TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 158, Number 1, July 1971

MATCHING THEORY FOR COMBINATORIAL GEOMETRIES(1)

BY
MARTIN AIGNER AND THOMAS A. DOWLING

Abstract. Given two combinatorial (pre-) geometries and an arbitrary binary relation between their point sets, a matching is a subrelation which defines a bijection between independent sets of the geometries. The theory of matchings of maximum cardinality is developed in two directions, one of an algorithmic, the other of a structural nature. In the first part, the concept of an augmenting chain is introduced to establish as principal results a min-max type theorem and a generalized Marriage Theorem. In the second part, Ore's notion of a deficiency function for bipartite graphs is extended to determine the structure of the set of critical sets, i.e. those with maximum deficiency. The two parts of the investigation are then connected using the theory of Galois connections.

1. **Introduction.** A *geometric relation* is defined as a triple $(G(S), R, G'(T))$, where $G(S)$, $G'(T)$ are pregeometries (matroids) on point sets S, T, respectively, and $R \subseteq S \times T$ is an arbitrary binary relation from S to T. The simplest example of a geometric relation is a bipartite graph, in which $G(S)$, $G'(T)$ are free geometries. In the present paper, we consider several questions which originated historically with finite bipartite graphs, or with their equivalent representation as a family of subsets of a finite set. Some classical results of matching theory for bipartite graphs are extended to geometric relations.

A *matching* in a geometric relation $(G(S), R, G'(T))$ is a subset M of R, the elements of which define a bijection ϕ_M from an independent set of $G(S)$ to an independent set of $G'(T)$. We assume that both $G(S)$, $G'(T)$ have finite rank, from which it follows that any matching is finite. A *maximum matching* is one of maximum cardinality.

By a *support* of $(G(S), R, G'(T))$, we understand a pair (C, D) of closed sets in $G(S)$, $G'(T)$, respectively, which *cover* R in the sense that, for all $(c, d) \in R$, either $c \in C$ or $d \in D$ holds. The flats C, D are called the *elements* of the support (C, D). The rank $\rho(C, D)$ is the sum of the ranks of its elements, and a *minimum support* is one of minimum rank.

Received by the editors June 4, 1970.

AMS 1969 *subject classifications*. Primary 0504, 0535, 0540; Secondary 0460, 0630.

Key words and phrases. Combinatorial geometry, matroid, geometric relation, matching, augmenting chain, König-Egerváry theorem, Marriage Theorem, deficiency function, geometric lattice, Galois connection, distributive lattice.

(1) Some of the results contained in this paper appeared previously as a research announcement. Research was partially supported by the U.S. Air Force under Grant No. AFOSR-68-1406 and the National Science Foundation under Grant No. GU-2059.

For bipartite graphs our definitions reduce to the usual ones. In this case, the concepts of maximum matching and minimum support are related by the well-known König-Egerváry theorem. We extend this theorem to geometric relations (Theorem 2) in §3. The proof rests on a characterization (Theorem 1) of a maximum matching in terms of the nonexistence of an "augmenting chain". The latter concept originated with bipartite graphs (cf. Berge [2]), where it is associated with the "Hungarian method" for finding a maximum matching. Our definition extends the notion of an augmenting chain to geometric relations by means of the MacLane-Steinitz exchange property.

In §4, Ore's [8] definition of a *deficiency function* on subsets of S, for the case of a bipartite graph, is generalized to a geometric relation. The sets of maximal deficiency, called critical sets, are shown to form a ring. The open sets in this ring form a distributive sublattice of the lattice of open sets (Theorem 3).

The notion of maximal deficiency is applied in §5 to obtain an expression for the cardinality of a maximum matching (Theorem 4), a result proved by Ore [8] for bipartite graphs. As a corollary to Theorem 4, we obtain a generalization of the Marriage Theorem of P. Hall (see e.g. [6]) and Rado [9]. The minimal critical open set is characterized in terms of the effect on the maximal deficiency when contracting $G(S)$ by a point.

In §6, we investigate the structure of minimum supports. The relation R induces a dual Galois connection between the lattices of closed sets in $G(S)$, $G'(T)$, for which the elements of *irredundant* supports (defined in §6) are the coclosed elements, with the canonical anti-isomorphism between the quotient lattices specifying the corresponding elements in each such support. Among these, the elements of *minimum* supports are shown to form anti-isomorphic distributive sublattices of the lattices of closed sets of $G(S)$, $G'(T)$. As a consequence, we show that the minimum supports exhibit a distributive lattice structure (Theorem 5).

2. **Preliminaries.** Our primary reference for notation, definitions, and terminology is Crapo and Rota [4]. We summarize some basic concepts in the present section which will be needed later.

A *pregeometry* $G(S)$ consists of a set S together with a closure operator J on subsets of S enjoying the following properties:

(2.1) *Exchange property.* For any elements $a, b \in S$, and for any subset $A \subseteq S$, if $a \in J(A \cup b)$, $a \notin J(A)$, then $b \in J(A \cup a)$.

