TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 157, June 1971

MAPPINGS ONTO THE PLANE

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Abstract. In this paper, we show that if X is a connected, locally connected, locally compact topological space and f is a 1-1 mapping of X onto E^2, then f is a homeomorphism. Using this result, we obtain theorems concerning the compactness of certain mappings onto E^2.

1. **Introduction.** Consider a 1-1 mapping f of a topological space X onto E^n (Euclidean n-space).

In [11], V. V. Proizvolov claimed to have proved that if X is connected, locally compact, and paracompact then f must be a homeomorphism. Later [12], he used this result to show that if X is connected, locally connected, and locally compact then f is a homeomorphism. There was, however, an error in the proof given in [11], and examples have been given by Kenneth Whyburn [20] and L. C. Glaser [7], [8], and [9] which show that neither of the above theorems is valid when $n \geqq 3$.

It is known (see [11, p. 1194] and [17, p. 1428]) that if $n=1$ and X is either locally connected or locally peripherally compact then f is a homeomorphism. (A topological space is said to be *locally peripherally compact* if for each point x of the space and each open neighborhood U of x there is an open neighborhood V of x with compact boundary such that $V \subset U$.)

The question as to whether either of Proizvolov's claimed theorems is true if $n=2$ has received considerable attention, and partial answers have been obtained by Glaser [7], Edwin Duda [3], R. F. Dickman, Jr. [1] and [2], and this author [10]. In [2], Dickman showed that if $n=2$ and X is a locally connected generalized continuum having no local separating point, then f is a homeomorphism (see §2 for definitions of *local separating point* and *generalized continuum*). In the present paper (§4) we make use of Dickman's result to show that the second of the above stated theorems of Proizvolov is valid when $n=2$, i.e., if X is a connected, locally connected, locally compact topological space and f is a 1-1 mapping of X onto E^2 then f is a homeomorphism.

Remark 1. To prove the above mentioned result in [2], Dickman first showed that if X is a locally connected generalized continuum with no local separating

Received by the editors May 29, 1970.

AMS 1970 *subject classifications.* Primary 54C05, 54C10; Secondary 54B15, 54B25, 54F25, 57A05.

Key words and phrases. One-to-one mapping, monotone mapping, compact mapping, locally connected generalized continuum, inverse system, upper-semicontinuous decomposition, reflexive compact mapping.

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point and if there is a 1-1 mapping of X onto E^2 then X must be a 2-manifold with boundary. It then follows from Theorem 5.1 of [10] that every 1-1 mapping of X onto E^2 is a homeomorphism. The first part of the proof may be shortened somewhat by using a theorem proved by Gail S. Young in [21]. If we observe that every simple closed curve in X must separate X, then [21, Theorem 1.1, p. 979] implies immediately that X is a 2-manifold with boundary.

REMARK 2. As the following example shows, it is not possible to obtain a theorem for the 2-dimensional case as strong as either of the above mentioned theorems which hold when $n=1$. Let X be the subset of the complex plane consisting of all numbers with positive imaginary parts, all negative irrational real numbers, and all nonnegative rational real numbers. Let f be defined as follows: for each $z \in X, f(z)=z^2$. Then X is a connected, locally connected, locally peripherally compact metric space and f is a 1-1 nontopological mapping of X onto the complex plane.

2. **Basic concepts and notation.** By a *mapping* we will mean a continuous function. A mapping f of a topological space X into a topological space Y is said to be *closed* if for each closed set H in X, $f(H)$ is closed in Y (or, equivalently, for each $y \in Y$ and each open set U in X with $f^{-1}(y) \subset U$, there is an open set V in Y such that $y \in V$ and $f^{-1}(V) \subset U$). We say that f is *monotone* if each point of Y has a compact connected inverse image in X. If each compact set in Y has a compact inverse image in X, then f is said to be *compact*.

Let f be a mapping of a metric space X into itself, and let ε be a positive number. We say that f is an ε-mapping if for each $x \in X$, $\rho(x, f(x)) < \varepsilon$.

By a *disc*, we will mean a closed 2-cell.

A subset of a topological space will be called *conditionally compact* if its closure in the space is compact.

A *generalized continuum* is a connected, locally compact metric space. It follows from [13, Corollary, p. 111] that such a space is always separable. If X is a locally connected generalized continuum, then every connected open set in X is arcwise connected [16, 5.3, p. 33].

A point x of a locally connected topological space X is called a *local separating point* of X if for some connected open set U in X, x separates U.

For the definition of *order* of a point in a topological space, see [16, p. 48] or [18, p. 35].

Let X be a locally connected generalized continuum. A collection $\mathscr{C}$ of ordered triples (V, p, q) will be called a *C-collection* for X provided that (1) $\mathscr{C}$ is countable, (2) for each $(V, p, q) \in \mathscr{C}$, V is a connected open set in X and $\{p, q\} \subset V$, and (3) if U is a connected open set in X and for some $x \in X$, U' and U'' are distinct components of $U - x$, then there is a member (V, p, q) of $\mathscr{C}$ such that $V \subset U$, $p \in U'$, and $q \in U''$.

For definitions and general concepts pertaining to *inverse systems* (sometimes called *inverse spectrums*), the reader is referred to [5, pp. 427–434] or [6, pp. 215–

220]. In this paper we shall be concerned only with inverse systems of topological spaces over the positive integers. If $\langle X_n, \mu_n^m \rangle$ is such a system, X_∞ will denote the inverse limit space and, for each n, μ_n will denote the projection mapping of X_∞ into X_n. If $\langle X_n, \mu_n^m \rangle$ and $\langle Y_n, \phi_n^m \rangle$ are inverse systems of topological spaces over the positive integers and $\langle f_n \rangle$ is a mapping of $\langle X_n, \mu_n^m \rangle$ into $\langle Y_n, \phi_n^m \rangle$, then f_∞ will denote the mapping from X_∞ into Y_∞ induced by $\langle f_n \rangle$.

