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CHARACTERISTIC SPHERES OF FREE DIFFERENTIABLE ACTIONS OF S^1 AND S^3 ON HOMOTOPY SPHERES

BY
HSU-TUNG KU AND MEI-CHIN KU

1. **Introduction.** Let (S^i, Σ^m, φ), $i=0$, 1 or 3, be a free differentiable action of S^i on the homotopy m-sphere Σ^m with orbit space Σ^m/S^i and a homotopy equivalence $f\colon \Sigma^m/S^i \to KP^k$ (K=real, complex or quaternionic according to whether $i=0$, 1 or 3). By a *characteristic homotopy* $(m-q)$*-sphere* of Σ^m we mean a homotopy sphere Σ^{m-q} which is S^i-invariant, that is $\varphi(S^i\times\Sigma^{m-q})\subset\Sigma^{m-q}\subset\Sigma^m$, and such that f is transverse regular on KP^l with $f^{-1}(KP^l)=\Sigma^{m-q}/S^i$ and $f|\Sigma^{m-q}/S^i$ is a homotopy equivalence. In this paper, we are concerned with the problem of finding characteristic homotopy spheres. The case $i=0$ and $q=1$ was studied by Browder and Livesay [3]. For the case $i=1$ and $q=2$, Montgomery and Yang have shown that the obstruction is preciesly the Browder-Livesay invariant which is obtained by restricting the action to the subgroup Z_2 [11]. We consider the case $i=3$, $m=4n+3$, and compare the obstructions between the S^3-action and S^1-action for $S^1\subset S^3$. We also give some interesting examples in dimensions 11, 13 and 15. Our methods will be based on the computation of the surgery obstruction by using a formula of Browder [2, 4.4].

Throughout the paper, Z denotes the ring of integers, $\sigma(M)$ the index of the smooth manifold M, $\tau(M)$ the tangent bundle of M. We let CP^n be the complex projective n-space and QP^n be the quaternion projective n-space.

2. **The invariants $I_{2k}(S^1, \Sigma^{2n+1})$ and $I_{4k}(S^3, \Sigma^{4n+3})$.** Suppose that S^1 (resp. S^3) acts freely and differentiably on a homotopy sphere Σ^{2n+1} (resp. Σ^{4n+3}), and let $N=\Sigma^{2n+1}/S^1$ (resp. Σ^{4n+3}/S^3) be the orbit space. Let $f\colon N\to CP^n$ (resp. $f\colon N\to QP^n$) be a homotopy equivalence which is transverse regular on the submanifold CP^{n-k} (resp. QP^{n-k}) with $n-k>2$ (resp. $n-k>1$), and let $M=f^{-1}(CP^{n-k})$ (resp. $f^{-1}(QP^{n-k})$). Furthermore we assume that dim $M=4q$. There is an obstruction to make f normally cobordant to $f'\colon N\to CP^n$ (resp. $f'\colon N\to QP^n$) a homotopy equivalence, such that if $M'=(f')^{-1}(CP^{n-k})$ (resp. $(f')^{-1}(QP^{n-k})$), $f'\colon (N, M')\to (CP^n, CP^{n-k})$ (resp. $f'\colon (N, M')\to (QP^n, QP^{n-k})$) is a homotopy equivalence on each term [2, 2.14]. The obstruction is simply the difference of two indices, namely $\sigma(M)-\sigma(CP^{n-k})$ (resp. $\sigma(M)-\sigma(QP^{n-k})$) which lies in the group $8Z$, and we shall denote it by $I_{2k}(S^1, \Sigma^{2n+1})$ (resp. $I_{4k}(S^3, \Sigma^{4n+3})$). It is precisely the obstruction of the free S^1 (resp. S^3) action on Σ^{2n+1} (resp. Σ^{4n+3}) having codimension $2k$ (resp. $4k$)

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characteristic homotopy sphere. For S^1 action with codimension 2 characteristic sphere, the obstruction is the invariant $I(S^1, \Sigma^{2n+1})$ defined by Montgomery and Yang [11], hence $I_2(S^1, \Sigma^{2n+1}) = I(S^1, \Sigma^{2n+1})$ if $n \geqq 4$. We restate a result of Browder in [2, 4.4] as the following theorem:

THEOREM 2.1 [2]. *Let (S^1, Σ^{2n+1}) or (S^3, Σ^{4n+3}) be a free differentiable action. Then*

$$I_{2k}(S^1, \Sigma^{2n+1}) = \langle L_q(\tau(N) \oplus k\rho^{-1}), \chi_{n-k}\rangle - 1,$$
$$I_{4k}(S^3, \Sigma^{4n+3}) = \langle L_q(\tau(N) \oplus k\rho^{-1}), \chi_{n-k}\rangle - \sigma(QP^{n-k}),$$

where ρ is the canonical bundle over $N = \Sigma^{2n+1}/S^1$ or Σ^{4n+3}/S^3, associated to the principal bundle

$$S^1 \to \Sigma^{2n+1} \to \Sigma^{2n+1}/S^1 \quad \text{or} \quad S^3 \to \Sigma^{4n+3} \to \Sigma^{4n+3}/S^3$$

respectively, ρ^{-1} the inverse of ρ, χ_{n-k} is the generator of $H_{4q}(N)$, $n-k=2q$ or $n-k=q$, and L_q the qth component of the Hirzebruch's L-genus [5],

$$L\colon (KO)^{\sim}(N) \to \sum_{i \geqq 0} H^{4i}(N, Q).$$

DEFINITION 2.2. Let $p_i(CP^n, Z) = r_i\bar{\alpha}^{2i}$ (resp. $p_i(QP^n, Z) = \bar{r}_i\bar{\beta}^i$). A homotopy complex (resp. quaternion) projective n-space M is called *semistandard* if the Pontrjagin classes $p_i(M) = r_i\alpha^{2i}$ for $i \leqq [(n-1)/4]$ (resp. $p_i(M) = \bar{r}_i\beta^i$, $i \leqq [n/2]$), where $\alpha, \bar{\alpha}$ (resp. $\beta, \bar{\beta}$) are the generators of $H^2(M, Z)$, $H^2(CP^n, Z)$ (resp. $H^4(M, Z)$, $H^4(QP^n, Z)$) respectively.