(2.2) *Finite basis property.* Any subset $A \subseteq S$ has a finite subset A_0 such that $J(A_0) = J(A)$.

We shall frequently denote the closure $J(A)$ of a subset $A \subseteq S$ simply by $\bar{A}$.

A set $A \subseteq S$ is *closed* if $A = \bar{A}$, and *open* if its complement in S is closed. A pregeometry $G(S)$ is *open* if the null set is closed, i.e. if S is open.

A *combinatorial geometry* (briefly, a geometry) is an open pregeometry $G(S)$ for which the elements $a \in S$, called *points*, are closed. Canonically associated with any pregeometry $G(S)$ is a geometry $G(S_0)$ whose points are equivalence classes of elements of $S-\bar{\phi}$, under the equivalence relation $a\sim b$ if and only if $\bar{a}=\bar{b}$.

Given a pregeometry $G(S)$ and subsets A, B of S with $A\subseteq B$ the *minor* $G_{[A,B]}$ is a pregeometry on the difference set $B-A$ with closure operator

$$J_{[A,B]}(C) = (J(C\cup A)\cap B)-A \quad \text{for } C\subseteq B-A. \tag{2.3}$$

Of particular importance among minors are the *restrictions* to sets $B\subseteq S$,

$$J_{[\varnothing,B]}(C) = \bar{C}\cap B \quad \text{for } C\subseteq B, \tag{2.4}$$

and the *contractions* to sets $S-A$,

$$J_{[A,S]}(C) = J(C\cup A)-A \quad \text{for } C\subseteq S-A. \tag{2.5}$$

The set $B\subseteq S$ is *independent* if it is a minimal set with given closure. By (2.2) any independent set is finite. The *rank* $r(A)$ of a set $A\subseteq S$ is defined as the cardinality of the largest independent subset of A. The rank function r satisfies the (upper) *semimodular inequality*

$$r(A_1\cup A_2)+r(A_1\cap A_2) \leqq r(A_1)+r(A_2). \tag{2.6}$$

For a minor $G_{[A,B]}$ of $G(S)$, the rank function is

$$r_{[A,B]}(C) = r(A\cup C)-r(A) \quad \text{where } C\subseteq B-A. \tag{2.7}$$

It follows from (2.7) that if C is independent in the minor $G_{[A,B]}$, then it is also independent in $G(S)$.

The closed sets, or *flats*, of a pregeometry $G(S)$, ordered by inclusion, form a geometric lattice $L(S)$ in which

$$C_1 \vee C_2 = J(C_1\cup C_2), \qquad C_1 \wedge C_2 = C_1\cap C_2. \tag{2.8}$$

The lattice $L(S)$ is anti-isomorphic to the lattice $M(S)$ of open sets. A canonical anti-isomorphism is provided by complementation $C\to S-C$ with respect to S.

The cardinality of any finite set A will be denoted by $\nu(A)$.

3. **Augmenting chains.** Throughout this paper, we consider an arbitrary (but fixed) geometric relation and denote it by $(G(S), R, G'(T))$. The rank functions of $G(S)$, $G'(T)$ will be denoted by r, r', respectively. The *converse geometric relation* of $(G(S), R, G'(T))$ is the relation $(G'(T), R', G(S))$ where $R'\subseteq T\times S$ is defined by $(b, a)\in R'$ if and only if $(a, b)\in R$. Most of the problems we consider will be symmetric with respect to $G(S)$ and $G'(T)$. The distinction between a geometric relation and its converse in such cases is unnecessary, but it will be convenient to distinguish the two on some occasions.

The relation R defines a function, which we also denote by R, from subsets of S to subsets of T, where, for any subset $A\subseteq S$,

$$R(A) = \{b\in T : (a, b)\in R \text{ for some } a\in A\}. \tag{3.1}$$

The function R is order-preserving between the Boolean algebras $B(S)$ and $B(T)$,

$$A_1 \subseteq A_2 \quad \text{implies} \quad R(A_1) \subseteq R(A_2), \tag{3.2}$$

preserves unions,

$$R(A_1 \cup A_2) = R(A_1) \cup R(A_2), \tag{3.3}$$

but not necessarily intersections,

$$R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2). \tag{3.4}$$

It follows from (3.2) to (3.4) and the semimodular inequality (2.6) for r' that the composite function $r'R$ is upper semimodular on subsets of S:

$$r'(R(A_1 \cup A_2)) + r'(R(A_1 \cap A_2)) \leqq r'(R(A_1)) + r'(R(A_2)). \tag{3.5}$$

The analogous definition and properties hold for the converse relation R'.