3. **Preliminary theorems.** The theorems of this section, many of which are well-known results, will be used in proving the results of §4.

THEOREM 3.1. *Let X and Y be topological spaces and f a mapping of X into Y. If f is closed and has compact point inverses, then f is compact.* (See [17, p. 1426] or [19, Corollary 2, p. 690].)

THEOREM 3.2. *If X is a topological space, Y is a metric space, and f is a compact mapping of X into Y, then f is a closed mapping.* (See [19, p. 690].)

THEOREM 3.3. *If X is a topological space and Y a locally compact topological space, and if there exists a compact mapping of X into Y, then X is locally compact.*

Proof. Suppose x is a point of X. Let f be a compact mapping of X onto Y, and let V be a conditionally compact open set in Y with $f(x) \in V$. Then $f^{-1}(\overline{V})$ is a closed and compact set in X. Since $f^{-1}(V) \subset f^{-1}(\overline{V})$, this implies that $f^{-1}(V)$ is conditionally compact in X. Hence, for each point x of X there is a conditionally compact open set containing x, i.e., X is locally compact.

THEOREM 3.4. *If $\langle X_n, \mu_n^m \rangle$ is an inverse system of Hausdorff spaces over the positive integers and if for each n $(n=1, 2, 3, \ldots)$ μ_n^{n+1} is a compact mapping, then each μ_n is a compact mapping.*

Proof. Let K be a compact set in X_n and consider the inverse system $\langle K_i, \theta_i^j \rangle$ where, for each i, $K_i = (\mu_n^{n+i})^{-1}(K)$ and, for each i and $j \geqq i$, $\theta_i^j = \mu_{n+i}^{n+j} | K_j$. We have an inverse system of compact Hausdorff spaces, and therefore [6, Theorem 3.6, p. 217], K_∞ is compact. But K_∞ is homeomorphic to $(\mu_n)^{-1}(K)$, so we conclude that $(\mu_n)^{-1}(K)$ is a compact set.

THEOREM 3.5. *Let X and Y be topological spaces and f a closed monotone mapping of X onto Y. Then for each connected set H in Y, $f^{-1}(H)$ is connected in X.* (For proof, see [17, p. 1427].)

THEOREM 3.6. *Suppose that $\langle X_n, \mu_n^m \rangle$ is an inverse system of metric spaces over the positive integers and that, for each n, μ_n^{n+1} is a closed monotone mapping of X_{n+1} onto X_n. Then, for each n and each connected set H in X_n, $(\mu_n)^{-1}(H)$ is a connected set in X_∞.*

Proof. By Theorem 3.1, each of the mappings μ_n^{n+1} is compact. Hence, it follows from Theorem 3.4 that each μ_n is a compact mapping.

Let x be a point of X_n and assume that $(\mu_n)^{-1}(x)$ is not connected. Then, since μ_n is compact, $(\mu_n)^{-1}(x)$ is the union of two disjoint nonempty compact sets K and K'. It follows from [6, Lemma 3.12, p. 218] that there exist finite open coverings $\mathscr{U}$ and $\mathscr{U}'$ of K and K', respectively, such that (1) no member of $\mathscr{U}$ intersects K' and no member of $\mathscr{U}'$ intersects K, and (2) for each $U \in \mathscr{U} \cup \mathscr{U}'$ there is a positive integer i and an open set V in X_i such that $U=(\mu_i)^{-1}(V)$. Since $\mathscr{U} \cup \mathscr{U}'$ is finite, it follows that for some $m \geqq n$ there exist finite collections $\mathscr{V}$ and $\mathscr{V}'$ of open sets in X_m such that $\mathscr{U}=\{(\mu_m)^{-1}(V) \mid V \in \mathscr{V}\}$ and $\mathscr{U}'=\{(\mu_m)^{-1}(V) \mid V \in \mathscr{V}'\}$. Then $\mathscr{V}$ covers $\mu_m(K)$ and $\mathscr{V}'$ covers $\mu_m(K')$. Now since no member of $\mathscr{U}$ intersects K', no member of $\mathscr{V}$ intersects $\mu_m(K')$. Similarly, no member of $\mathscr{V}'$ intersects $\mu_m(K)$. Therefore, $\mu_m(K) \cup \mu_m(K')$ is not connected. But μ_m is a mapping of X_∞ *onto* X_m (see [6, p. 216]), and therefore $\mu_m(K) \cup \mu_m(K')=(\mu_n^m)^{-1}(x)$. Since Theorem 3.5 implies that $(\mu_n^m)^{-1}(x)$ is connected, we have a contradiction.

Hence, for each n and each $x \in X_n$, $(\mu_n)^{-1}(x)$ is compact and connected, i.e., μ_n is a monotone mapping. Since μ_n is closed (Theorem 3.2), it now follows from Theorem 3.5 that, for each connected set H in X_n, $(\mu_n)^{-1}(H)$ is connected.

THEOREM 3.7. *Suppose that $\langle X_n, \mu_n^m\rangle$ is an inverse system of metric spaces over the positive integers, that each X_n is locally connected, and that each μ_n^{n+1} is a closed monotone mapping of X_{n+1} onto X_n. Then X_∞ is locally connected.* (This result follows immediately from [6, Lemma 3.12, p. 218] and Theorem 3.6.)

