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THE STRUCTURE OF PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICES. I: SUBDIRECT DECOMPOSITION

BY
H. LAKSER[1]

Abstract. In this paper all subdirectly irreducible pseudocomplemented distributive lattices are found. This result is used to establish a Stone-like representation theorem conjectured by G. Grätzer and to find all equational subclasses of the class of pseudocomplemented distributive lattices.

1. **Introduction.** A pseudocomplemented distributive lattice is an algebra $\langle L;\ \wedge,\ \vee, {}^*, 0, 1\rangle$ where $\langle L;\ \wedge,\ \vee, 0, 1\rangle$ is a distributive lattice with 0, 1 and the unary operation * is pseudocomplementation; $x \wedge y = 0$ if and only if $y \leqq x^*$. That the class of pseudocomplemented distributive lattices is equational was first observed by P. Ribenboim [10]. The following four identities are useful in calculations:

(1) $x \leqq x^{**}$;

(2) $x^{***} = x^*$;

(3) $(x \vee y)^* = x^* \wedge y^*$;

(4) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$.

In this paper the unary operation * is one of the fundamental operations of pseudocomplemented distributive lattices; thus a *congruence of pseudocomplemented distributive lattices*, also called a **-congruence*, preserves the operation *, and similarly for a *homomorphism of pseudocomplemented distributive lattices*, or **-homomorphism*. We also note that *-homomorphisms preserve 0 and 1.

The main purpose of this paper is to characterize the congruences of pseudocomplemented distributive lattices (Theorem 1) and thereby to find all the subdirectly irreducible pseudocomplemented distributive lattices (Theorem 2). We present two applications of these results. First we prove that any pseudocomplemented distributive lattice can be *-embedded in the ideal lattice of an atomic Boolean algebra (Theorem 3), answering a question of G. Grätzer [3]. Secondly, we show how Theorem 2 can be used to find the lattice of equational subclasses of

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the class of all pseudocomplemented distributive lattices, obtaining a result of K. B. Lee [8], [8a].

2. **Congruences.** Let $\langle L; \wedge, \vee, *, 0, 1\rangle$, henceforth more briefly L, be a pseudocomplemented distributive lattice. An element $u \in L$ is said to be *dense* if $u^*=0$; the set $D(L)$ of the dense elements of L is a dual ideal of L. The set $S(L) = \{x^* \mid x \in L\}$ is the *skeleton of* L, and if $a \in S(L)$ we say that a is *skeletal*. Define the operation $\cup$ on $S(L)$ by $a \cup b = (a^* \wedge b^*)^*$. Then, by the theorem of Glivenko [1, V, Theorem 26], $\langle S(L); \wedge, \cup, *, 0, 1\rangle$ is a Boolean algebra. Let $\mathfrak{C}(S(L))$ denote the lattice of congruences of $S(L)$, and let $\mathfrak{C}(D(L))$ denote the lattice of congruences of $D(L)$. A pair $\langle \Theta_1, \Theta_2\rangle \in \mathfrak{C}(S(L)) \times \mathfrak{C}(D(L))$ is said to be a *congruence pair of* L if

(5) $a \in S(L), u \in D(L), u \geqq a, a \equiv 1\,(\Theta_1)$ imply $u \equiv 1\,(\Theta_2)$.

LEMMA 1. *The set $\mathfrak{C}_P(L)$ of congruence pairs of L is a sublattice of*

$$\mathfrak{C}(S(L)) \times \mathfrak{C}(D(L)).$$

Proof. We need only show that if $\langle \Theta_1, \Theta_2\rangle, \langle \Phi_1, \Phi_2\rangle \in \mathfrak{C}_P(L)$ then

$$\langle \Theta_1 \wedge \Phi_1, \Theta_2 \wedge \Phi_2\rangle \quad \text{and} \quad \langle \Theta_1 \vee \Phi_1, \Theta_2 \vee \Phi_2\rangle$$