Suppose that a free differentiable S^3-action on a homotopy sphere Σ^{4n+3} is given ($n \geqq 3$) such that the orbit space Σ^{4n+3}/S^3 is a semistandard quaternion projective space. If we restrict the action to the subgroup S^1 of S^3, we have the fibre bundle $\eta\colon S^2 \to \Sigma^{4n+3}/S^1 \xrightarrow{\pi} \Sigma^{4n+3}/S^3$ with $p(\eta) = 1 + 4\bar{\alpha}$, $\bar{\alpha}$ the generator of $H^4(\Sigma^{4n+3}/S^3, Z)$ and $p(QP^n) = (1+\alpha)^{2(n+1)}(1+4\alpha)^{-1}$, α the generator of $H^4(QP^n, Z)$ [1]. Hence we see that Σ^{4n+3}/S^1 is also semistandard. In [6], W. C. Hsiang has shown the existence of infinitely many nonhomeomorphic semistandard complex $(2n+1)$-spaces and quaternion n-spaces such that their Pontrjagin classes p_i are distinct for $i \geqq [n/2]+1$.

THEOREM 2.3. *Suppose that S^3 acts freely and differentiably on a homotopy $(4n+3)$-sphere Σ^{4n+3} $(n \geqq 3)$ such that the orbit space Σ^{4n+3}/S^3 is a semistandard homotopy quaternion projective space. Then*

$$\begin{aligned} &I_{4n-4[n/2]-4j}(S^3, \Sigma^{4n+3}) \\ (1) \quad &= \sum_{i=0}^{j-1} \{\langle L_{[n/2]+j-i}(\tau(\Sigma^{4n+3}/S^3)), \bar{\chi}'_{[n/2]+j-i}\rangle - \langle L_{[n/2]+j-i}(\tau(QP^n)), \bar{\chi}_{[n/2]+j-i}\rangle\} \\ &\quad \times \langle L_i((n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_i\rangle \quad \text{for } j \geqq 1, \end{aligned}$$

$$I_{4n-4[n/2]-4j+2}(S^1, \Sigma^{4n+3})$$

$$(2)\quad = \sum_{i=0}^{j-1} \{\langle L_{[n/2]+j-i}(\tau(\Sigma^{4n+3}/S^3)), \bar{\chi}'_{[n/2]+j-i}\rangle - \langle L_{[n/2]+j-i}(\tau(QP^n)), \bar{\chi}_{[n/2]+j-i}\rangle\} \times \langle L_i(\pi^*\eta \oplus (2n-2[n/2]-2j+1)\rho^{-1}), \chi_i\rangle \quad \text{for } j \geqq 1,$$

$$(3)\quad \begin{aligned} I_{4n-4[n/2]-4}(S^3, \Sigma^{4n+3}) &= I_{4n-4[n/2]-2}(S^1, \Sigma^{4n+3}) \\ &= s_{[n/2]+1}\{\langle p_{[n/2]+1}(\Sigma^{4n+3}/S^3), \bar{\chi}'_{[n/2]+1}\rangle \\ &\quad - \langle p_{[n/2]+1}(QP^n), \bar{\chi}_{[n/2]+1}\rangle\}, \end{aligned}$$

$$(4)\quad \begin{aligned} &I_{4n-4[n/2]+4j}(S^3, \Sigma^{4n+3}) \\ &= I_{4n-4[n/2]+4j+2}(S^1, \Sigma^{4n+3}) = 0 \quad \text{for } [n/2]-2 \geqq j \geqq 0, \end{aligned}$$

where $\rho, \bar{\rho}$ *denote the canonical bundles over* CP^{2n+1} *and* QP^n, χ_d, $\bar{\chi}_d$ *and* $\bar{\chi}'_d$ *the generators of* $H_{4d}(CP^{2n+1}, Z)$, $H_{4d}(QP^n, Z)$ *and* $H_{4d}(\Sigma^{4n+3}/S^3, Z)$, *respectively* (*cf. Theorem* 2.1), $s_{[n/2]+1}$ *the coefficient of* $\rho_{[n/2]+1}$ *in* $L_{[n/2]+1}$ [5, p. 12], *and* $[n/2]$ *is the largest integer less than or equal to* $n/2$.

Notation. In the proof, we will denote the canonical bundles over Σ^{4n+3}/S^1 and Σ^{4n+3}/S^3 by ρ' and $\bar{\rho}'$ respectively, and χ'_d the generator of $H_{4d}(\Sigma^{4n+3}/S^1, Z)$. Let $\tau = \tau(CP^{2n+1})$, $\tau' = \tau(\Sigma^{4n+3}/S^1)$, $\bar{\tau} = \tau(QP^n)$ and $\bar{\tau}' = \tau(\Sigma^{4n+3}/S^3)$.

Proof. Since Σ^{4n+3} admits free S^3-action, we have fibre bundle

$$\eta' : S^2 \longrightarrow \Sigma^{4n+3}/S^1 \xrightarrow{\pi'} \Sigma^{4n+3}/S^3$$

which is homotopically equivalent to the standard fibration $\eta: S^2 \to CP^{2n+1} \xrightarrow{\pi} QP^n$ by $(f, \bar{f})$, that is, we have the following commutative diagram:

$$\begin{array}{ccccc} \eta' : S^2 & \longrightarrow & \Sigma^{4n+3}/S^1 & \xrightarrow{\pi'} & \Sigma^{4n+3}/S^3 \\ & & \downarrow f & & \downarrow \bar{f} \\ \eta : S^2 & \longrightarrow & CP^{2n+1} & \xrightarrow{\pi} & QP^n \end{array}$$

Notice that $\rho' = f^*(\rho)$, $\bar{\rho}' = \bar{f}^*(\bar{\rho})$ and $\eta' = \bar{f}^*(\eta)$. By assumption we have

$$(5)\quad \begin{aligned} \bar{f}^*(p_i(QP^n)) &= p_i(\Sigma^{4n+3}/S^3) \quad \text{and} \\ f^*(p_i(CP^{2n+1})) &= p_i(\Sigma^{4n+3}/S^1) \quad \text{for } i \leqq [n/2]. \end{aligned}$$