THEOREM 3.8. *Suppose that Y is a complete metric space and that $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n<\infty$. Suppose, furthermore, that $\langle Y_n, \phi_n^m\rangle$ is an inverse system of metric spaces over the positive integers such that for each n $(n=1, 2, 3, \ldots)$*

(1) *$Y_n=Y$,*

(2) *ϕ_n^{n+1} is an ε_n-mapping, and*

(3) *for $y, z \in Y$ and for each positive integer $i \leqq n$, $\rho(\phi_i^n(y), \phi_i^n(z))<1/n$ whenever $\rho(y, z)<3 \sum_{j=n}^{\infty} \varepsilon_j$.*

Then the inverse limit space Y_∞ is homeomorphic to Y.

Proof. Since, for each n, ϕ_n^{n+1} is an ε_n-mapping of Y into Y, and since $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ is a summable sequence, it follows that each element of Y_∞ is a Cauchy sequence in Y. For each $\langle y_n\rangle \in Y_\infty$, let $F(\langle y_n\rangle)$ denote the point of Y to which $\langle y_n\rangle$ converges. We will now show that F is a homeomorphism of Y_∞ onto Y.

(i) *F is continuous.*

Proof of (i). Let $\langle y_n\rangle \in Y_\infty$ and let y denote the point $F(\langle y_n\rangle)$. Suppose that V is an open set in Y with $y \in V$. Then we can choose a positive number δ such that $N(y, 2\delta) \subset V$ (where $N(y, 2\delta)=\{z \mid \rho(y, z)<2\delta\}$) and a positive integer m such that $\sum_{j=m}^{\infty} \varepsilon_j<\delta$. Let $V'=(\phi_m)^{-1}(N(y, \delta))$. Then V' is an open set in Y_∞. Since

$$\rho(y, y_m) < \sum_{j=m}^{\infty} \varepsilon_j < \delta,$$

we have $y_m \in N(y, \delta)$ and, consequently, $\langle y_n \rangle \in V'$. If $\langle z_n \rangle$ is a point of V', then $\rho(y, z_m) < \delta$; therefore, letting $z = F(\langle z_n \rangle)$, we have

$$\rho(y, z) \leqq \rho(y, z_m) + \rho(z_m, z) < \delta + \sum_{j=m}^{\infty} \varepsilon_j < 2\delta,$$

which implies that $z \in V$. We conclude, then, that $F(V') \subset V$. Thus, F is continuous at $\langle y_n \rangle$.

(ii) *F is a* 1-1 *mapping.*

Proof of (ii). Let $\langle y_n \rangle$ and $\langle z_n \rangle$ be distinct points of Y_∞. Choose a positive integer k such that $y_k \neq z_k$, and then choose a positive integer m such that $m \geqq k$ and $\rho(y_k, z_k) > 1/m$. Since $y_k = \phi_k^m(y_m)$ and $z_k = \phi_k^m(z_m)$, we have (using our hypothesis), $\rho(y_m, z_m) \geqq 3 \sum_{j=m}^{\infty} \varepsilon_j$. For each positive integer $i \geqq m$, then, we have

$$\rho(y_i, z_i) + \rho(y_i, y_m) + \rho(z_i, z_m) \geqq \rho(y_m, z_m) \geqq 3 \sum_{j=m}^{\infty} \varepsilon_j,$$

and since each of the distances $\rho(y_i, y_m)$ and $\rho(z_i, z_m)$ is less than $\sum_{j=m}^{\infty} \varepsilon_j$ this means that $\rho(y_i, z_i) > \sum_{j=m}^{\infty} \varepsilon_j$. Hence, the two sequences $\langle y_n \rangle$ and $\langle z_n \rangle$ cannot converge to the same point of Y, i.e., $F(\langle y_n \rangle) \neq F(\langle z_n \rangle)$.

(iii) *F takes Y_∞ onto Y.*

Proof of (iii). Let y be a point of Y. For each ordered pair (m, n) of positive integers, let $y_n^m = \phi_n^{\max(m,n)}(y)$.

We assert that, for each n, $\langle y_n^m \rangle_{m=1}^{\infty}$ is a Cauchy sequence. For suppose that $\varepsilon > 0$. Let k be an integer such that $k \geqq n$ and $1/k < \varepsilon$. Then if m and r are positive integers such that $m \geqq r \geqq k$, we have

$$\rho(y_r^m, y) = \rho(\phi_r^m(y), y) < \sum_{j=r}^{\infty} \varepsilon_j < 3 \sum_{j=r}^{\infty} \varepsilon_j,$$

which implies (because of our hypothesis) that

$$\rho(y_n^m, y_n^r) = \rho(\phi_n^r(y_r^m), \phi_n^r(y)) < 1/r \leqq 1/k < \varepsilon.$$

Thus, $\langle y_n^m \rangle_{m=1}^{\infty}$ is a Cauchy sequence and must converge to some point of Y.

For each positive integer n, let $y_n = \lim_{m \to \infty} y_n^m$. Since $y_n^m = \phi_n^{n+1}(y_{n+1}^m)$ whenever m and n are positive integers and $m > n$, it follows from the continuity of the ϕ_n^{n+1}'s that each y_n is the image, under ϕ_n^{n+1}, of y_{n+1}. Therefore, $\langle y_n \rangle \in Y_\infty$.

We now have left to show that $\langle y_n \rangle$ converges to y in Y. Suppose that $\varepsilon > 0$. Choose a positive integer k such that $\sum_{j=k}^{\infty} \varepsilon_j < \varepsilon/2$. If $n \geqq k$ then, letting m be an integer such that $m > n$ and $\rho(y_n^m, y_n) < \varepsilon/2$, we have

$$\rho(y_n, y) \leqq \rho(y_n^m, y_n) + \rho(y_n^m, y) = \rho(y_n^m, y_n) + \rho(\phi_n^m(y), y)$$
$$< \varepsilon/2 + \sum_{j=n}^{\infty} \varepsilon_j \leqq \varepsilon/2 + \sum_{j=k}^{\infty} \varepsilon_j < \varepsilon.$$

Hence, $\langle y_n \rangle$ converges to y.