satisfy (5). That $\langle \Theta_1 \wedge \Phi_1, \Theta_2 \wedge \Phi_2\rangle$ satisfies (5) is clear. To establish the same for $\langle \Theta_1 \vee \Phi_1, \Theta_2 \vee \Phi_2\rangle$ we observe first that a congruence on a Boolean algebra is determined by the dual ideal of all elements congruent to 1. Let $a \in S(L)$ and let $a \equiv 1\,(\Theta_1 \vee \Phi_1)$; then there are $b, c \in S(L)$ such that $b \equiv 1\,(\Theta_1)$, $c \equiv 1\,(\Phi_1)$, and $a = b \wedge c$. Let $u \in D(L), u \geqq a$. Since $D(L)$ is a dual ideal of L we conclude that $u \vee b, u \vee c \in D(L)$ and, applying (5) to $\langle \Theta_1, \Theta_2\rangle$ and $\langle \Phi_1, \Phi_2\rangle$, we see that $u \vee b \equiv 1\,(\Theta_2)$ and $u \vee c \equiv 1\,(\Phi_2)$. Thus $u = (u \vee b) \wedge (u \vee c) \equiv 1\,(\Theta_2 \vee \Phi_2)$, establishing (5) for the pair $\langle \Theta_1 \vee \Phi_1, \Theta_2 \vee \Phi_2\rangle$ and completing the proof of the lemma.

Let $\mathfrak{C}^*(L)$ denote the lattice of $*$-congruences of the pseudocomplemented distributive lattice L. If X is a set, R a relation on X, and $Y \subseteq X$, let R_Y denote the relation induced by R on Y.

THEOREM 1. *Let $\varphi: \mathfrak{C}^*(L) \to \mathfrak{C}(S(L)) \times \mathfrak{C}(D(L))$ be defined by $\Theta\varphi = \langle \Theta_{S(L)}, \Theta_{D(L)}\rangle$. Then φ determines a lattice isomorphism between $\mathfrak{C}^*(L)$ and $\mathfrak{C}_P(L)$; if $\langle \Theta_1, \Theta_2\rangle \in \mathfrak{C}_P(L)$ the corresponding $*$-congruence Θ on L is determined by $x \equiv y\,(\Theta)$ if and only if*

(i) $x^* \equiv y^*\,(\Theta_1)$ *and*

(ii) $x \vee u \equiv y \vee u\,(\Theta_2)$ *for all* $u \in D(L)$.

Proof. Let $\Theta \in \mathfrak{C}^*(L)$. Since $D(L)$ is a sublattice of L, $\Theta_{D(L)}$ is a congruence on $D(L)$. Since $a \cup b = (a^* \wedge b^*)^*$ in $S(L)$ it follows that Θ preserves all operations in $S(L)$ and so that $\Theta_{S(L)}$ is a congruence on $S(L)$. Clearly, $\langle \Theta_{S(L)}, \Theta_{D(L)}\rangle$ satisfies (5) and so $\mathfrak{C}^*(L)\varphi \subseteq \mathfrak{C}_P(L)$.

We first show that φ is one-to-one. Let $x \in L$. Then, by distributivity and (1), $x = x^{**} \wedge (x \vee x^*)$ where $x^{**} \in S(L)$ and, by (3), $x \vee x^* \in D(L)$. Thus $\Theta \in \mathfrak{C}^*(L)$ is determined by $\Theta_{S(L)}$ and $\Theta_{D(L)}$; that is, φ is one-to-one.

We now show that $\mathfrak{C}^*(L)\varphi = \mathfrak{C}_P(L)$. Let $\langle \Theta_1, \Theta_2 \rangle \in \mathfrak{C}_P(L)$ and define $x \equiv y(\Theta)$ if and only if

(i) $x^* \equiv y^*(\Theta_1)$, and

(ii) $x \vee u \equiv y \vee u(\Theta_2)$ for all $u \in D(L)$.

(Note that if $u \in D(L)$ then $x \vee u \in D(L)$ for all $x \in L$, since $D(L)$ is a dual ideal of L.)