Now we consider the standard free action of S^3 on S^{4n+3}. This action has characteristic $(4[n/2]+4j+3)$-sphere $S^{4[n/2]+4j+3}$. Hence by Theorem 2.1, we have

$$\langle L_{[n/2]+j}(\bar{\tau} \oplus (n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_{[n/2]+j}\rangle = \sigma(QP^{[n/2]+j})$$

or

$$(6)\quad \begin{aligned} &\langle L_{[n/2]}(\bar{\tau})L_j((n-[n/2]-j)\bar{\rho}^{-1}) + \cdots + L_{[n/2]+j}((n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_{[n/2]+j}\rangle \\ &= \sigma(QP^{[n/2]+j}) - \langle L_{[n/2]+j}(\bar{\tau}) + \cdots + L_{[n/2]+1}(\bar{\tau}) \\ &\quad \times L_{j-1}((n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_{[n/2]+j}\rangle. \end{aligned}$$

Since $\bar{f}^*(p_i(QP^n))=p_i(\Sigma^{4n+3}/S^3)$ for $i\leqq[n/2]$, $L_i(\bar{\tau}')=\bar{f}^*L_i(\bar{\tau})$ for $i\leqq[n/2]$. Thus we obtain

$$\begin{aligned}(7)\qquad &\langle L_{[n/2]}(\bar{\tau}')L_j((n-[n/2]-j)(\bar{\rho}')^{-1})+\cdots+L_{[n/2]+j}((n-[n/2]-j)(\bar{\rho}')^{-1}), \bar{\chi}'_{[n/2]+j}\rangle \\ &= \langle \bar{f}^*\{L_{[n/2]}(\bar{\tau})L_j((n-[n/2]-j)\bar{\rho}^{-1}) \\ &\qquad +\cdots+L_{[n/2]+j}((n-[n/2]-j)\bar{\rho}^{-1})\}, \bar{\chi}'_{[n/2]+j}\rangle \\ &= \sigma(QP^{[n/2]+j})-\langle L_{[n/2]+j}(\bar{\tau})+\cdots+L_{[n/2]+1}(\bar{\tau}) \\ &\qquad \times L_{j-1}((n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_{[n/2]+j}\rangle\end{aligned}$$

by (6). Again by Theorem 2.1,

$$\begin{aligned}&I_{4n-4[n/2]-4j}(S^3, \Sigma^{4n+3}) \\ &= \langle L_{[n/2]+j}(\bar{\tau}')+\cdots+L_{[n/2]+1}(\bar{\tau}')L_{j-1}((n-[n/2]-j)(\bar{\rho}')^{-1}), \bar{\chi}'_{[n/2]+j}\rangle \\ &\quad +\langle L_{[n/2]}(\bar{\tau}')L_j((n-[n/2]-j)(\bar{\rho}')^{-1}) \\ &\qquad +\cdots+L_{[n/2]+j}((n-[n/2]-j)(\bar{\rho}')^{-1}), \bar{\chi}'_{[n/2]+j}\rangle-\sigma(QP^{[n/2]+j}) \\ &= \sum_{i=0}^{j-1}\{\langle L_{[n/2]+j-i}(\tau(\Sigma^{4n+3}/S^3)), \bar{\chi}'_{[n/2]+j-i}\rangle \\ &\qquad -\langle L_{[n/2]+j-i}(\tau(QP^n)), \bar{\chi}_{[n/2]+j-i}\rangle\}\langle L_i((n-[n/2]-j)\bar{\rho}^{-1}), \bar{\chi}_i\rangle\end{aligned}$$

by (7), and (1) is proved.

The proof of (2) is similar. We have $\tau'=\pi'^*\bar{\tau}'\oplus\pi'^*\eta'$ and $\tau=\pi^*\bar{\tau}\oplus\pi^*\eta$. As we remarked before, $f^*p_i(\tau)=p_i(\tau')$ for $i\leqq[n/2]$. The standard free S^1-action on S^{4n+3} has characteristic sphere $S^{4[n/2]+4j+1}$, hence

$$\begin{aligned}1 &= \langle L_{[n/2]+j}(\tau\oplus(2n-2[n/2]-2j+1)\rho^{-1}), \chi_{[n/2]+j}\rangle \\ &= \langle L_{[n/2]+j}(\pi^*\bar{\tau}\oplus\pi^*\eta\oplus(2n-2[n/2]-2j+1)\rho^{-1}), \chi_{[n/2]+j}\rangle.\end{aligned}$$

For simplicity, let $k=2n-2[n/2]-2j+1$, then

$$\begin{aligned}(8)\qquad &\langle L_{[n/2]}(\pi^*\bar{\tau})L_j(\pi^*\eta\oplus k\rho^{-1})+\cdots+L_{[n/2]+j}(\pi^*\eta\oplus k\rho^{-1}), \chi_{[n/2]+j}\rangle \\ &= 1-\langle L_{[n/2]+j}(\pi^*\bar{\tau})+\cdots+L_{[n/2]+1}(\pi^*\bar{\tau})L_{j-1}(\pi^*\eta\oplus k\rho^{-1}), \chi_{[n/2]+j}\rangle \\ &= 1-\langle L_{[n/2]+j}(\pi^*\bar{\tau}), \chi_{[n/2]+j}\rangle \\ &\qquad -\cdots-\langle L_{[n/2]+1}(\pi^*\bar{\tau}), \chi_{[n/2]+1}\rangle\langle L_{j-1}(\pi^*\eta\oplus k\rho^{-1}), \chi_{j-1}\rangle \\ &= 1-\langle L_{[n/2]+j}(\bar{\tau}), \bar{\chi}_{[n/2]+j}\rangle \\ &\qquad -\cdots-\langle L_{[n/2]+1}(\bar{\tau}), \bar{\chi}_{[n/2]+1}\rangle\langle L_{j-1}(\pi^*\eta\oplus k\rho^{-1}), \chi_{j-1}\rangle.\end{aligned}$$

