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ON THE EXISTENCE OF TRIVIAL INTERSECTION SUBGROUPS

BY
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Abstract. Let G be a transitive nonregular permutation group acting on a set X, and let H be the subgroup of G fixing some element of X. Suppose each nonidentity element of H fixes exactly b elements of X. If $b=1$, G is a Frobenius group, and it is well known that H has only trivial intersection with its conjugates. If $b>1$, it is shown that this conclusion still holds, provided H satisfies certain natural conditions. Applications to the study of Hall subgroups and certain simple groups related to Zassenhaus groups are given.

1. A transitive permutation group G in which every nonidentity element fixes either one or no letters is called a Frobenius group. If H is the stabilizer of a single letter in such a group, it is easy to see that H is a trivial intersection (t.i.) subgroup of G, that is $H \cap H^x = 1$ unless $H = H^x$. Theorem A gives a condition for the existence of t.i. subgroups in permutation groups which are generalizations of Frobenius groups.

If π is a set of primes, x is a π-element of a group G if $o(x)$, the order of x, is divisible only by primes in π. A subgroup H of G is a π-subgroup if every element of H is a π-element; if also the order $|H|$ of H is prime to the index $|G:H|$ of H in G, H is a π-Hall subgroup. Let π' denote the collection of all primes not in π. A group G is called π-isolated if no π-element commutes with a π'-element.

Our main theorem is

THEOREM A. *Let G be transitive on X. Suppose that the subgroup H of G of all elements fixing one letter in X is π-Hall in G, and G is π-isolated. Then if every nonidentity element of H fixes exactly b letters, H is a trivial intersection subgroup of G. If $b>1$, H is also nilpotent, and $|N_G(H):H|=b$.*

The proof of this result depends on the description of partitioned groups, given in §2. In §3, the proof of Theorem A is given, and we conclude with some applications of this result.

All groups considered are finite. Basic notation, and statements of theorems quoted by name, may be found in [7]. We note that a group G is a Frobenius group if and only if G contains a proper trivial intersection subgroup H which is its own normalizer in G. A classical theorem of Frobenius' asserts that in this case, G

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contains a normal subgroup K, with the properties $K \cap H = 1$, $KH = G$. We will call K the Frobenius kernel and H the Frobenius complement of G.

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2. A partition of a group G is a collection σ of subgroups of G such that every nonidentity element of G occurs in one and only one subgroup in the set σ. A partition σ is called nontrivial if it does not consist of $\{G\}$ alone; σ is normal if, for each U in σ, U^x is in σ, for all x in G. If σ, τ are partitions of G and g is in G, then $\sigma^g = \{U^g : U \in \sigma\}$ and $\sigma \cap \tau = \{U \cap V : U \in \sigma, V \in \tau\}$ are also partitions of G. It is thus evident that any group admitting a nontrivial partition admits a nontrivial normal partition. If U is contained in σ, a partition of G, we call U a component of σ. The components of a normal partition are t.i. subgroups.

Combining results due to Baer, Kegel, and Suzuki, we may give the following description of groups admitting nontrivial normal partitions.

THEOREM 1. *Let G be a finite group having a nontrivial normal partition. Then one of the following cases holds*:

(a) *G is isomorphic to a Suzuki group $Sz(q)$ or a linear fractional group $PSL(2, q)$.*

(b) *G is isomorphic to S_4, the symmetric group on four letters.*

(c) *G is a Frobenius group.*

(d) *G is a p-group.*

(e) *G contains a nilpotent normal subgroup K of prime index p in G, every element of G not in K has order p and is centralized only by p-elements, and p divides $|K|$.*

Proof. Let N be the maximal nilpotent normal subgroup of G. If $N=1$, G is nonsolvable; then by Suzuki [13], case (a) holds. If N is nontrivial but contains no Sylow subgroup of G, Theorem B and Theorem A of Baer [2] give case (b). Finally, if N contains a Sylow subgroup of G, then one of the remaining cases must hold, by Baer [1, Satz 5.1].

We will also need the following lemmas from [1].

LEMMA 1. *If G is a Frobenius group, then a Frobenius complement is a component in any nontrivial normal partition of G.*

LEMMA 2. *If σ is a partition of G, and x and y are commuting elements of G of distinct orders, then x, y are in the same component of σ.*

Proof. We may assume by symmetry that $o(x) < o(y)$; let $m = o(x)$. Then $(xy)^m = y^m \neq 1$. Since xy and y have a common power $y^m \neq 1$, xy, y and hence x lie in the same component of σ.

