_{v_\alpha p}$ (on the right) is faithful. It follows that $f^{\alpha}_{pq}(x_{pq}) = 0$

for every α. By 3.1(5) (and 3.6(4)), $x_{pq} \in \bigcap_\alpha \ker f^\alpha_{pq} = 0$. Hence $x=0$, that is, T has zero right annihilator. According to Gordon [4, Theorem 3.1], $T=S$ and S is an essential, projective submodule of $_RR$ as was to be shown.

We remark that since the e_{v_α}'s are local idempotents contained in S, every Re_{v_α} is simple. Thus it is clear that the $e_{v_\alpha}R$'s are just the homogeneous components of S.

(4.5) COROLLARY. *Let R be the ring defined by the special $\mathscr{R}$-set* 4.2 *and let* $1=\sum_{i=1}^n e_i$ *be the canonical decomposition* 3.6 *of the identity of R as a sum of orthogonal, local idempotents e_i. Then, in the notation of* 4.2, *the left socle of R is the sum of its κ homogeneous components $S_\alpha = e_{v_\alpha}R$. Furthermore, the following hold.*

(1) *The dimension*(5) *of the completely reducible left R-module $S_\alpha \cap Re_i$ is the dimension χ^α_i of $e_{v_\alpha}Re_i$ as a left vector space over the division ring $e_{v_\alpha}Re_{v_\alpha} \simeq L^\alpha$.*

(2) *The dimension of S_α is given by*

$$\sum_{i\in V_\alpha} \chi^\alpha_i = \sum_{1\leq i\leq n;\, v_\alpha \rho i} \chi^\alpha_i.$$

Proof. Since R is reduced and Re_{v_α} is simple, the map $\eta_\alpha: R \to e_{v_\alpha}Re_{v_\alpha}$ defined by $\eta_\alpha(x) = e_{v_\alpha}xe_{v_\alpha}$ is a ring-epimorphism. So the diagrams

$$\begin{tikzcd}
R \arrow[d, "\eta_\alpha"'] & \times & e_{v_\alpha}Re_i \arrow[r] \arrow[d, equal] & e_{v_\alpha}Re_i \arrow[d, equal] \\
e_{v_\alpha}Re_{v_\alpha} \arrow[d] & \times & e_{v_\alpha}Re_i \arrow[r] \arrow[d] & e_{v_\alpha}Re_i \arrow[d] \\
L^\alpha & \times & {[L^\alpha]_{v_\alpha i}} \arrow[r] & {[L^\alpha]_{v_\alpha i}}
\end{tikzcd}$$

are commutative. Moreover, the vertical maps in the bottom row are isomorphisms. Thus (1) follows from the fact that $S_\alpha \cap I = S_\alpha I$ for any left ideal I of R [4, Lemma 1.1]. Then, since $[L^\alpha]_{v_\alpha i}$ is just the $L^\alpha - L^\alpha$ bimodule of row-finite $1\times\chi^\alpha_i$ matrices over L^α, the rest of (1) is obvious. (2) is clear since $S_\alpha = \bigoplus \sum_{i\in V_\alpha} e_{v_\alpha}Re_i$ and $e_{v_\alpha}Re_i \neq 0$ when $v_\alpha \,\rho\, i$.

(4.6) REMARK. From 4.4 and 4.5 we see that the special $\mathscr{R}$-set $(n, L_{ij}, t_{ikj}, \Omega, R^\alpha, f^\alpha_{ij}, \chi^\alpha_i)$ in 4.2 may be denoted in slightly simpler notation by $(n, L_{ij}, t_{ikj}, \rho, f^\alpha_{ij})$.

5. Some characterization theorems.

(5.1) THEOREM. *Let R be a semiperfect ring with projective, essential left socle. Then there exist a special $\mathscr{R}$-set*(6) $(n, L_{ij}, t_{ikj}, \rho, f^\alpha_{ij})$ *and positive integers $m_1, \dots, m_n$ such that R is isomorphic to the ring of $n\times n$ blocked matrices in which the i,jth block of a typical matrix is an $m_i \times m_j$ matrix with arbitrary entries in L_{ij}. Furthermore, the following statements hold.*

(5) By the dimension of a completely reducible module $_RM$ we mean the cardinal number of simple summands in a direct sum decomposition of $_RM$ into simples.

(6) For the definition, see 3.1, 4.1, 4.2 and 4.6.