If we use the fact that $\pi'^*\bar{f}^*=f^*\pi^*$, we can see that

$$\begin{aligned}(9)\qquad &\langle L_{[n/2]}(\pi'^*\bar{\tau}')L_j(\pi'^*\eta'\oplus k\rho'^{-1})+\cdots+L_{[n/2]+j}(\pi'^*\eta'\oplus k\rho'^{-1}), \chi'_{[n/2]+j}\rangle \\ &= 1-\langle L_{[n/2]+j}(\bar{\tau}), \bar{\chi}_{[n/2]+j}\rangle \\ &\qquad -\cdots-\langle L_{[n/2]+1}(\bar{\tau}), \bar{\chi}_{[n/2]+1}\rangle\langle L_{j-1}(\pi^*\eta\oplus k\rho^{-1}), \chi_{j-1}\rangle.\end{aligned}$$

Therefore, if we substitute (9) into the formula for $I_{4n-4[n/2]-4j+2}(S^1, \Sigma^{4n+3})$, the conclusion of (2) follows. The proof of (4) is essentially the same but a little simpler. The statement (3) follows from (1), (2) and (5). This completes the proof of Theorem 2.3.

As a simple consequence, we state the following:

COROLLARY 2.4. (i) *If* $I_{4n-4[n/2]-4}(S^3, \Sigma^{4n+3}) \neq 0$, *then*

$$I_{4n-4[n/2]-8}(S^3, \Sigma^{4n+3}) \neq I_{4n-4[n/2]-6}(S^1, \Sigma^{4n+3}).$$

(ii) $I_2(S^1, \Sigma^{15}) = I_6(S^1, \Sigma^{15})$ *and* $I_2(S^1, \Sigma^{19}) = I_6(S^1, \Sigma^{19})$, *since* $L_3(\Sigma^{15}/S^3) = 0$, $L_4(\Sigma^{19}/S^3) = 1$ *and* $\langle L_1(\pi^*\eta \oplus \rho^{-1}), \chi_1 \rangle = 1$.

(iii) *For* $n \geqq 5$, $I_{4n-4[n/2]-2}(S^1, \Sigma^{4n+3}) = I_{4n-4[n/2]-6}(S^1, \Sigma^{4n+3})$ *if and only if*

$$\begin{aligned} 3\{\langle L_{[n/2]+2}(\tau(\Sigma^{4n+3}/S^3)), \bar{\chi}'_{[n/2]+2}\rangle - \langle L_{[n/2]+2}(\tau(QP^n)), \bar{\chi}_{[n/2]+2}\rangle\} \\ = (2n-2[n/2]-4)\{\langle L_{[n/2]+1}(\tau(\Sigma^{4n+3}/S^3)), \bar{\chi}'_{[n/2]+1}\rangle \\ - \langle L_{[n/2]+1}(\tau(QP^n)), \bar{\chi}_{[n/2]+1}\rangle\}. \end{aligned}$$

We may derive the similar results in other cases.

To conclude this section we prove the following characterization theorem.

THEOREM 2.5. *Let* S^3 *act freely and differentiably on a homotopy* $(4n+3)$-*sphere* Σ^{4n+3} $(n \geqq 4)$. *Then the orbit space* Σ^{4n+3}/S^3 *is a semistandard homotopy quaternion projective space if and only if* (4) *is satisfied.*

Proof. We use the notation of Theorem 2.3. Suppose (4) holds. It suffices to show that the following relations are satisfied:

$$\langle L_j(\bar{\tau}'), \bar{\chi}'_j\rangle = \langle L_j(\bar{\tau}), \bar{\chi}_j\rangle \quad \text{for } 1 \leqq j \leqq [n/2]. \tag{10}$$

From $I_{4n-8}(S^3, \Sigma^{4n+3}) = I_{4n-6}(S^1, \Sigma^{4n+3}) = 0$, we obtain

$$\begin{aligned} \langle L_2(\bar{\tau}' \oplus (n-2)(\bar{\rho}')^{-1}), \bar{\chi}'_2\rangle &= \langle L_2(\tau' \oplus (2n-3)(\rho')^{-1}), \chi'_2\rangle \\ &= \langle L_2(\pi'^*\bar{\tau}' \oplus \pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi'_2\rangle \end{aligned}$$

because $\tau' = \pi'^*\bar{\tau}' \oplus \pi'^*\eta'$. Simplifying this equation we get

$$\begin{aligned} &\langle L_1(\bar{\tau}'), \bar{\chi}'_1\rangle\{\langle L_1((n-2)(\bar{\rho}')^{-1}), \bar{\chi}'_1\rangle - \langle L_1(\pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi'_1\rangle\} \\ &\quad = \langle L_2(\pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi'_2\rangle - \langle L_2((n-2)(\bar{\rho}')^{-1}), \bar{\chi}'_2\rangle. \end{aligned} \tag{11}$$

Similarly if we consider the standard free S^3 action on S^{4n+3}, we have

$$I_{4n-8}(S^3, S^{4n+3}) = I_{4n-6}(S^1, S^{4n+3}) = 0;$$

thus the same argument implies

$$\begin{aligned} &\langle L_1(\bar{\tau}), \bar{\chi}_1\rangle\{\langle L_1((n-2)(\bar{\rho})^{-1}), \bar{\chi}_1\rangle - \langle L_1(\pi^*\eta \oplus (2n-3)\rho^{-1}), \chi_1\rangle\} \\ &\quad = \langle L_2(\pi^*\eta \oplus (2n-3)\rho^{-1}), \chi_2\rangle - \langle L_2((n-2)(\bar{\rho})^{-1}), \bar{\chi}_2\rangle. \end{aligned} \tag{12}$$

By comparing (11) and (12) we have

$$\langle L_1(\bar{\tau}'), \bar{\chi}'_1\rangle = \langle L_1(\bar{\tau}), \bar{\chi}_1\rangle \tag{13}$$

since

$$\langle L_1((n-2)(\bar\rho')^{-1}), \bar\chi_1'\rangle = \langle L_1((n-2)\bar\rho^{-1}), \bar\chi_1\rangle,$$
$$\langle L_1(\pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi_1'\rangle = \langle L_1(\pi^*\eta \oplus (2n-3)\rho^{-1}), \chi_1\rangle,$$
$$\langle L_2(\pi'^*\eta' \oplus (2n-3)(\rho')^{-1}), \chi_2'\rangle = \langle L_2(\pi^*\eta \oplus (2n-3)\rho^{-1}), \chi_2\rangle,$$

and

$$\langle L_2((n-2)(\bar\rho')^{-1}), \bar\chi_2'\rangle = \langle L_2((n-2)\bar\rho^{-1}), \bar\chi_2\rangle.$$