A direct computation establishes that Θ is an equivalence relation on L. That Θ preserves $\vee$ is equally direct, using (3) to establish condition (i). To establish the preservation of $\wedge$ under Θ we note that (ii) follows by distributivity and that Θ_1 preserves * since * is the complementation in $S(L)$. Thus condition (i) is established by using (2) and (4):

$$(x \wedge z)^* = (x \wedge z)^{***} = (x^{**} \wedge z^{**})^*.$$

We now prove that * is preserved by Θ. Let $x, y \in L$ and let $x \equiv y(\Theta)$. As noted above, $x^* \equiv y^*(\Theta_1)$ implies that $x^{**} \equiv y^{**}(\Theta_1)$, establishing (i) for x^*, y^*. To establish (ii) for x^*, y^* we note that, since $x^* \equiv y^*(\Theta_1)$ and Θ_1 is a Boolean congruence, there is an $a \in S(L)$ such that $a \equiv 1(\Theta_1)$ and $x^* \wedge a = y^* \wedge a$. Let $u \in D(L)$; by (5) $a \vee u \equiv 1(\Theta_2)$ and so $x^* \vee u \equiv (x^* \vee u) \wedge (a \vee u) = (x^* \wedge a) \vee u = (y^* \wedge a) \vee u = (y^* \vee u) \wedge (a \vee u) \equiv y^* \vee u(\Theta_2)$. Consequently, if the pair x, y satisfies conditions (i) and (ii) then so does the pair x^*, y^*, proving that Θ preserves *.

To conclude the proof that $\mathfrak{C}^*(L)\varphi = \mathfrak{C}_P(L)$ we need only show that $\Theta_{S(L)} = \Theta_1$, $\Theta_{D(L)} = \Theta_2$. If $x, y \in D(L)$ then condition (i) always holds since $x^* = 0 = y^*$. Thus $x \equiv y(\Theta)$ if and only if $x \vee u \equiv y \vee u(\Theta_2)$ for all $u \in D(L)$ if and only if $x \equiv y(\Theta_2)$, the last equivalence following by letting $u = x \wedge y$. Consequently $\Theta_{D(L)} = \Theta_2$. Now let $x, y \in S(L)$. Then $x \equiv y(\Theta)$ implies $x^* \equiv y^*(\Theta_1)$ implies $x = x^{**} \equiv y^{**} = y(\Theta_1)$. Conversely, $x \equiv y(\Theta_1)$ implies $x^* \equiv y^*(\Theta_1)$ and $x \vee u \equiv y \vee u(\Theta_2)$ for all $u \in D(L)$, the latter implication as above in the proof that Θ preserves *. Thus $\Theta_{S(L)} = \Theta_1$.

Consequently φ is a one-to-one correspondence between $\mathfrak{C}^*(L)$ and $\mathfrak{C}_P(L)$, and to conclude that it is an isomorphism we need only observe that φ preserves the partial order. Thus the proof of the theorem is concluded.

For Stone algebras (pseudocomplemented distributive lattices satisfying the identity $x^* \vee x^{**} = 1$), the isomorphism between $\mathfrak{C}^*(L)$ and $\mathfrak{C}_P(L)$ is implicit in C. C. Chen and G. Grätzer [2] and was first used explicitly by the author in [7]. The idea of approaching problems concerning Stone algebras L by investigating $S(L)$ and $D(L)$ was initiated by C. C. Chen and G. Grätzer [2] in their "triple" characterization which was extended to pseudocomplemented distributive lattices in general by T. Katriňak [6]. Although we do not make use of any results of the aforementioned papers, we feel it proper to acknowledge our indebtedness to their authors for the basic idea.