(iv) *F^{-1} is continuous.*

Proof of (iv). Suppose that $y \in Y$, and let $\langle y_n \rangle = F^{-1}(y)$. Let W be an open set in Y_∞ such that $\langle y_n \rangle \in W$. It follows from [6, Lemma 3.12, p. 218] that there is a positive integer m and an open set W_m in Y_m such that

$$\langle y_n \rangle \in (\phi_m)^{-1}(W_m) \subset W.$$

Choose an integer $k \geqq m$ such that $N(y_m, 1/k) \subset W_m$, and let $\delta = \sum_{j=k}^{\infty} \varepsilon_j$.

Now suppose that $z \in N(y, \delta)$. Letting $\langle z_n \rangle = F^{-1}(z)$, we have

$$\begin{aligned} \rho(y_k, z_k) &\leqq \rho(y_k, y) + \rho(y, z) + \rho(z, z_k) \\ &< \sum_{j=k}^{\infty} \varepsilon_j + \delta + \sum_{j=k}^{\infty} \varepsilon_j = 3 \sum_{j=k}^{\infty} \varepsilon_j, \end{aligned}$$

and this implies that

$$\rho(y_m, z_m) = \rho(\phi_m^k(y_k), \phi_m^k(z_k)) < 1/k.$$

Therefore, $z_m \in W_m$ and $\langle z_n \rangle \in W$. Hence, for each point z of $N(y, \delta)$, $F^{-1}(z) \in W$.

We conclude that F^{-1} is continuous at y.

THEOREM 3.9 (V. V. PROIZVOLOV). *If X is a locally connected, locally peripherally compact topological space and Y a metric space, and if there is a* 1-1 *mapping of X onto Y, then X is metrizable.* (See [12, Theorem 1, p. 1321].)

4. **1-1 mappings onto the plane.** The main result of this paper is the last theorem of this section (Theorem 4.4). We begin the section by proving two lemmas which, in turn, will be used in the proof of Theorem 4.3. Theorem 4.4 is obtained as an easy generalization of Theorem 4.3.

LEMMA 4.1. *If X is a locally connected generalized continuum, then there is a C-collection for X.*

Proof. Let S_1 denote the set of all local separating points of X of order 2 in X. Since X is separable, we can choose a countable collection $\mathscr{V}$ of connected open sets in X such that

(1) if $x \in S_1$ and U is an open set containing x, then there is a member of V of $\mathscr{V}$ such that $x \in V$ and $V \subset U$, and

(2) for each $V \in \mathscr{V}$, bd V consists of exactly two points.

For each $V \in \mathscr{V}$, let $\mathscr{W}(V)$ denote the collection of all $W \in \mathscr{V}$ such that $\overline{W} \subset V$, and let $\mathscr{J}(V)$ denote the collection of ordered triples of the form (V, p, q) where $\{p, q\} = \text{bd } W$ for some $W \in \mathscr{W}(V)$. Now let $\mathscr{C}_1 = \bigcup_{V \in \mathscr{V}} \mathscr{J}(V)$. Since $\mathscr{V}$ is countable, each $\mathscr{W}(V)$ is countable and, consequently, each $\mathscr{J}(V)$ is countable. Hence, $\mathscr{C}_1$ is a countable collection.

We now assert that if U is a connected open set and, for some $x \in S_1$, U' and U'' are distinct components of $U - x$, then there is a member (V, p, q) of $\mathscr{C}_1$ such that $V \subset U$, $p \in U'$, and $q \in U''$. Let V be an element of $\mathscr{V}$ such that $x \in V \subset U$. Next choose an element W of $\mathscr{W}(V)$ such that $x \in W$ and such that

$$\text{diam } W < \min \{\text{diam } U', \text{diam } U''\}.$$

Then neither U' nor U'' is a subset of W, and this implies that bd W intersects each of U' and U''. If $p \in U' \cap \operatorname{bd} W$ and $q \in U'' \cap \operatorname{bd} W$ then $\operatorname{bd} W=\{p, q\}$; this implies that (V, p, q) is a member of $\mathscr{J}(V)$ and, therefore, of $\mathscr{C}_1$. Thus, our assertion is established.

Now let S_2 denote the set of all local separating points of X which are not in S_1. By [16, Theorem 9.2, p. 61], S_2 is countable. Let $y_1, y_2, y_3, \ldots$ denote the points of S_2 and, for each n $(n=1, 2, 3, \ldots)$, let $V_{n1}, V_{n2}, V_{n3}, \ldots$ be a sequence of connected open neighbourhoods of y_n such that $\lim_{i\to\infty} \operatorname{diam} V_{ni}=0$. It follows from the separability and local connectedness of X that, for each ordered pair (n, i) of positive integers, $V_{ni}-y_n$ has only countably many components; hence, we can choose a countable collection $\mathscr{P}_{ni}$ of ordered pairs of points of V_{ni} such that for each ordered pair (V', V'') of components of $V_{ni}-y_n$ there is a member of $\mathscr{P}_{ni}$ having its first element in V' and its second element in V''. Let $\mathscr{C}_2$ denote the collection of all ordered triples (V, p, q) such that for some (n, i), $V=V_{ni}$ and $(p, q) \in \mathscr{P}_{ni}$. Since each $\mathscr{P}_{ni}$ is countable, $\mathscr{C}_2$ is countable.