(1) *$m_1, \ldots, m_n$ are uniquely determined by R up to order and $(n, L_{ij}, t_{ikj}, \rho, f_{ij}^{\alpha})$ is determined by R up to equivalent*(7) *$\mathcal{R}$-sets.*

(2) *R is indecomposable if and only if the undirected graph defined by ρ is connected.*

(3) *If Ω is the set of ρ-maximal elements, then R is indecomposable if and only if given indices i and j with $1 \leqq i, j \leqq n$ there exist a sequence $\beta_0 = i, \beta_1, \ldots, \beta_p = j$ and a sequence $\alpha_1, \ldots, \alpha_p$ with $\alpha_k \in \Omega$ such that $\alpha_k \,\rho\, \beta_{k-1}$ and $\alpha_k \,\rho\, \beta_k$ for $1 \leqq k \leqq p$.*

(4) *R is left* (*right*) *perfect if and only if every L_{ii} is a left* (*right*) *perfect local ring.*

(5) *The left socle of R is finitely generated if and only if $L_{\alpha i}$ is a finite dimensional left vector space over the division ring $L_{\alpha\alpha}$ for every ρ-maximal element α and every i. If this is the case and every L_{ii} is a left* (*or right*) *perfect local ring, then R is semiprimary.*

(6) *R is semiprimary if and only if the L_{ii}'s are semiprimary local rings.*

(7) *R is left* (*resp. right*) *artinian if and only if every L_{ij} is a left artinian L_{ii}-module* (*resp. right artinian L_{jj}-module*).

(8) (7) *holds with artinian replaced by noetherian.*

Proof. Evidently, the ring defined by $(n, L_{ij}, t_{ikj}, \rho, f_{ij}^{\alpha})$ is formally the reduced ring of the blocked matrix ring in the statement of the theorem (see 1.1 and 3.9). Thus §1 and the assumption that R is semiperfect allow us to assume without loss of generality that R is reduced.

Write $R = \bigoplus \sum_{i=1}^{n} Re_i$ where the e_i are orthogonal local idempotents. Let ρ be the relation on $\{1, \ldots, n\}$ defined by $i \,\rho\, j$ if $e_i R e_j \neq 0$ and let $v_1, \ldots, v_\kappa$ be the full set of distinct ρ-maximal elements. Given v_α, the assumption that the left socle S is essential implies that Re_{v_α} contains a simple. Since S is projective and semiperfect rings are Krull-Schmidt, there exists an index i such that $e_i R e_{v_\alpha} \neq 0$—i.e., $i \,\rho\, v_\alpha$. So $i = v_\alpha$ and Re_{v_α} is simple. In particular, $e_{v_\alpha} R e_{v_\alpha}$ is a division ring.

Conversely, suppose Re_i is simple and let $j \,\rho\, i$. This implies the existence of an epimorphism $Re_j \to Re_i$. Since this epimorphism trivially splits, we have $Re_j \simeq Re_i$ by the indecomposability of Re_j. But R is reduced. This forces $j = i$ implying that i is ρ-maximal. Thus $Re_{v_1}, \ldots, Re_{v_\kappa}$ constitute a full set of isomorphism types of minimal left ideals of R.

We remark that the above argument shows in particular that v_α's exist.

Next, since R is reduced, it follows that a given e_{v_α} acts like a left identity on any simple isomorphic to Re_{v_α}. So, by projectivity of S, each $e_{v_\alpha} R$ is a homogeneous component of S (and every homogeneous component has this form). This enables us to show that ρ is a special relation on $\{1, \ldots, n\}$: Assume $i \,\rho\, j$. Then the preceding argument implies via Theorem 3.1 in [4] that $e_{v_\alpha} R e_i R e_j \neq 0$ for some v_α. Since $e_{v_\alpha} R e_i R e_j \subseteq e_{v_\alpha} R e_j$, we have $v_\alpha \,\rho\, i$ and $v_\alpha \,\rho\, j$.