Next we see from $I_{4n-8}(S^3, \Sigma^{4n+3}) = \langle L_2(\bar\tau' \oplus (n-2)\bar\rho^{-1}), \bar\chi_2'\rangle - 1 = 0$ and $I_{4n-8}(S^3, S^{4n+3}) = 0$ that

$$\text{(14)} \qquad \langle L_2(\bar\tau'), \bar\chi_2'\rangle = \langle L_2(\bar\tau), \bar\chi_2\rangle.$$

Now we suppose that

$$\langle L_i(\bar\tau'), \bar\chi_i'\rangle = \langle L_i(\bar\tau), \bar\chi_i\rangle \quad \text{for } 1 \leqq i < [n/2].$$

But

$$I_{4n-4i-4}(S^3, \Sigma^{4n+3}) = \langle L_{i+1}(\bar\tau' \oplus (n-i-1)(\bar\rho')^{-1}), \bar\chi_{i+1}'\rangle - \sigma(QP^{i+1}) = 0$$

and

$$I_{4n-4i-4}(S^3, S^{4n+3}) = \langle L_{i+1}(\bar\tau \oplus (n-i-1)\bar\rho^{-1}), \bar\chi_{i+1}\rangle - \sigma(QP^{i+1}) = 0.$$

Thus,

$$\begin{aligned}\langle L_{i+1}(\bar\tau'), \bar\chi_{i+1}'\rangle &= \sigma(QP^{i+1}) - \langle L_i(\bar\tau'), \bar\chi_i'\rangle\langle L_1((n-i-1)(\bar\rho')^{-1}), \bar\chi_1'\rangle \\ &\quad - \cdots - \langle L_{i+1}((n-i-1)(\bar\rho')^{-1}), \bar\chi_{i+1}'\rangle \\ &= \sigma(QP^{i+1}) - \langle L_i(\bar\tau), \bar\chi_i\rangle\langle L_1((n-i-1)\bar\rho^{-1}), \bar\chi_1\rangle \\ &\quad - \cdots - \langle L_{i+1}((n-i-1)\bar\rho^{-1}), \bar\chi_{i+1}\rangle \\ &= \langle L_{i+1}(\bar\tau), \bar\chi_{i+1}\rangle.\end{aligned}$$

This completes the proof of the theorem.

3. **Lower dimensional examples.** This section contains some results on the free differentiable actions of Z_2, S^1 and S^3 on homotopy spheres of dimensions 11, 13 and 15.

I. *Actions on homotopy* 11-*spheres.* In [8] we studied the differentiable actions of S^3 on homotopy 11-spheres and proved

THEOREM 3.1. *Let Σ_M^{11} denote the Milnor sphere which represents the generator of θ_{11}* [9]. *Then a homotopy* 11-*sphere Σ^{11} admits a free differentiable S^3-action if and only if $\Sigma^{11} \approx 32k\Sigma_M^{11}$ for some $k \equiv 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 10, \pm 11, \pm 12, \pm 14, \pm 16$ (mod 31). These all admit infinitely many topologically distinct actions which can be distinguished by the first Pontrjagin classes of the orbit spaces.*

The proof of this theorem is based on the examples constructed by the Hsiang brothers [7] and the following fact [8]: Let S^3 act freely and differentiably on a homotopy 11-sphere Σ^{11}. Then either

(i) $p_1(\Sigma^{11}/S^3)=(672k+2)\bar{\alpha}$ for some integer k, and

$$\mu(\Sigma^{11}) \equiv -(13k+1)(k+1)k/31 \pmod 1,$$

or

(ii) $p_1(\Sigma^{11}/S^3)=(672k+194)\bar{\alpha}$ for some integer k, and

$$\mu(\Sigma^{11}/S^3) \equiv -7(-k+1)(7k+2)(k-7)/31 \pmod 1,$$

where $\bar{\alpha}$ denotes the generator of $H^4(\Sigma^{11}/S^3, Z)$ and $\mu(\Sigma^{11})$ the Eells-Kuiper μ-invariant [4].

It is said in [8] that the homotopy sphere $k\Sigma_M^{11}$, k odd, admits no free differentiable S^1-actions. Since we made some errors in computation, this does not follow from Lemma 2.1 of [8]. We conjecture that this result remains true. We correct this by proving the following Lemma 3.2. The proof being essentially the same, we only sketch the proof.

LEMMA 3.2. *Let S^1 act freely and differentiably on a homotopy 11-sphere Σ^{11}. Then*

$$\mu(\Sigma^{11}) \equiv (9i+102i^2+288i^3+148j+720ij+496t)/992 \pmod 1,$$

for some integers i, j and t. Moreover,

$$p_1(\Sigma^{11}/S^1) = (24i+6)\alpha^2,$$
$$p_2(\Sigma^{11}/S^1) = (1008i^2+264i+15+1440j)\alpha^4,$$

where α denotes the generator of $H^2(\Sigma^{11}/S^1, Z)$.

Proof. Let $D^2 \to W \xrightarrow{\pi} \Sigma^{11}/S^1$ be the associated disk bundle of the principal bundle $\xi: S^1 \to \Sigma^{11} \to \Sigma^{11}/S^1$. We see that $p(\xi)=1+\alpha^2$, $\omega_2(W)\neq 0$ and the index $\sigma(W)=1$. Let

$$p_1(\Sigma^{11}/S^1) = r_1\alpha^2, \qquad p_2(\Sigma^{11}/S^1) = r_2\alpha^4,$$
$$p_1(W) = r_3\beta^2, \qquad p_2(W) = r_4\beta^4.$$

$\beta=\pi^*\alpha$ and $r_1, r_2, r_3, r_4 \in Z$. We have $r_3=r_1+1$, $r_4=r_1+r_2$. Since β reduction mod 2 is $\omega_2(W)$, the invariant $\nu(\Sigma^{11})$ is well defined [10]. By substituting the above data into the formula for ν, and using the fact that $\nu(\Sigma^{11})\equiv 2\mu(\Sigma^{11}) \pmod 1$ [10], we have

$$\begin{aligned} 496\nu(\Sigma^{11}) &\equiv 992\mu(\Sigma^{11}) \\ &\equiv \{-312+80r_1-64r_2+142r_1^2+60r_1r_2-45r_1^3\}/2^6\cdot 3^2\cdot 5 \pmod{496}. \end{aligned}$$

The rest of the proof is just repeating the same argument as in the proof of Lemma 2.1 of [8].