3. **Subdirectly irreducible pseudocomplemented distributive lattices.** Given an algebra A, the *trivial congruence* on A, denoted ω or, more explicitly, ω_A, is defined as $x \equiv y(\omega)$ if and only if $x=y$. A congruence Θ on A is said to be *nontrivial* if $\Theta \neq \omega$. An algebra A is said to be *subdirectly irreducible* if A has a smallest nontrivial congruence. This definition agrees with that in G. Birkhoff [1, p. 141] and R. S. Pierce [9, p. 43] and differs trivially from the definition in G. Grätzer [4, p. 114] in that G. Grätzer considers the one-element algebra to be subdirectly irreducible while we do not. The fundamental Subdirect Representation Theorem of G. Birkhoff (see [9, p. 51] or [4, p. 114]) states that any algebra A is the subdirect product of a family $(B_\gamma \mid \gamma \in \Gamma)$ of subdirectly irreducible algebras, where each B_γ is a homomorphic image of A. Parenthetically, it should be observed that the direct product of an empty family of algebras is the one-element algebra; thus if $\Gamma = \varnothing$ we have a subdirect representation of the one-element algebra.

Let $\langle B; \wedge, \vee, ', 0, e\rangle$ be a Boolean algebra, where the greatest element is denoted e. Adjoin a greatest element 1 to B and denote the resulting lattice by $\bar{B}$. That is, $\bar{B} = B \cup \{1\}$ where $x < 1$ for all $x \in B$. Then $\bar{B}$ is a pseudocomplemented distributive lattice where the pseudocomplementation * satisfies

$$
\begin{aligned}
x^* &= 1 \quad \text{if } x = 0,\\
&= x' \quad \text{if } x \in B - \{0\},\\
&= 0 \quad \text{if } x = 1.
\end{aligned}
\tag{6}
$$

THEOREM 2. *A pseudocomplemented distributive lattice is subdirectly irreducible if and only if it is isomorphic to $\bar{B}$ for some Boolean algebra B.*

Proof. We first show that $\bar{B}$ is a subdirectly irreducible pseudocomplemented distributive lattice for any Boolean algebra B. Let Θ_0 on $\bar{B}$ be defined by $x \equiv y(\Theta_0)$ if and only if $x=y$ or $\{x, y\} = \{e, 1\}$; Θ_0 is clearly a *-congruence. Now let Φ be a nontrivial *-congruence on $\bar{B}$. Then there are $x, y \in \bar{B}$, $x<y$, such that $x \equiv y(\Phi)$. If $y=1$ then $x \leqq e < y$ and so $e \equiv 1(\Phi)$. If $y \neq 1$ and $x=0$ then $y' = y^* \equiv 0^* = 1(\Phi)$, and again $e \equiv 1(\Phi)$. If $\{x, y\} \neq \{0, 1\}$ then $y \wedge x' = y \wedge x^* \equiv x \wedge x^* = 0(\Phi)$, and again $e \equiv 1(\Phi)$. Thus we conclude that if Φ is nontrivial then $e \equiv 1(\Phi)$, that is, that $\Theta_0 \leqq \Phi$. Consequently Θ_0 is the smallest nontrivial congruence on $\bar{B}$, proving that $\bar{B}$ is subdirectly irreducible.

Now let L be a subdirectly irreducible pseudocomplemented distributive lattice. If $D(L) = \{1\}$ then L is a subdirectly irreducible Boolean algebra, and so L is isomorphic to the two-element Boolean algebra $\mathbf{2} = \bar{\mathbf{1}}$, $\mathbf{1}$ denoting the one-element Boolean algebra. Otherwise, let Θ be the smallest nontrivial *-congruence on L, and let Θ correspond to the congruence pair $\langle \Theta_1, \Theta_2\rangle$. Let Φ be any nontrivial lattice congruence on $D(L)$. The pair $\langle \omega_{S(L)}, \Phi\rangle$ satisfies (5) and so is a nontrivial congruence pair on L. By Theorem 1 we conclude that $\langle \Theta_1, \Theta_2\rangle \leqq \langle \omega_{S(L)}, \Phi\rangle$; that is, $\Theta_1 = \omega_{S(L)}$ and $\Theta_2 \leqq \Phi$. Thus Θ_2 is included in each nontrivial congruence on $D(L)$. Since $\langle \Theta_1, \Theta_2\rangle$ is nontrivial and $\Theta_1 = \omega_{S(L)}$, we conclude that Θ_2 is nontrivial, and so Θ_2 is the smallest nontrivial lattice congruence on $D(L)$. Thus $D(L)$ is a subdirectly irreducible distributive lattice, that is, $D(L)$ is the two element chain, which we can write $\{e, 1\}$, $e<1$.