3. The following theorem, due to Suzuki, is needed for the proof of Theorem A.

THEOREM 2. *Let P be a p-Sylow subgroup of G. Suppose that for any $x \neq 1$ in P, $C_G(x)$ is a p-group, and $x^p = 1$; then P is a t.i. subgroup of G.*

Proof. If P is abelian, P is t.i. since for each $x \neq 1$ in P, $P = C_G(x)$. For $p=2$, $x^2=1$ for all x implies P is abelian, so we need only consider p odd, and P nonabelian.

If P contains any nonidentity element conjugate to its inverse, P is abelian, by Theorem (4D) of Brauer and Fowler [4]. Let D be maximal among all nontrivial intersections of P with one of its distinct conjugates P', and define $M = N_G(D)$. Since P is nonabelian, M has odd order, and is solvable by the Theorem of Feit and Thompson [6]. Since the p'-elements of M centralize no nonidentity element of D, the q-Sylow subgroups of M are cyclic for $q \neq p$ [7, p. 200]. Theorem B of Hall and Higman [8] now applies to M; thus there is a normal p'-subgroup K of M such that M/K has a normal p-Sylow subgroup. Since K must then centralize D, $K=1$, and M has a normal p-Sylow subgroup. However, P and P' can be chosen so that $P \cap M$ and $P' \cap M$ are p-Sylow subgroups of M. Thus $P \cap M = P' \cap M = D$, and $D = N_P(D)$, contrary to $D \neq P$. Thus P is a t.i. subgroup of G.

Throughout the rest of this section we will assume the following hypothesis:

(H1) G is a transitive permutation group acting on the set X. The subgroup H of all elements fixing some element α of X is a π-Hall subgroup of G, and G is π-isolated. Every element $x \neq 1$ of H fixes exactly b elements of X, $b > 1$.

LEMMA 3. *If* (H1) *holds for G, b is a π'-number, and every π-element of G fixes b letters of X.*

Proof. Let p be a prime in π, and let x be an element of order p in H. The set X contains $|G:H|$ elements, and x fixes b elements and permutes the rest in orbits of length p. Thus p divides $|G:H|-b$. Since p is relatively prime to $|G:H|$, p is relatively prime to b.

For the second part, suppose x is a π-element fixing no points in X. Then x is not of prime-power order, for then a conjugate of x would lie in a Sylow subgroup of H. Thus there are commuting π-elements u, v of coprime order with $x = uv$. By induction on $o(x)$, we may assume u and v both fix points of X. If u has no fixed points in common with v, u permutes the fixed points of v in orbits of length $o(u)$, and thus $o(u)$ divides b, a contradiction. Thus u and v have a common fixed point, which is a fixed point of $x = uv$, proving the lemma.

The next lemma allows us to apply §2 to our problem.

LEMMA 4. *Suppose G satisfies* (H1). *If H is not a t.i. subgroup of G, G contains a collection of t.i. subgroups which induce a nontrivial normal partition of H.*

Proof. Let $x \neq 1$ be any π-element of G, and let D_x denote the subgroup of all elements of G fixing the fixed points of x. Let x, and y be π-elements of G. Then if $t \neq 1$ is in both D_x and D_y, x and y have the same b fixed points, so $D_x = D_y$. Thus the subgroups D_x are t.i. subgroups of G. If $H \neq D_x$ for some x, H is nontrivially partitioned by the subgroups D_h, $h \neq 1$ in H.

Proof of Theorem A. If $b=1$, the theorem is the classical case; so we assume $b > 1$. If $H=1$, H is a t.i. subgroup; the last paragraph of the proof below applies.

We adopt (H1) on G; suppose H is not a t.i. subgroup of G. Then H is partitioned, and one of the cases of Theorem 1 describes H. We will show that none of the cases (a)–(e) can occur.

(i) *H is not $Sz(q)$, $PSL(2, q)$, or S_4.* The possible partitions of these groups are given in Suzuki [13] and Baer [2]. If H is one of these groups, a cyclic Hall subgroup K of odd order is a component of the partition of H, and $N_H(K)$ is a dihedral group. By a theorem of Jordan [15, p. 6], $N_G(K)$ is transitive on the b fixed points of K, so $b=|N_G(K):N_H(K)|$. Since the automorphism group of an odd-order cyclic group is abelian, and $b\neq 1$, this contradicts the π-isolation of G.