Now suppose that U is a connected open set and that for some $y \in S_2$, U' and U'' are distinct components of $U-y$. For some n, $y=y_n$, and for some i, $V_{ni}\subset U$. There must exist components V' and V'' of $V_{ni}-y_n$ such that $V'\subset U'$ and $V''\subset U'$. Therefore, there is a member (p, q) of $\mathscr{P}_{ni}$ such that $p \in V'\subset U''$ and $q \in V''\subset U''$. But $(V_{ni}, p, q) \in \mathscr{C}_2$. Thus, we have shown that there is a member of $\mathscr{C}_2$ having as its first element, a subset of U, as its second element a point of U', and as its third element a point of U''.

We now obtain a C-collection $\mathscr{C}$ for X by letting $\mathscr{C}=\mathscr{C}_1 \cup \mathscr{C}_2$.

LEMMA 4.2. *Suppose that X is a locally connected generalized continuum and that f is a* 1-1 *mapping of X onto E^2. If A is an arc in X and ε is a positive number, then there exist*

(1) *a locally connected generalized continuum X',*

(2) *a closed monotone mapping μ of X' onto X,*

(3) *a* 1-1 *mapping g of X' onto E^2, and*

(4) *a compact, uniformly continuous ε-mapping ϕ of E^2 onto E^2,*

such that $f\mu=\phi g$ and $\mu^{-1}(A)$ is a disc in X'.

Proof. Let B denote the straight line interval $\{(x, y) \mid -1\leqq x\leqq 1, y=0\}$ in E^2, and let Q denote the square disc $\{(x, y) \mid -1\leqq x\leqq 1, 0\leqq y\leqq 2\}$. Since f is 1-1 and continuous, A is taken homeomorphically onto an arc in E^2. We shall first consider the special case in which $f(A)=B$ and then proceed to the more general case.

Case 1. $f(A)=B$. Choose a positive number ε_0 such that $\varepsilon_0<\min\{1, \varepsilon\}$. Let D_1 denote the disc in E^2 bounded by B and the curve $y=\varepsilon_0(1-x^2)$ $(-1\leqq x\leqq 1)$, and let D_2 denote the disc bounded by B and $y=2\varepsilon_0(1-x^2)$. Now define the function ϕ of E^2 onto E^2 as follows:

$$\begin{aligned}\phi(x, y) &= (x, y) && \text{if } (x, y) \notin D_2,\\ &= (x, 0) && \text{if } (x, y) \in D_1,\\ &= (x, 2(y-\varepsilon_0[1-x^2])) && \text{if } (x, y) \in D_2-D_1.\end{aligned}$$

The function is clearly continuous and, since $\varepsilon_0 < \varepsilon$, it follows that ϕ is an ε-mapping. Since the restriction of ϕ to $E^2 - D_2$ is the identity mapping, ϕ is uniformly continuous; and since $\phi^{-1}(H)$ is bounded whenever H is a bounded set in E^2, ϕ is a compact mapping. The set $E^2 - D_1$ is taken homeomorphically onto $E^2 - B$ and $\phi^{-1}(B)$ is the disc D_1. For each $(x, y) \in E^2$, $\phi^{-1}(x, y)$ is either a point or a vertical arc. Since $\varepsilon_0 < 1$, D_2 is a subset of Q; hence, $\phi(Q) = Q$ and $\phi|(E^2 - Q)$ is the identity mapping.

Because of the local compactness and local connectedness of X, we can choose a conditionally compact, connected open set U in X with $A \subset U$. Then $f|\bar{U}$ is a homeomorphic embedding of $\bar{U}$ into E^2; and $\phi^{-1}f$ takes $\bar{U} - A$ homeomorphically onto $\phi^{-1}f(\bar{U}) - D_1$. The set $\phi^{-1}f(U)$ is connected, locally connected, and locally compact. Now let T be a topological space homeomorphic to $\phi^{-1}f(\bar{U})$ and such that $T \cap X = \phi$. Let θ be a homeomorphism of T onto $\phi^{-1}f(\bar{U})$. We now define X' to be the topological space obtained from the topological sum of $X - U$ and T by identifying each point of bd U with its image under $\theta^{-1}\phi^{-1}f$. Then X' is connected, locally connected, and locally compact.

We define the function μ of X' onto X as follows:

$$\begin{aligned} \mu(p) &= f^{-1}\phi\theta(p) && \text{if } p \in T, \\ &= p && \text{if } p \notin T. \end{aligned}$$

Then μ is continuous and $\mu^{-1}(A)$ is the disc $\theta^{-1}(D_1)$. For each $q \in X$, $\mu^{-1}(q)$ is either a point or an arc; hence, μ is monotone. For each point q of X and each open set V in X' with $\mu^{-1}(q) \subset V$, there is an open set W in X such that $q \in W$ and $\mu^{-1}(W) \subset V$. Thus, μ is a closed mapping.

Now, define the function g of X' onto E^2 in the following manner:

$$\begin{aligned} g(p) &= \theta(p) && \text{if } p \in T, \\ &= \phi^{-1}f(p) && \text{if } p \notin T. \end{aligned}$$

Then g is a 1-1 mapping of X' onto E^2 such that $f\mu = \phi g$.

By Theorem 3.9, X' is metrizable and, therefore, may be regarded as being a locally connected generalized continuum.