We now claim that if $a \in S(L)$ and $a \neq 1$ then $a \leqq e$. Assume, on the contrary, that there is an $a \in S(L)$ satisfying $a \nleqq e$ and $a \neq 1$. Let Ψ be the congruence on $S(L)$ determined by the dual ideal $[a)$. Let $b \in S(L)$, $u \in D(L)$, $b \leqq u$, and $b \equiv 1\,(\Psi)$. Then $b \geqq a$ and, since $e \ngeqq a$, $u=1$. Thus the pair $\langle \Psi, \omega_{D(L)} \rangle$ is a nontrivial congruence pair on L, contradicting the fact that $\langle \omega_{S(L)}, \iota_{D(L)} \rangle$ is the smallest nontrivial congruence pair on L. ($\iota_{D(L)}$ is the congruence $x \equiv y\,(\iota_{D(L)})$ for any $x, y \in D(L)$.) Thus if $a \in S(L)$ and $a \neq 1$ then $a<e$. Since each element of L is the meet of a dense element and a skeletal element we conclude that $L=\bar{B}$ where $B \cong S(L)$ or, more explicitly, $B=(S(L)-\{1\}) \cup \{e\}$. Thus the theorem is proved.

It should be noted that we permit a Boolean algebra to consist of only one element, that is, we permit $0=1$.

K. B. Lee [8] proved Theorem 2 for *finite* subdirectly irreducible pseudocomplemented distributive lattices using a different approach.

4. **The embedding theorem.** Let $(A_\gamma \mid \gamma \in \Gamma)$ be a family of sets. If $x \in \prod (A_\gamma \mid \gamma \in \Gamma)$ and $\gamma \in \Gamma$ we denote the projection of x onto the factor A_γ by x_γ. If each A_γ is a pseudocomplemented distributive lattice then so is $\prod (A_\gamma \mid \gamma \in \Gamma)$ and if $x \in \prod (A_\gamma \mid \gamma \in \Gamma)$ then $(x^*)_\gamma = (x_\gamma)^*$ for all $\gamma \in \Gamma$; thus there is no ambiguity in writing x^*_γ.

If L is a lattice we denote the lattice of ideals of L by $\Im(L)$. If L is a distributive lattice with 0 then $\Im(L)$ is a pseudocomplemented distributive lattice (see [1, p. 129]); if $I \in \Im(L)$ then $I^* = \{x \in L \mid (x] \wedge I = (0]\}$.

It is well known that the lattice of ideals of a complete atomic Boolean algebra is a Stone algebra. In [3] G. Grätzer proved conversely that any Stone algebra can be embedded as a *-subalgebra of the Stone algebra of ideals of a complete atomic Boolean algebra. He also raised the question whether the same theorem can be proved for pseudocomplemented distributive lattices if we dispense with the requirement that the atomic Boolean algebra be complete(2). In this section we answer the question in the affirmative. We first prove the result for subdirectly irreducible pseudocomplemented lattices and then apply the Subdirect Representation Theorem to derive the general theorem.

LEMMA 2. *Let* $(L_\gamma \mid \gamma \in \Gamma)$ *be a family of distributive lattices with* 0 *and let the mapping* $f\colon \prod (\Im(L_\gamma) \mid \gamma \in \Gamma) \to \Im(\prod (L_\gamma \mid \gamma \in \Gamma))$ *be defined by*

$$If = \prod (I_\gamma \mid \gamma \in \Gamma) \subseteq \prod (L_\gamma \mid \gamma \in \Gamma) \quad \text{for all } I \in \prod (\Im(L_\gamma) \mid \gamma \in \Gamma).$$

*Then f is a *-embedding.*

(2) A (not necessarily complete) Boolean algebra is said to be atomic if every non-0 element is the supremum of a set of atoms.