THEOREM 3.3. *Let S^1 act freely and differentiably on a homotopy 11-sphere Σ^{11}. Then $I_2(S^1, \Sigma^{11})=16(9i^2+2i+14j)$ for some integers i and j. Hence if $I_2(S^1, \Sigma^{11})=0$, then i is even and $\Sigma^{11}\approx m\Sigma_M^{11}$ for some even integers m.*

Proof. It is known that $p(\rho^{-1})=(1+\alpha^2)^{-1}$. We apply Theorem 2.1 to obtain

$$\begin{aligned} I_2(S^1, \Sigma^{11}) &= \langle L_2(\tau(\Sigma^{11}/S^1) \oplus \rho^{-1}), \chi_4\rangle - 1 \\ &= \langle L_2(\Sigma^{11}/S^1)+L_1(\Sigma^{11}/S^1)L_1(\rho^{-1})+L_2(\rho^{-1}), \chi_4\rangle - 1 \\ &= \frac{1}{45}[7(1008i^2+264i+15+1440j)-(24i+6)^2] \\ &\quad -\frac{1}{3}(24i+6)\cdot\frac{1}{3}+\frac{1}{45}(7-1)-1 \\ &= 16(9i^2+2i+14j). \end{aligned}$$

Corollary 3.4. *There exists free differentiable Z_2-action on $k\Sigma_M^{11}$ for some k such that this action cannot be extended to the free differentiable action of S^1.*

Proof. Let $I(Z_2, \Sigma^{11})$ be the Browder-Livesay invariant [3], [11]. Montgomery and Yang showed that $I(Z_2, \Sigma^{11})=I_2(S^1, \Sigma^{11})$ [11]. Santiago has constructed the free involution (Z_2, Σ_k^{11}) for all $k \in Z$ such that

$$I(Z_2, \Sigma_k^{11}) = \sigma(W_k) = 8k,$$

with Σ_k^{11} bounding a π-manifold W_k^{12} [12], [14]. Thus for k odd, or k is not of the form $2(9i^2+2i+14j)$, the free Z_2-action on $k\Sigma_M^{11}$ cannot be extended to free S^1-action.

Theorem 3.5. *Let S^3 act freely on homotopy 11-spheres Σ^{11} and $Z_2 \subset S^1$ are subgroups of S^3. Then*

(i) $p_1(\Sigma^{11}/S^3)=(672k+2)\bar{\alpha}$ *for some integer k, and* $I(Z_2, \Sigma^{11})=I_2(S^1, \Sigma^{11})=224k$ *or*

(ii) $p_1(\Sigma^{11}/S^3)=(672k+194)\bar{\alpha}$ *for some integer k, and* $I(Z_2, \Sigma^{11})=I_2(S^1, \Sigma^{11}) =224k+64$.

In particular, Σ^{11} has S^1-invariant characteristic 9-sphere S^9 if and only if $\Sigma^{11}\approx S^{11}$.

The involutions in (i) for different k are all differentiably distinct. This answers a question of the Hsiangs [7].

Proof. If $p_1(\Sigma^{11}/S^3)=(672k+2)\bar{\alpha}$, then

$$p_1(\Sigma^{11}/S^1) = (672k+6)\beta^2, \quad \text{and} \quad p_2(\Sigma^{11}/S^1) = (64512k^2+3072k+15)\beta^4,$$

where $\beta^2=\pi^*\bar{\alpha}$, $\pi: \Sigma^{11}/S^1 \to \Sigma^{11}/S^3$ the natural projection [7]. Similarly, if $p_1(\Sigma^{11}/S^3)=(672k+194)\bar{\alpha}$, then

$$p_1(\Sigma^{11}/S^1) = (672k+198)\beta^2, \quad \text{and} \quad p_2(\Sigma^{11}/S^1) = (6412k^2+39936k+6159)\beta^4.$$

Thus (i) and (ii) are easily computed as in the proof of Theorem 3.3. We note that $I_2(S^1, \Sigma^{11})=0$ if and only if $p_1(\Sigma^{11}/S^3)=(672k+2)\bar{\alpha}$ for $k=0$. Hence $\mu(\Sigma^{11}) \equiv -(13k+1)(k+1)k/31 \equiv 0 \pmod 1$, and so $\Sigma^{11}\approx S^{11}$ by [4].

II. *Actions on homotopy* 13-*spheres*. Let S^1 act on a homotopy 13-sphere Σ^{13} with $p_1(\Sigma^{13}/S^1)=7\alpha^2$, α the generator of $H^2(\Sigma^{13}/S^1, Z)$. W. C. Hsiang has shown that there are infinitely many nonequivalent actions of S^1 on some Σ^{13} with $p_1(\Sigma^{13}/S^1)=7\alpha^2$ and different second and third Pontrjagin classes [6]. Let $p_2(\Sigma^{13}/S^1)=r_2\alpha^4$ and $p_3(\Sigma^{13}/S^1)=r_3\alpha^6$. Since the index $\sigma(\Sigma^{13}/S^1)=1$, we have

$$\begin{aligned} 1 &= \langle L_3(\Sigma^{13}/S^1), [\Sigma^{13}/S^1]\rangle \\ &= \left\langle \frac{1}{3^3\cdot 5\cdot 7}(62p_3(\Sigma^{13}/S^1)-13p_2(\Sigma^{13}/S^1)p_1(\Sigma^{13}/S^1)+p_1(\Sigma^{13}/S^1)^3), [\Sigma^{13}/S^1]\right\rangle \\ &= \frac{1}{3^3\cdot 5\cdot 7}(62r_3-91r_2+686) \end{aligned}$$

where $[\Sigma^{13}/S^1]$ denotes the fundamental class of Σ^{13}/S^1. It is not difficult to see that

$$r_2 = 62\bar{\imath}+21 \quad \text{and} \quad r_3 = 91\bar{\imath}+35 \qquad \text{for some } \bar{\imath}\in Z.$$