Let x_i^α be the dimension of $e_{v_\alpha} R e_i$ as a left vector space over $L^\alpha = e_{v_\alpha} R e_{v_\alpha}$. Notice that $x_{v_\alpha}^\alpha$ is trivially unity. Also, the natural action of $e_i R e_j$ on $e_{v_\alpha} R e_i$ (i.e., on the

(7) See 3.7 for the definition.

right) induces a map $f_{ij}^{\alpha}: e_iRe_j \to [L^{\alpha}]_{ij}$; $[L^{\alpha}]_{ij}$ being the $L^{\alpha}-L^{\alpha}$ bimodule of row-finite $x_i^{\alpha} \times x_j^{\alpha}$ matrices over L^{α}. Evidently, every nonzero $f_{v_{\alpha}i}^{\alpha}$ is epic and the nonzero f_{ii}^{α}'s map the identity to the identity. The first half of the condition in 4.2(4) is also easily verified. Setting $L_{ij}=e_iRe_j$, a standard matrix calculation shows that the diagrams

$$\begin{tikzcd} L_{ik} \otimes L_{kj} \arrow[r, "t_{ikj}"] \arrow[d, "f_{ik}^{\alpha} \otimes f_{kj}^{\alpha}"'] & L_{ij} \arrow[d, "f_{ij}^{\alpha}"'] \\ {[L^{\alpha}]_{ik} \otimes [L^{\alpha}]_{kj}} \arrow[r, "\text{nat}"] & {[L^{\alpha}]_{ij}} \end{tikzcd}$$

commute for t_{ikj} the natural maps induced by multiplication in R. Since R is reduced semiperfect, t_{pqp} is nonepic for $p \neq q$. Finally, let $x \in \bigcap_{\alpha} \ker f_{ij}^{\alpha}$. The definition of the f_{ij}^{α}'s implies, as we have seen above, that x right annihilates the left socle. So $x=0$. This completes the proof of the representation part of the theorem.

Statement (1) comes from 1.1 and 3.8.

(2) follows from 2.1 in [4]. (In fact, (2) holds in any (say) semiperfect ring, ρ being defined as in the proof of the theorem.)

(4) is immediate by 3.11(2) and [1, Theorem 2.1].

The first statement in (5) is 4.5(2). The second statement follows from the first by (4) and a theorem of Gordon [4, Theorem 3.4].

(6) is implied by 3.11(1) together with [1, Theorem 2.1(b)].

(7) follows, for example, from (6) plus 3.11(3).

(8) can be verified (as can (7)) in a straightforward manner. We leave this to the reader.

It remains to show (3): Suppose $i \rho j$ and $i \rho k$. Then, since ρ is special, there are ρ-maximal elements v_{α} and v_{β} such that $v_{\alpha} \rho j$, $v_{\alpha} \rho i$, $v_{\beta} \rho i$ and $v_{\beta} \rho k$. So (3) follows from [4, Theorem 2.1].

(5.2) THEOREM. *Let R be a semiperfect ring with projective, essential left socle and a unique isomorphism class of minimal left ideals. Then there exist positive integers $m_1, \ldots, m_n$, nonzero cardinal numbers $\chi_1, \ldots, \chi_n$, a division ring D and n^2 additive groups L_{ij} satisfying*

(1) *L_{ij} is a subgroup of $[D]_{ij}$(8), L_{ii} being a local subring of $[D]_{ii}$ (such that the identity of L_{ii} is the identity matrix);*

(2) *$L_{ik}L_{kj} \subseteq L_{ij}$ and $L_{ij}L_{ji} \subset L_{ii}$ for $i \neq j$;*

(3) *$\chi_n = 1$, $L_{ni}=[D]_{ni}$ for $1 \leqq i \leqq n$ and $L_{in}=0$ for $1 \leqq i < n$;*

such that R is isomorphic to the ring of $n \times n$ blocked matrices, a typical block being an $m_i \times m_j$ matrix with entries in L_{ij}.

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(8) $[D]_{ij}$ denotes the $D-D$ bimodule of $\chi_i \times \chi_j$ row-finite matrices over D.

Proof. The extra hypothesis implies via 5.1 the existence of an "extra special" $\mathscr{R}$-set $(n, L_{ij}, t_{ikj}, \rho, f_{ij})$ defining R where ρ has a unique maximal element and every nonzero f_{ij} is monic.

REMARK. Any semiperfect ring R with essential, projective left socle is a canonical finite subdirect sum of rings of the type characterized in 5.2 (see 3.12, 3.13 and 3.14). Thus many of the properties of R may be easily deduced from properties of the subdirect summands (or from the corresponding "extra special" $\mathscr{R}$-sets).