Proof. Since $If=\{x\in\prod(L_\gamma\mid\gamma\in\Gamma)\mid x_\gamma\in I_\gamma \text{ for all } \gamma\}$ it is clear that If is an ideal of $\prod(L_\gamma\mid\gamma\in\Gamma)$ and that f is one-to-one. It is also evident that f preserves 0, 1 and, since meet is identical with set intersection, meet; consequently f preserves the partial order, a fact evident in any event. Thus in order to complete the proof of the lemma we need only show that if $I, J\in\prod(\mathfrak{I}(L_\gamma)\mid\gamma\in\Gamma)$ then $(I\vee J)f\subseteq If\vee Jf$ and that if $If\wedge Jf=(0]$ then $Jf\subseteq I^*f$.

Let $x\in(I\vee J)f$. Then $x_\gamma\in(I\vee J)_\gamma=I_\gamma\vee J_\gamma$ for each $\gamma\in\Gamma$. Thus, for each $\gamma\in\Gamma$, there are $y^\gamma\in I_\gamma$, $z^\gamma\in J_\gamma$ such that $x_\gamma=y^\gamma\vee z^\gamma$. Let $y, z\in\prod(L_\gamma\mid\gamma\in\Gamma)$ be defined by $y_\gamma=y^\gamma$, $z_\gamma=z^\gamma$ for all $\gamma\in\Gamma$. Then $y\in If$, $z\in Jf$, and $x=y\vee z$; consequently $x\in If\vee Jf$, showing that f preserves $\vee$.

Now let $If\wedge Jf=(0]$ and let $x\in Jf$. Then, for each $\gamma\in\Gamma$, $x_\gamma\wedge y_\gamma=0$ for all $y\in If$. Let $\gamma\in\Gamma$ and let $z\in I_\gamma$. Let $y\in\prod(L_\gamma\mid\gamma\in\Gamma)$ be defined by $y_\gamma=z$ and $y_\delta=0$ if $\delta\neq\gamma$. Then $y\in If$ and so $x_\gamma\wedge z=0$. Thus $x_\gamma\in I_\gamma^*$. Since $\gamma\in\Gamma$ was arbitrary we conclude that $x\in I^*f$; thus $Jf\subseteq I^*f$, showing that f preserves * and completing the proof of the lemma.

LEMMA 3. *Let L be a subdirectly irreducible pseudocomplemented distributive lattice. Then there is an atomic Boolean algebra B and a *-embedding $\varphi: L\to\mathfrak{I}(B)$.*

Proof. By Theorem 2 there is a Boolean algebra B_0 such that $L\cong\bar{B}_0$. There is then a set X and an embedding of Boolean algebras $i: B_0\to\mathfrak{B}_X$, where $\mathfrak{B}_X$ denotes the Boolean algebra of all subsets of X. We thus obtain an embedding $\bar{\imath}: \bar{B}_0\to\bar{\mathfrak{B}}_X$ by requiring that $\bar{\imath}|B_0=i$ and $(1)\bar{\imath}=1$; by (6) $\bar{\imath}$ is a *-embedding(3). Consequently we need only prove the lemma for $\bar{\mathfrak{B}}_X$.