Case 2. *$f(A)$ is any arc in E^2.* Let h be a homeomorphism of E^2 onto E^2 such that $(hf)(A) = B$. Then the restriction of h^{-1} to Q is a uniformly continuous mapping. Choose a positive number δ such that, for $z_1, z_2 \in Q$, $\rho(h^{-1}(z_1), h^{-1}(z_2)) < \varepsilon$ if $\rho(z_1, z_2) < \delta$. Now, using the same procedure as was used in Case 1, we can find

(1) a locally connected generalized continuum X',

(2) a closed monotone mapping μ of X' onto X,

(3) a 1-1 mapping g_* of X' onto E^2, and

(4) a compact, uniformly continuous δ-mapping ϕ_* of E^2 onto E^2,

such that $(hf)\mu = \phi_* g_*$, $\mu^{-1}(A)$ is a disc in X', $\phi_*(Q) = Q$, and $\phi_*|(E^2 - Q)$ is the identity mapping. Let $g = h^{-1}g_*$ and let $\phi = h^{-1}\phi_* h$. Clearly g is a 1-1 mapping of X' onto E^2 and ϕ is a mapping of E^2 onto E^2. We also have

$$f\mu = h^{-1}(hf)\mu = h^{-1}\phi_* g_* = (h^{-1}\phi_* h)(h^{-1}g_*) = \phi g.$$

It only remains to be shown that ϕ is a compact, uniformly continuous ε-mapping. The compactness of ϕ follows from the compactness of ϕ_* and the fact that h is a homeomorphism. Since ϕ_* is the identity on E^2-Q, ϕ must be the identity on $h^{-1}(E^2-Q)$, i.e., on $E^2-h^{-1}(Q)$; thus, ϕ is uniformly continuous. If $p\in h^{-1}(Q)$ then $h(p)\in Q$, $\phi_*h(p)\in Q$, and (since ϕ_* is a δ-mapping) $\rho(h(p), \phi_*h(p))<\delta$; hence, $\rho(h^{-1}h(p), h^{-1}\phi_*h(p))<\varepsilon$, i.e., $\rho(p, \phi(p))<\varepsilon$. Since $\phi(p)=p$ for each p in $E^2-h^{-1}(Q)$, we conclude that ϕ is an ε-mapping.

THEOREM 4.3. *If X is a locally connected generalized continuum and f is a* 1-1 *mapping of X onto E^2 then f is a homeomorphism.*

Proof. To prove this theorem we will construct two inverse systems, $\langle X_n, \mu_n^m\rangle$ and $\langle Y_n, \phi_n^m\rangle$, of topological spaces over the positive integers and a mapping $\langle f_n\rangle$ of $\langle X_n, \mu_n^m\rangle$ into $\langle Y_n, \phi_n^m\rangle$ such that

(1) $X_1=X$, $Y_1=E^2$, and $f_1=f$,

(2) the induced mapping f_∞ of X_∞ into Y_∞ is a homeomorphism of X_∞ onto Y_∞, and

(3) the projection mapping ϕ_1 (of Y_∞ into Y_1) is compact, and the projection mapping μ_1 takes X_∞ onto X_1.

We will then be able to conclude that f is a compact mapping and, therefore, a homeomorphism.

(i) *Construction of* $\langle X_n, \mu_n^m\rangle$, $\langle Y_n, \phi_n^m\rangle$, *and* $\langle f_n\rangle$. We will define the required spaces and mappings inductively, beginning with X_1, μ_1^1, Y_1, ϕ_1^1, and f_1. Each Y_n will be E^2, each ϕ_n^m will be uniformly continuous, and each X_n will be a locally connected generalized continuum. In order to continue at each stage, it will be necessary that for each n we choose an infinite C-collection $\{(V_{nj}, p_{nj}, q_{nj}) \mid j=1, 2, 3, \dots\}$ for X_n immediately after defining X_n. We proceed as follows.

Let Z^+ denote the set of positive integers, and let σ be a 1-1 function from Z^+ onto $Z^+\times Z^+$ such that $\sigma(1)=(1, 1)$ and such that, for each $n\in Z^+$, n is not less than the first element of $\sigma(n)$.

Now let $X_1=X$, $Y_1=E^2$, and $f_1=f$. Choose an infinite C-collection $\{(V_{1j}, p_{1j}, q_{1j}) \mid j=1, 2, 3, \dots\}$ for X_1 (Lemma 4.1). Let μ_1^1 and ϕ_1^1 be, respectively, the identity mapping on X_1 and the identity mapping on Y_1.

Let A_1 be an arc from p_{11} to q_{11} in V_{11}. Choose a positive number $\varepsilon_1<1/6$. By Lemma 4.2 there exist

(1) a locally connected generalized continuum X_2,

(2) a closed, monotone mapping μ_1^2 of X_2 onto X_1,

(3) a 1-1 mapping f_2 of X_2 onto E^2, and

(4) a compact, uniformly continuous ε_1-mapping ϕ_1^2 of E^2 onto E^2,

such that $f_1\mu_1^2=\phi_1^2 f_2$ and such that $(\mu_1^2)^{-1}(A_1)$ is a disc in X_2. Let

$$\{(V_{2j}, p_{2j}, q_{2j}) \mid j=1, 2, 3, \dots\}$$

be an infinite C-collection for X_2. Let $Y_2=E^2$, and let μ_2^2 be the identity mapping on X_2 and ϕ_2^2 the identity mapping on Y_2.

At each $(n+1)$th stage $(n \geqq 2)$, we proceed as follows.