(ii) *H is not a Frobenius group.* If H is a Frobenius group, a Frobenius complement C is a component of the partition of H, by Lemma 1. Let K be the Frobenius kernel of H; by a theorem of Thompson [14], K is nilpotent. The partition of H induces a partition of K. We will show that there is a unique component U of the partition containing $Z(K)$. Let $x\neq 1$ be contained in $Z(K)$. If there is an element $y\neq 1$ in K with $o(x)\neq o(y)$, x and y lie in the same component U by Lemma 2. Then since U contains two elements of unequal order, $Z(K)\leqq U$. Otherwise every element in K has order p, and K is a p-Sylow subgroup of H and hence of G. If any p'-element centralized a p-element, K would contain p'-elements. Thus the hypotheses of Theorem 2 hold for K, and K is a t.i. subgroup of G. Now choose $k\neq 1$ in K, and regarding the points of X as right cosets of H, let Ht be a point fixed by k. Then $Htk=Ht$, so $tkt^{-1}\in H$. Since K is the unique p-Sylow subgroup of H, $tkt^{-1}\in K$, and since K is a t.i. subgroup, $tKt^{-1}=K$. It now is clear that $HtK=Ht$, so any fixed point of k is a fixed point for K, that is $K=U$ is a component of H containing $Z(K)$.

Now consider the (unique) component U containing $Z(K)$. Since $Z(K)$ is a characteristic subgroup of K, U is normal in H. Thus U is normalized by the complement C of H. By the previously-cited theorem of Jordan, $M=N_G(U)$ is transitive on the fixed points of U, so $H<M$, and M contains π'-elements. If $C=N_M(C)$, then M is a Frobenius group with complement C, since C is a t.i. subgroup of G. The Frobenius kernel of M is nilpotent by Thompson's Theorem, so there are π'-elements commuting with π-elements in G, contrary to π-isolation. However, if there is a π'-element x of M normalizing C, x normalizes CU, and since x centralizes no π-element, CU is nilpotent, contrary to hypothesis. Finally suppose $N_M(C)$ contains a π-element x of prime power order. Then x permutes the b fixed points of C, and by Lemma 3, must fix one of them. Thus $\langle x, C\rangle$ is isomorphic to a subgroup of H; since $N_H(C)=C$, we have x in C. Thus $N_M(C)\neq C$ is impossible; this disposes with the present case.

(iii) *H is a t.i. subgroup.* We have shown that if this is false, H is partitioned, and only cases (d) and (e) of Theorem 1 can possibly occur. In both these cases, $Z(H)\neq 1$. Let x be an element of prime order p in $Z(H)$. By π-isolation, $H=C(x)$, so we may regard X as the set of all conjugates x^t of x, t ranging over G. The permutation condition in (H1) now becomes

(H2) Every element $y \neq 1$ of H centralizes b conjugates of x.

In particular, x satisfies (H2), so H contains b conjugates of x. Let D be the component of the partition of H containing x. We will show that H is partitioned by conjugates of D. Let $y \neq 1$ be any element of H. If $o(y) \neq p$, then y is in D, by Lemma 2. Suppose $o(y)=p$. If $C_G(y)$ is not a p-group, there is a p'-element z with $yz=zy$. By π-isolation, z is a π-element, and hence yz has fixed points, which must be the same as those of y and z. Thus z centralizes x, and y, z, and x all lie in the same component D. If $o(y)=p$, and $C_G(y)$ is a p-group, $C_G(y)$ lies in some p-Sylow subgroup P of G. Since H is π-Hall, for some t, $C_G(y) \leqq P \leqq H^t$. By (H2), $X \cap C_G(y)$ and $X \cap H^t$ both contain b conjugates of x. Thus y and x^t have the same fixed points, and y lies in D^t.

Since H is partitioned by isomorphic subgroups, from Theorem 1 and Lemma 3 we see that H is a p-group which satisfies $z^p=1$ for all z in H. Now H satisfies the hypothesis of Theorem 2, so H is a t.i. subgroup of G.

We have proven that H is a t.i. subgroup. It then follows that all elements of H fix the same b letters, so by the theorem of Jordan, $b=|N_G(H):H|$. As $b \neq 1$, and H is a π-Hall subgroup, H is normalized by π'-elements, which centralize no elements of H. Thus $N_G(H)$ is a Frobenius group, and H is nilpotent by Thompson's Theorem. This completes the proof of Theorem A.

The hypotheses of Theorem A cannot be weakened significantly without invalidating the conclusion. For the following examples, let G be the group $PSL(2, 11)$, the structure of which can be found in [10, II.8]. Consider the transitive representation of G on right cosets $\{Hx\}$ of a 2-Sylow subgroup H. Each element of H fixes 9 cosets, but H is not a t.i. subgroup, since G contains a subgroup D isomorphic to a dihedral group of order 12.