Let Y be an infinite set and let B be the Boolean algebra of all finite and cofinite subsets of $X\times Y$; B is an atomic Boolean algebra. For each $A\subseteq X$ let $I_A=\{Z\in B\mid Z\subseteq A\times Y, Z \text{ finite}\}$; then I_A is an ideal of B. We note that $I_X=\{Z\in B\mid Z \text{ finite}\}$ and that $I_\varnothing=(\varnothing]$, $\varnothing$ denoting the empty set. That $I_{A\cup B}=I_A\vee I_B$ and $I_{A\cap B}=I_A\wedge I_B$ follows by direct computation. Define $\varphi: \bar{\mathfrak{B}}_X\to\mathfrak{I}(B)$ by $A\varphi=I_A$ if $A\in\mathfrak{B}_X$ and $1\varphi=B$. Then φ is a one-to-one lattice homomorphism. Let $A\in\mathfrak{B}_X$; then $(I_A)^*=\{Z\in B\mid Z\wedge W=\varnothing \text{ for all } W\in I_A\}=\{Z\in B\mid Z\subseteq(X-A)\times Y\}$, where $X-A$ denotes the complement in X of the set A. Thus $A=\varnothing$ implies that $(I_A)^*=B$. However, if $A\neq\varnothing$ then, since Y is infinite, $(X-A)\times Y$ has an infinite complement; thus if $Z\in B$ and $Z\subseteq(X-A)\times Y$ then Z is finite. Consequently $A\neq\varnothing$ implies that $(I_A)^*=I_{X-A}$. Clearly $(1\varphi)^*=B^*=(\varnothing]=I_\varnothing$. Thus, since the pseudocomplementation in $\bar{\mathfrak{B}}_X$ satisfies (6), it follows that $\varphi: \bar{\mathfrak{B}}_X\to\mathfrak{I}(B)$ preserves *, completing the proof of the lemma.

THEOREM 3. *Let L be a pseudocomplemented distributive lattice. Then there is an atomic Boolean algebra B and a *-embedding $\varphi: L\to\mathfrak{I}(B)$.*

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(3) We note that if $\langle B_0;\wedge,\vee,',0,e_0\rangle$, $\langle B_1;\wedge,\vee,',0,e_1\rangle$ are Boolean algebras and $i: B_0\to B_1$ then $\bar{\imath}: \bar{B}_0\to\bar{B}_1$ is a *-homomorphism if and only if i is one-to-one; if $x\in B_0$, $x\neq 0$, and $xi=0$, then $x^*=x'<e_0$ and thus $x^*\bar{\imath}\leqq e_1<1=(x\bar{\imath})^*$.

Proof. By the Subdirect Decomposition Theorem [4, Theorem 3, p. 124], or [9, Theorem 3.4, p. 51], there is a family $(L_\gamma \mid \gamma \in \Gamma)$ of subdirectly irreducible pseudocomplemented distributive lattices and a *-embedding $i: L \to \prod (L_\gamma \mid \gamma \in \Gamma)$. By Lemma 3, for each $\gamma \in \Gamma$ there is an atomic Boolean algebra B_γ and a *-embedding $\varphi_\gamma: L_\gamma \to \mathfrak{I}(B_\gamma)$. Let $B = \prod (B_\gamma \mid \gamma \in \Gamma)$; then B is atomic. By Lemma 2 there is a *-embedding $f: \prod (\mathfrak{I}(B_\gamma) \mid \gamma \in \Gamma) \to \mathfrak{I}(B)$. Composing i, $\prod (\varphi_\gamma \mid \gamma \in \Gamma)$, and f, we get the desired embedding, establishing the theorem.

We observe in the above proof that if L is the one-element lattice then $\Gamma = \varnothing$ and so B is the one-element Boolean algebra.

5. **Equational subclasses.** In [8], [8a] K. B. Lee derived the structure of the lattice of equational subclasses of the equational class of all pseudocomplemented distributive lattices by investigating the appropriate identities. In this section we present an alternate proof based on Theorem 2.

For each integer $n \geqq 0$ let B_n denote the n-atom Boolean algebra $\mathbf{2}^n$, and let $\mathscr{B}_n$ denote the equational class of pseudocomplemented distributive lattices generated by $\bar{B}_n$. Denote the class of all one-element pseudocomplemented distributive lattices by $\mathscr{B}_{-1}$ and the class of all pseudocomplemented distributive lattices by $\mathscr{B}_\omega$.