As a special case of 5.2, we obtain an interesting generalization of a theorem of Zaks [9, Theorem 1.4, p. 67]:

(5.3) THEOREM. *Let R be an indecomposable semiperfect ring with projective, essential left socle and the additional property that every indecomposable direct summand of ${}_RR$ has a unique simple submodule. Then there exist positive integers $m_1, \dots, m_n$ and a division ring D possessing n^2 additive subgroups L_{ij} satisfying*

(1) *$L_{ik}L_{kj} \subseteq L_{ij}$ and the L_{ii}'s are local subrings of D;*

(2) *$L_{ij}L_{ji} \subset L_{ii}$ for $i \neq j$, $L_{ni} = D$ for $1 \leqq i \leqq n$ and $L_{in} = 0$ for $1 \leqq i < n$;*

(3) *$L_{ij} \neq 0$ for $i < j$ implies $L_{ji} \neq 0$;*

such that R is isomorphic to the ring of $n \times n$ blocked matrices in which a typical block is an $m_i \times m_j$ matrix with arbitrary entries in L_{ij}.

Proof. Theorem 5.1(3) implies immediately that R has a unique isomorphism class of minimal left ideals. So, with the exception of the normalization condition (3), everything follows from 5.2 and 4.5(1).

To show (3), let ρ be the reflexive relation on $\{1, \dots, n\}$ defined by $i \rho j$ if $L_{ij} \neq 0$. Since (1) holds, ρ is transitive. So (3) is a consequence of Lemma 2.7 and Lemma 3.8.

COROLLARY. *A semiperfect ring R with projective, essential left socle in which indecomposable direct summands of ${}_RR$ have unique simple submodules is a unique ring-direct sum of rings of the type characterized in Theorem* 5.3.

Proof. This is an instance of Theorem 2.8 in [4].

(5.4) REMARKS. Let ρ be the reflexive relation on $\{1, \dots, n\}$ defined in 5.2 by $i \rho j$ if $L_{ij} \neq 0$. If ρ happens to be a partial ordering (transitive and antisymmetric), then the rings in 5.2 become blocked, triangular matrix rings. This follows from 2.7 (or Szpilrajn's Theorem). Note that a sufficient condition for ρ to be a partial ordering in 5.3 is for each L_{ii} to be a division ring. This is the case when, for example, R is left perfect.

Even for rings of the type characterized in 5.3, ρ may fail to be antisymmetric: Let P be the ring of p-adic integers, J its radical and Q its quotient field. Then the reflexive relation defined by the ring of matrices of the form

$$\begin{bmatrix} P & J & 0 \\ J & P & 0 \\ Q & Q & Q \end{bmatrix}$$

is not antisymmetric.

Unfortunately, also, one cannot transport the normalization condition (3) of 5.3 to 5.2: Consider the nontransitive, antisymmetric, reflexive relation

$$\rho = \{(1, 1), (1, 3), (2, 1), (2, 2), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4)\}.$$

ρ is not isomorphic to a relation ρ^* on $\{1, \ldots, 4\}$ satisfying (b) in 2.7. We leave the reader to find his own example of a ring of the type in 5.2 which defines the relation ρ.

We conclude with the following proposition, the proof of which is obvious.

(5.5) PROPOSITION. *Let R be a semiperfect ring with projective, essential left socle. Then R has projective, essential right socle if and only if there exists a special $\mathscr{R}$-set of the form $(n, L_{ij}, t_{ikj}, \breve{\rho}, g_{ij}^{\beta})$ where the reduced ring of R is the ring defined by the special $\mathscr{R}$-set $(n, L_{ij}, t_{ikj}, \rho, f_{\alpha}^{ij})$ and $\breve{\rho}$ is the reflexive relation defined on $\{1, \ldots, n\}$ by $i \breve{\rho} j$ if $j \rho i$.*

REFERENCES

1. H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. **95** (1960), 466–488. MR **28** #1212.

2. A. Goldie, *Torsion-free modules and rings*, J. Algebra **1** (1964), 268–287. MR **29** #2282.

3. R. Gordon, *Rings faithfully represented on their left socle*, J. Algebra **7** (1967), 303–342. MR **36** #1483.

4. ———, *Rings in which minimal left ideals are projective*, Pacific J. Math. **31** (1969), 679–692.

5. N. Jacobson, *Structure of rings*, 2nd rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R. I., 1964. MR **36** #5158.

6. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966. MR **34** #5857.

7. K. Morita, *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **6** (1958), 83–142. MR **20** #3183.

8. M. Osima, *Notes on basic rings*, Math. J. Okayama Univ. **2** (1953), 103–110. MR **14**, 941.

9. A. Zaks, *Semiprimary rings of generalized triangular type*, J. Algebra **9** (1968), 54–78. MR **36** #6455.

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