Now α reduction modulo 2 is the second Stiefel-Whitney class $\omega_2(\Sigma^{13}/S^1)$ which is different from zero. Thus by [5], $A(\Sigma^{13}/S^1, \alpha/2)$ is an integer. A simple calculation shows that

$$A(\Sigma^{13}/S^1, \alpha/2) = \frac{-259+14r_2-r_3}{2^6\cdot 3^3\cdot 5\cdot 7} = \frac{37\bar{\imath}}{2^6\cdot 3^2\cdot 5}$$

hence $\bar{\imath}=2^6\cdot 3^2\cdot 5\cdot i$ for some $i\in Z$. Thus,

$$\begin{aligned} I_4(S^1, \Sigma^{13}) &= \langle L_2(\tau(\Sigma^{13}/S^1)\oplus 2\rho^{-1}), \chi_4\rangle-1 \\ &= \frac{7\cdot 62\bar{\imath}}{45} = 2^7\cdot 7\cdot 31 i. \end{aligned}$$

Therefore we complete the proof of

THEOREM 3.6. *There are infinitely many free differentiable S^1-actions on homotopy 13-spheres so that none of them has a characteristic homotopy 9-sphere.*

III. *Actions on homotopy 15-spheres.*

THEOREM 3.7. *Let S^3 act freely and differentiably on Σ^{15}.*

(i) *Σ^{15}/S^3 is a semistandard homotopy quaternion projective 3-space if and only if $I_4(S^3, \Sigma^{15})=I_6(S^1, \Sigma^{15})$.*

(ii) *If Σ^{15}/S^3 is a semistandard quaternion projective 3-space, then $I_2(S^1,\Sigma^{15})=I_6(S^1,\Sigma^{15})=2^7\cdot 217i$ for some $i\in Z$, and $\Sigma^{15}\in 32bP_{16}\oplus Z_2$. Hence there are infinitely many free S^3-actions on some Σ^{15} so that none of them has a S^1-invariant characteristic 13-sphere S^{13} and S^1-invariant characteristic homotopy 9-spheres.*

Proof. Let $p_i(\Sigma^{15}/S^3)=r_i\alpha^i$, $i=1, 2, 3$, α the generator of $H^4(\Sigma^{15}/S^3, Z)$. Then

$$I_4(S^3, \Sigma^{15}) = \frac{1}{45}(7r_2-r_1^2-10r_1+17)-1$$

and

$$I_6(S^1, \Sigma^{15}) = \frac{1}{45}(7r_2-r_1^2+5r_1-43)-1,$$

whence $I_4(S^3, \Sigma^{15})=I_6(S^1, \Sigma^{15})$ if and only if $r_1=4$. Repeating the argument used in the proof of Theorem 3.6, since $\sigma(\Sigma^{15}/S^3)=0$, we see that $r_2=31\bar{i}+12$ and $r_3=26\bar{i}+8$ for some $\bar{i}\in Z$. But Σ^{15}/S^3 is a spin manifold, hence $\hat{A}_3(\Sigma^{15}/S^3)$ is an even integer and $\hat{A}_3(\Sigma^{15}/S^3)=\bar{i}/(2^6\cdot 3)$. Thus $\bar{i}=2^7\cdot 3\cdot \underline{i}$ for some $\underline{i}\in Z$. By Theorem 2.3 (3),

$$\begin{aligned} I_4(S^3, \Sigma^{15}) &= \frac{7}{45}\{\langle p_2(\Sigma^{15}/S^3), \bar{\chi}_2'\rangle - \langle p_2(QP^2), \bar{\chi}_2\rangle\} \\ &= \frac{7}{45}\{(31\bar{i}+12)-12\} \\ &= \frac{217\bar{i}}{45} = \frac{217\cdot 2^7\cdot 3\underline{i}}{45} = 2^7\cdot 217 i \quad \text{for some } \underline{i} = 15 i\in Z. \end{aligned}$$

Now let W^{16} be the total space of the disk bundle $D^4\to W\to \Sigma^{15}/S^3$ associated to the principal bundle $S^3\to\Sigma^{15}\to\Sigma^{15}/S^3$. By using standard technique (cf. proof of Lemma 3.2), we obtain the Eells-Kuiper μ-invariant as follows:

$$\begin{aligned} &\mu(\Sigma^{15}) \\ &= \frac{12096p_3(W)p_1(W)+5040p_2(W)^2-22680p_2(W)p_1(W)^2+9639p_1(W)^4-181440}{2^{15}\cdot 3^4\cdot 5\cdot 7\cdot 127} \\ &= \frac{3i(32i-3)}{2\cdot 127} \quad (\text{mod } 1). \end{aligned}$$

Hence $\Sigma^{15}\in 32bP_{16}\oplus Z_2$ by [4].

4. **Free S^3-actions with codimension 4 characteristic homotopy spheres.** In this section we like to compare the invariants $I_2(S^1, \Sigma^{4n+3})$ and $I_4(S^3, \Sigma^{4n+3})$ for any free differentiable action of S^3 on a homotopy sphere Σ^{4n+3}, where $S^1\subset S^3$. The arguments are similar to those used in §2, so we shall use the notation of Theorem 2.3.