THEOREM 4 [8], [8a]. *The equational subclasses of $\mathscr{B}_\omega$ are precisely the $\mathscr{B}_n$, $-1 \leqq n \leqq \omega$, and*

$$\mathscr{B}_{-1} \subseteq \mathscr{B}_0 \subseteq \cdots \subseteq \mathscr{B}_n \subseteq \cdots \subseteq \mathscr{B}_\omega,$$

where all the inclusions are proper.

Proof. Since, for each integer $n \geqq 0$, B_n is a subalgebra of B_{n+1}, it follows that $\bar{B}_n$ is a *-sublattice of $\bar{B}_{n+1}$; thus $\mathscr{B}_n \subseteq \mathscr{B}_{n+1}$ for all n, $0 \leqq n < \omega$. Since the lattice of *-congruences of a pseudocomplemented distributive lattice is a sublattice of the lattice of lattice congruences, it is distributive. It then follows by a result of B. Jónsson [5, Corollary 3.5] that the inclusion $\mathscr{B}_n \subseteq \mathscr{B}_{n+1}$ is proper([4]). Clearly $\mathscr{B}_{-1} \subseteq \mathscr{B}_0$, and $\mathscr{B}_{-1} \neq \mathscr{B}_0$ because $\bar{B}_0 = \mathbf{2}$.

Now let L be any subdirectly irreducible pseudocomplemented distributive lattice; then $L = \bar{B}$ for some Boolean alegbra B. Any finitely generated subalgebra of B is isomorphic to B_n for some integer n. Since B is the direct limit of its finitely generated subalgebras it follows that L is the direct limit of a family of $\bar{B}_n$, n integral. Since any equational class is closed under direct limits it follows that $L \in \bigvee (\mathscr{B}_n \mid n < \omega)$. Now an equational class is determined by the subdirectly irreducible algebras that it contains; thus $\mathscr{B}_\omega = \bigvee (\mathscr{B}_n \mid n < \omega)$.

It remains only to prove that the family $(\mathscr{B}_n \mid -1 \leqq n \leqq \omega)$ exhausts all equational subclasses. Let $\mathscr{B}$ be an equational subclass of $\mathscr{B}_\omega$; we may assume that $\mathscr{B} \neq \mathscr{B}_{-1}$.

([4]) This observation was communicated to the author by R. A. Day. That the inclusion is proper also follows from the observation that $\bar{B}_n$ satisfies $(x_1 \wedge \cdots \wedge x_n)^* \vee (x_1^* \wedge \cdots \wedge x_n)^* \vee \cdots \vee (x_1 \wedge \cdots \wedge x_n^*)^* = 1$ and that $\bar{B}_{n+1}$ does not (K. B. Lee [8]).

If $\mathscr{B}_n \subseteq \mathscr{B}$ for all integers n then $\mathscr{B}_\omega = \bigvee (\mathscr{B}_n \mid n < \omega) \subseteq \mathscr{B}$, that is, $\mathscr{B} = \mathscr{B}_\omega$. Otherwise there is a largest integer N such that $\mathscr{B}_N \subseteq \mathscr{B}$. Then $\bar{B}_{N+1} \notin \mathscr{B}$. Let $\bar{B}$, B Boolean, be a subdirectly irreducible member of $\mathscr{B}$. Since $\bar{B}_{N+1}$ is not a *-sublattice of $\bar{B}$ it follows that B is finite and so $B = B_n$ for some $n \leqq N$. Thus $\bar{B} \in \mathscr{B}_N$, establishing that $\mathscr{B} = \mathscr{B}_N$. We have thus established that the set $\{\mathscr{B}_{-1}, \mathscr{B}_0, \ldots, \mathscr{B}_\omega\}$ is the set of all equational subclasses of $\mathscr{B}_\omega$, concluding the proof of the theorem.

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UNIVERSITY OF MANITOBA,
WINNIPEG, CANADA