Let $\sigma(n)=(i, k)$. Then $i \leqq n$, which implies that the collection

$$\{(V_{ij}, p_{ij}, q_{ij}) \mid j = 1, 2, 3, \ldots\}$$

has been chosen. From Theorem 3.5, it follows that $(\mu_i^n)^{-1}(V_{ik})$ is a connected open set in X_n. Let A_n be an arc in $(\mu_i^n)^{-1}(V_{ik})$ having one endpoint in $(\mu_i^n)^{-1}(p_{ik})$ and the other in $(\mu_i^n)^{-1}(q_{ik})$. Now choose a positive number $\varepsilon_n < \frac{1}{2}\varepsilon_{n-1}$ such that, for $z, z' \in E^2$ and $1 \leqq j \leqq n$, $\rho(\phi_j^n(z), \phi_j^n(z')) < 1/n$ whenever $\rho(z, z') < 6\varepsilon_n$. (The uniform continuity of each ϕ_j^n makes the choosing of such an ε_n possible.) Then, by Lemma 4.2, there exist

(1) a locally connected generalized continuum X_{n+1},

(2) a closed, monotone mapping μ_n^{n+1} of X_{n+1} onto X_n,

(3) a 1-1 mapping f_{n+1} of X_{n+1} onto E^2, and

(4) a compact, uniformly continuous ε_n-mapping ϕ_n^{n+1} of E^2 onto E^2,

such that $f_n\mu_n^{n+1}=\phi_n^{n+1}f_{n+1}$ and such that $(\mu_n^{n+1})^{-1}(A_n)$ is a disc in X_{n+1}. Choose an infinite C-collection $\{(V_{(n+1)j}, p_{(n+1)j}, q_{(n+1)j}) \mid j=1, 2, 3, \ldots\}$ for X_{n+1}. Let $Y_{n+1}=E^2$ and let μ_{n+1}^{n+1} be the identity mapping on X_{n+1} and ϕ_{n+1}^{n+1} the identity mapping on Y_{n+1}. For $1 \leqq j \leqq n$ let $\mu_j^{n+1}=\mu_j^n\mu_n^{n+1}$ and $\phi_j^{n+1}=\phi_j^n\phi_n^{n+1}$.

(ii) *X_∞ is a locally connected generalized continuum having no local separating point.*

Proof of (ii). The connectedness and local connectedness of X_∞ follow, respectively, from Theorems 3.6 and 3.7. By Theorem 3.1, each μ_n^{n+1} is a compact mapping, and therefore, by Theorem 3.4, μ_1 is a compact mapping of X_∞ into X_1. Since X_1 is locally compact, then, X_∞ is locally compact (Theorem 3.3).

Since X_∞ is a subspace of the product of countably many metric spaces, X_∞ is metrizable (see [5, Corollary 7.3, p. 191]). Hence, X_∞ may be regarded as being a locally connected generalized continuum.

Now assume that some point $\langle x_n \rangle$ of X_∞ is a local separating point of X_∞. Let W be a connected open set in X_∞ such that $W-\langle x_n \rangle$ is not connected. It follows from [6, Lemma 3.12, p. 218] that there is a positive integer m and a connected open set U_m in X_m such that

$$\langle x_n \rangle \in (\mu_m)^{-1}(U_m) \subset W.$$

Let $U=(\mu_m^{-1})(U_m)$ and for each integer $i>m$ let $U_i=(\mu_m^i)^{-1}(U_m)$. By Theorem 3.6, U must be connected and, therefore, separated by $\langle x_n \rangle$. Let $\langle x_n' \rangle$ and $\langle x_n'' \rangle$ be points of $U-\langle x_n \rangle$ such that $\langle x_n \rangle$ separates $\langle x_n' \rangle$ from $\langle x_n'' \rangle$ in U. Since $\langle x_n \rangle$, $\langle x_n' \rangle$, and $\langle x_n'' \rangle$ are distinct points of X_∞, we can choose a positive integer $r \geqq m$ such that x_r, x_r', and x_r'' are distinct points of X_r. Now U_r is a connected open set (Theorem 3.5), and x_r' cannot be in the same component of U_r-x_r as is x_r'' (for otherwise, by Theorem 3.6, $\langle x_n' \rangle$ and $\langle x_n'' \rangle$ would be in the same component of the subset $(\mu_r)^{-1}(U_r-x_r)$ of $U-\langle x_n \rangle$). Hence, x_r separates x_r' from x_r'' in U_r. Similarly, for each $i>r$, U_i is a connected open set in X_i and x_i separates x_i' from x_i'' in U_i. Let

U_r' and U_r'' denote the components of $U_r - x_r$ which contain, respectively, x_r' and x_r''. Letting $U_i' = (\mu_r^i)^{-1}(U_r')$ and $U_i'' = (\mu_r^i)^{-1}(U_r'')$ for each $i > r$, we have (for each $i \geqq r$) that each of U_i' and U_i'' is a connected set in X_i (Theorem 3.5). Since $\{(V_{rj}, p_{rj}, q_{rj}) \mid j = 1, 2, 3, \ldots\}$ is a C-collection for X_r, there is a positive integer k such that $V_{rk} \subset U_r$, $p_{rk} \in U_r'$, and $q_{rk} \in U_r''$. For some integer $s \geqq r$, $\sigma(s) = (r, k)$ and A_s is an arc in $(\mu_r^s)^{-1}(V_{rk})$ with one endpoint in U_s' and the other in U_s''. This means that $(\mu_s^{s+1})^{-1}(A_s)$ is a disc in U_{s+1} which intersects each of the connected sets U_{s+1}' and U_{s+1}''. But, since no point separates a disc and since $x_{s+1} \notin U_{s+1}' \cup U_{s+1}''$, this implies that x_{s+1} does not separate U_{s+1}' from U_{s+1}'' in U_{s+1}. We have already shown, however, that, for each $i \geqq r$, x_i separates x_i' from x_i'' in U_i. Since $x_{s+1}' \in U_{s+1}'$ and $x_{s+1}'' \in U_{s+1}''$, this gives us a contradiction. We conclude that X_∞ has no local separating point.