THEOREM 4.1. *Suppose that a free differentiable action of S^3 on a homotopy $(4n+3)$-sphere Σ^{4n+3} is given, $n\geqq 3$, and let S^1 be the subgroup of S^3. Then $I_2(S^1, \Sigma^{4n+3})=I_4(S^3, \Sigma^{4n+3})$.*

Proof. According to Theorem 2.1, we have

$$\begin{aligned} I_2(S^1, \Sigma^{4n+3}) &= \langle L_n(\tau'\oplus\rho'^{-1}), \chi_n'\rangle - 1 \\ &= \langle L_n(\pi'^*\bar{\tau}'\oplus\pi'^*\eta'\oplus\rho'^{-1}), \chi_n'\rangle - 1 \\ &= \langle L_{n-1}(\pi'^*\bar{\tau}'), \chi_{n-1}'\rangle\langle L_1(\pi'^*\eta'\oplus\rho'^{-1}), \chi_1'\rangle \\ &\quad + \langle L_{n-2}(\pi'^*\bar{\tau}'), \chi_{n-2}'\rangle\langle L_2(\pi'^*\eta'\oplus\rho'^{-1}), \chi_2'\rangle \\ &\quad + \cdots + \langle L_n(\pi'^*\eta'\oplus\rho'^{-1}), \chi_n'\rangle + \{\langle L_n(\pi'^*\bar{\tau}'), \chi_n'\rangle - 1\} \\ &= \langle L_{n-1}(\bar{\tau}'), \bar{\chi}_{n-1}'\rangle + \langle L_{n-2}(\bar{\tau}'), \bar{\chi}_{n-2}'\rangle\langle L_2(\pi^*\eta\oplus\rho^{-1}), \chi_2\rangle \\ &\quad + \cdots + \langle L_n(\pi^*\eta\oplus\rho^{-1}), \chi_n\rangle + \{\langle L_n(\bar{\tau}'), \bar{\chi}_n'\rangle - 1\} \end{aligned}$$

because $\langle L_1(\pi'^*\eta' \oplus \rho'^{-1}), \chi_1' \rangle = 1$ and $\pi'^*\bar{f}^* = f^*\pi^*$. Similarly,

$$\begin{aligned} I_4(S^3, \Sigma^{4n+3}) &= \langle L_{n-1}(\bar{\tau}' \oplus \bar{\rho}'^{-1}), \bar{\chi}_{n-1}' \rangle - \sigma(QP^{n-1}) \\ &= \langle L_{n-1}(\bar{\tau}'), \bar{\chi}_{n-1}' \rangle + \langle L_{n-2}(\bar{\tau}'), \bar{\chi}_{n-2}' \rangle \langle L_1(\bar{\rho}'^{-1}), \bar{\chi}_1' \rangle \\ &\quad + \cdots + \langle L_{n-1}(\bar{\rho}'^{-1}), \bar{\chi}_{n-1}' \rangle - \sigma(QP^{n-1}). \end{aligned}$$

But $\langle L_n(\pi'^*\bar{\tau}'), \chi_n' \rangle - 1 = -\sigma(QP^{n-1})$. Thus it suffices to show that

$$\langle L_k(\pi^*\eta \oplus \rho^{-1}), \chi_k \rangle = \langle L_{k-1}(\bar{\rho}^{-1}), \bar{\chi}_{k-1} \rangle \quad \text{for } k \geqq 1$$

since $\langle L_{k-1}(\bar{\rho}'^{-1}), \bar{\chi}_{k-1}' \rangle = \langle L_{k-1}(\bar{\rho}^{-1}), \bar{\chi}_{k-1} \rangle$ for $k \geqq 1$. By direct computation, we see easily that the above relation holds for $k = 1, 2$. Suppose it is true for $k < m \leqq n$. We consider the standard free action of S^1 and S^3 on S^{4m+3}. The same computation gives

$$\begin{aligned} 0 &= I_2(S^1, S^{4m+3}) \\ &= \langle L_{m-1}(\bar{\tau}), \bar{\chi}_{m-1} \rangle + \langle L_{m-2}(\bar{\tau}), \bar{\chi}_{m-2} \rangle \langle L_2(\pi^*\eta \oplus \rho^{-1}), \chi_2 \rangle \\ &\quad + \cdots + \langle L_m(\pi^*\eta \oplus \rho^{-1}), \chi_m \rangle + \{\langle L_m(\bar{\tau}), \bar{\chi}_m \rangle - 1\}. \\ 0 &= I_4(S^3, S^{4m+3}) \\ &= \langle L_{m-1}(\bar{\tau}), \bar{\chi}_{m-1} \rangle + \langle L_{m-2}(\bar{\tau}), \bar{\chi}_{m-2} \rangle \langle L_1(\bar{\rho}^{-1}), \bar{\chi}_1 \rangle \\ &\quad + \cdots + \langle L_{m-1}(\bar{\rho}^{-1}), \bar{\chi}_{m-1} \rangle - \sigma(QP^{m-1}). \end{aligned}$$

Again $\langle L_m(\bar{\tau}), \bar{\chi}_m \rangle - 1 = -\sigma(QP^{m-1})$ and so

$$\langle L_m(\pi^*\eta \oplus \rho^{-1}), \chi_m \rangle = \langle L_{m-1}(\bar{\rho}^{-1}), \bar{\chi}_{m-1} \rangle$$

follows from induction hypotheses. This completes the proof of the theorem.

ADDED. In the paper *Differentiable S^1 actions on homotopy spheres* (to appear), G. Brumfiel has found all possible homotopy spheres in dimensions 9, 11 and 13 which admit free differentiable actions of S^1. He also studied the free differentiable actions of S^1 on homotopy spheres which do not bound π-manifolds.

The authors were informed that the action on homotopy spheres not in bP_{2n} were also studied by D. Frank.

Concerning the problem related to the existence of characteristic homotopy spheres, the authors also showed the following results: There are homotopy $(4n+1)$- or $(4n+3)$-spheres which admit infinitely many topologically distinct free differentiable actions of S^1 or S^3 with characteristic homotopy spheres in certain dimensions and without characteristic homotopy spheres in some other dimensions (cf. *Free differentiable actions of S^1 and S^3 on homotopy spheres*, Proc. Amer. Math. Soc. **25** (1970), 864–869). We remark that Theorem 4.1 can be generalized as follows with the same proof:

$$I_{4k+2}(S^1, \Sigma^{4n+3}) = I_{4k}(S^3, \Sigma^{4n+3}) + I_{4k+4}(S^3, \Sigma^{4n+3}) \quad \text{for } 0 \leqq k < n-2.$$

See also a paper of B. Conrad (*Extending free circle actions on spheres to S^3 actions*, mimeographed, Temple University, Philadelphia, Pa., 1970).

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University of Massachusetts,
Amherst, Massachusetts 01002