For $\pi=\{2, 3\}$, D is a π-Hall subgroup and G is π-isolated, but D is not a t.i. subgroup. Even the requirement that G is π-isolated, contains a π-Hall subgroup, and contains subgroups partitioning the π-elements is not enough. The subgroups B isomorphic to $PSL(2, 5)$ contained in $PSL(2, 11)$ illustrate this. In spite of these examples, the use of §2 in studying permutation groups in which any element $x \neq 1$ fixing one letter fixes b letters might lead to a partial description of these groups.

4. As an application of our previous results, we give a criterion for the existence of abelian Hall subgroups.

THEOREM B. *If G is a group in which the centralizer of every nonidentity π-element is a π-Hall subgroup, G has an abelian π-Hall subgroup which is the centralizer of each of its nonidentity elements.*

Proof. It is enough to show G has an abelian π-Hall subgroup, as the remainder of the conclusion follows easily. Suppose first that π contains more than one prime. Let x be a π-element of prime order p, and let Q be any q-Sylow subgroup of $C(x)$, $q \neq p$. If $y \neq 1$ is any element in Q, $C(xy)=C(x)=C(y)$, since x and y are powers of xy. Thus $Q \leqq Z(C(x))$, and the conclusion follows, by interchanging p

and q. If $\pi=\{p\}$, every p-element of G is central in some p-Sylow subgroup of G. If b is the number of conjugates of some p-element $x\neq 1$ occurring in any p-Sylow subgroup P, then every nonidentity p-element centralizes b conjugates of x. Thus Theorem A applies, and P is a t.i. subgroup of G. Now P centralizes each of its nonidentity elements, and the conclusion follows.

Theorem A also has applications in the study of certain simple groups. We call a subgroup A of a group G a special subgroup if A is the centralizer of each of its nonidentity elements, and $|N_G(A):A|=2$.

Groups having special subgroups have been studied in [5], [9], and [11]; the known simple groups with this property are the groups $Sz(q)$, $PSL(3,4)$, and $PSL(2,q)$. Feit and Thompson determined those simple groups having a special subgroup of order 3. We will describe our results here, referring the reader to [9] for the necessary preliminaries.

Let G be a simple group containing a special subgroup of order $a\geqq 5$. Applying the method of exceptional characters to G, we obtain $(a-1)/2$ "exceptional" irreducible characters, and one "distinguished" irreducible character θ of G.

Employing knowledge of these characters and an idea of Suzuki's [12], a formula for $|G|$ may be given. This formula is $|G|=a\theta(1)(\theta(1)+\varepsilon)r^2$, where $\varepsilon=1$ or -1, and r is an integer with $r^2\equiv 1$ modulo a. If $r=1$ (this is the case in the known simple groups listed above), then much more information can be obtained on the conjugacy classes of G. In particular, every element has order dividing a, $\theta(1)$ or $\theta(1)+\varepsilon$. In this context, we offer the following generalizations of Theorem 1 of [9].

THEOREM C. *Let G be a simple group having a special subgroup of order $a\geqq 5$. If in the group order formula $|G|=a\theta(1)(\theta(1)+\varepsilon)r^2$, $r=\varepsilon=1$, and G has a subgroup of order $\theta(1)$, then G is isomorphic to either $Sz(q)$ or $PSL(2,q)$, with $q=\theta(1)$.*

REMARK. If $\theta(1)$ is even, or $\theta(1)$ is odd and $\theta(1)-\varepsilon\leqq 22a$, it is easy to show that G contains an element x with $|C_G(x)|=\theta(1)$, using the information on conjugacy classes given in [9].

Description of proof. Let D be the subgroup of order $\theta(1)$ of G. Employing the known values of the exceptional characters of G, it can be seen that G satisfies the hypotheses of Theorem A with respect to the permutation representation of G on cosets of D, with $b=|A|$. It is then easy to show that the permutation character of G acting on right cosets of $N_G(D)$ is $1+\theta$, so that G is doubly transitive. As the values of θ are known, we can show that in this representation, G is a Zassenhaus group. The theorem then follows from the classification of these groups.

THEOREM D. *If G is a simple group with a special subgroup of order a, and for some involution x in G, $|C_G(x)|\leqq 9a/2$, then G is isomorphic to $PSL(2,q)$, for some q.*

Description of proof. Using an idea of Brauer's [3], we can show that $|C_G(x)|\leqq 9a/2$ forces $r=1$ in the group order formula, aside from cases involving small values of a, which can be eliminated by separate arguments. Then by considering the distribution of conjugacy classes, we can see that every element of order dividing $\theta(1)$ has a centralizer of order $\theta(1)$, and that Theorem B applies to G. By considering elementary divisibility properties, we see that $|G|=a(2a+\varepsilon)(2a+2\varepsilon)$, and the conclusion follows from Theorem 1 of [9].

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