(iii) *Y_∞ is a topological plane.*

Proof of (iii). We will show that the inverse system $\langle Y_n, \phi_n^m \rangle$ and the sequence $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ satisfy the hypothesis of Theorem 3.8.

Since for each n, $\varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$, we have $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. And for each n, $Y_n = E^2$ and ϕ_n^{n+1} is an ε_n-mapping.

Let n be a positive integer and let z and z' be points of E^2 such that $\rho(z, z') < 3 \sum_{j=n}^{\infty} \varepsilon_j$. Then

$$\rho(z, z') < 3 \sum_{j=n}^{\infty} (\tfrac{1}{2})^{j-n} \varepsilon_n = 3\varepsilon_n \sum_{j=0}^{\infty} (\tfrac{1}{2})^j = 6\varepsilon_n.$$

Because of the way in which ε_n was chosen, this implies that $\rho(\phi_i^n(z), \phi_i^n(z')) < 1/n$ for each positive integer $i \leqq n$.

Hence, by Theorem 3.8, Y_∞ is homeomorphic to E^2.

(iv) *The induced mapping f_∞ is a homeomorphism of X_∞ onto Y_∞.*

Proof of (iv). Since, for each n, f_n is a 1-1 mapping of X_n onto Y_n, it follows from [6, Theorem 3.15, p. 219] that f_∞ is a 1-1 mapping of X_∞ onto Y_∞. But X_∞ is a locally connected generalized continuum having no local separating point, and Y_∞ is a topological plane. Hence, by Dickman's theorem in [2], f_∞ is a homeomorphism.

(v) *The induced mapping ϕ_1 (of Y_∞ into Y_1) is compact and the induced mapping μ_1 takes X_∞ onto X_1.*

Proof of (v). For each n, ϕ_n^{n+1} is a compact mapping of Y_{n+1} onto Y_n. Therefore, by Theorem 3.4, ϕ_n is a compact mapping.

For each n, μ_n^{n+1} is a mapping of X_{n+1} onto X_n. Hence, μ_1 takes X_∞ onto X_1 (see [6, Remark, p. 216]).

(vi) *The mapping f is a homeomorphism.*

Proof of (vi). By [6, Lemma 3.11, p. 218], we have $f_1\mu_1 = \phi_1 f_\infty$. Since ϕ_1 is a compact mapping and f_∞ is a homeomorphism of X_∞ onto Y_∞, $\phi_1 f_\infty$ is a compact mapping, i.e., $f_1\mu_1$ is a compact mapping.

Let K be a compact set in E^2. Then $(f_1\mu_1)^{-1}(K)$ is compact and, since μ_1 is continuous, the image of $(f_1\mu_1)^{-1}(K)$ under μ_1 is compact. But, since μ_1 takes

X_∞ *onto* X_1, the image of $(f_1\mu_1)^{-1}(K)$ under μ_1 is simply $f_1^{-1}(K)$. Thus, K has a compact inverse image with respect to f_1. We conclude that f_1 is a compact mapping.

Since f_1 is compact, it is also closed (Theorem 3.2). Hence, $f\,(=f_1)$ is a closed 1-1 mapping of X onto E^2; i.e., f is a homeomorphism of X onto E^2.

THEOREM 4.4. *If X is a connected, locally connected, locally compact topological space and f is a* 1-1 *mapping of X onto E^2 then f is a homeomorphism.*

Proof. By Theorem 3.9, X is metrizable and, therefore, may be considered to be a locally connected generalized continuum. Hence, by Theorem 4.3, f is a homeomorphism.

5. **Compactness of mappings onto the plane.** In this section we shall be concerned with mappings which generate upper-semicontinuous decompositions. For the necessary definitions and further references the reader is referred to [4], [14] and [16, pp. 122–136].

THEOREM 5.1. *Suppose that X is a connected, locally connected, locally compact topological space and that f is a mapping of X onto E^2. If f has compact point inverses and generates an upper-semicontinuous decomposition of X, then f is a compact mapping.*

Proof. By [14, Theorem 5, p. 71], f factors into the form $h\phi$, where ϕ is a closed mapping and h is a 1-1 mapping. Since local connectedness is invariant under closed mappings (see [5, 1.4, p. 121 and 3.5, p. 125] or [18, p. 91]), and since local compactness is invariant under closed mappings with compact point inverses (see [5, 6.6, p. 240]), it follows that $\phi(X)$ is a connected, locally connected, locally compact topological space. By Theorem 4.4, then, h is a homeomorphism. Since ϕ is compact (Theorem 3.1), we conclude that f is compact.

In [4], Duda defines a mapping f to be *reflexive compact* provided that, for each compact set K in the domain space, $f^{-1}f(K)$ is compact. He then shows [4, Theorem 3, p. 689] that a mapping with compact point inverses generates an upper-semicontinuous decomposition if and only if it is reflexive compact. In light of this result, then, Theorem 5.1 can be equivalently restated as follows.

THEOREM 5.2. *If X is a connected, locally connected, locally compact topological space and f is a reflexive compact mapping of X onto E^2 then f is a compact mapping.*

G. T. Whyburn has shown [15, Theorem 5.1, p. 312] that every monotone mapping of E^2 onto itself is a compact mapping. Since every monotone mapping generates an upper-semicontinuous decomposition (see [4, p. 688], [16, p. 127]), the following more general result is an immediate corollary to Theorem 5.1.

THEOREM 5.3. *If X is a connected, locally connected, locally compact topological space and f is a monotone mapping of X onto E^2 then f is a compact mapping.*

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