A matching M in $(G(S), R, G'(T))$ will be denoted alternatively by its corresponding bijection $\phi_M: A \to B$ when it is necessary to specify the independent sets A, B which are matched by M. Thus

$$M = \{(a, \phi_M(a)) : a \in A\}.$$

Definition. Given a matching $\phi_M: A \to B$ in $(G(S), R, G'(T))$, an *augmenting chain* with respect to M is a sequence

$$(a_0', b_1'), (b_1, a_1), (a_1', b_2'), \ldots, (b_n, a_n), (a_n', b_{n+1}') \tag{3.6}$$

of $2n+1$ $(n \geqq 0)$ distinct ordered pairs such that

$$\begin{aligned} &(a_i, b_i) \in M, && 1 \leqq i \leqq n. \\ &(a_i', b_{i+1}') \in R - M, && 0 \leqq i \leqq n. \end{aligned} \tag{3.7}$$

$$a_0' \in S - \bar{A}, \qquad b_{n+1}' \in T - \bar{B}. \tag{3.8}$$

$$\left.\begin{aligned} &a_i' \in \bar{A}, && a_i' \notin J\Big(\Big(A - \bigcup_{j=1}^{i} a_j\Big) \cup \bigcup_{j=1}^{i-1} a_j'\Big), \\ &b_i' \in \bar{B}, && b_i' \notin J\Big(\Big(B - \bigcup_{j=1}^{i} b_j\Big) \cup \bigcup_{j=1}^{i-1} b_j'\Big), \end{aligned}\right. \qquad 1 \leqq i \leqq n. \tag{3.9}$$

Note that if both $G(S)$, $G'(T)$ are free geometries, (3.9) implies that $a_i' = a_i$, $b_i' = b_i$ for $1 \leqq i \leqq n$, so that our definition reduces to that of an augmenting chain in a bipartite graph. We shall prove that M is a maximum matching in $(G(S), R, G'(T))$ if and only if there does not exist an augmenting chain with respect to M. The first step in the proof is

Proposition 1. *If a matching M admits an augmenting chain, it is not maximum.*

Proof. Let the chain be given by (3.6), and define

$$P = \{(a_i, b_i) : 1 \leqq i \leqq n\}, \qquad P' = \{(a_i', b_{i+1}') : 0 \leqq i \leqq n\}.$$

A straightforward inductive argument, using (3.9) and the exchange property, shows that

$$A'_i = \left(A - \bigcup_{j=1}^{i} a_j\right) \cup \bigcup_{j=1}^{i} a'_j$$

and

$$B'_i = \left(B - \bigcup_{j=1}^{i} b_j\right) \cup \bigcup_{j=1}^{i} b'_j$$

are independent sets with closure $\bar{A}$, $\bar{B}$, respectively, for $1 \leqq i \leqq n$. Thus by (3.8), the sets $A' = A'_n \cup a'_0$, $B' = B'_n \cup b'_{n+1}$ are independent sets, each of cardinality $\nu(M)+1$. It follows that $M' = (M-P) \cup P'$ is a matching of cardinality $\nu(M)+1$, so M is not maximum.

PROPOSITION 2. *If M is a matching and (C, D) is a support, then $\nu(M) \leqq \rho(C, D)$.*

Proof. By definition, $R(S-C) \subseteq D$. Thus

$$\begin{aligned}\nu(M) &= \nu(A) = \nu(A \cap C) + \nu(A \cap (S-C)) \\ &= \nu(A \cap C) + \nu(\phi_M(A \cap (S-C))) = r(A \cap C) + r'(\phi_M(A \cap (S-C))) \\ &\leqq r(A \cap C) + r'(R(A \cap (S-C))) \leqq r(C) + r'(R(S-C)) \\ &\leqq r(C) + r'(D) = \rho(C, D).\end{aligned}$$

To prove the converse of Proposition 1, we require several lemmas valid for any pregeometry.

LEMMA 1. *If B_1, B_2 are subsets of an independent set B, then*

$$\bar{B}_1 \cap \bar{B}_2 = J(B_1 \cap B_2).$$

The proof is straightforward.

LEMMA 2. *If B is an independent set, and $D \subseteq \bar{B}$, then the set*

$$B_1 = \{b \in B : D \nsubseteq J(B-b)\}$$

is the unique minimal subset of B whose closure contains D.

Proof. Suppose $D \subseteq \bar{B}_2$, where $B_2 \subseteq B$. If $B_1 \nsubseteq B_2$, there exists a point $b \in B_1$ such that $B_2 \subseteq B - b$. But then $D \subseteq J(B-b)$, contradicting the definition of B_1. Thus $D \subseteq \bar{B}_2$, $B_2 \subseteq B$ imply $B_2 \supseteq B_1$. To prove that $D \subseteq \bar{B}_1$, we apply Lemma 1:

$$D \subseteq \bigcap_{b' \in B - B_1} J(B-b') = J\left(\bigcap_{b' \in B - B_1} (B-b')\right) = \bar{B}_1.$$

LEMMA 3. *Let B be an independent set and*

$$B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n$$

a strictly increasing sequence of subsets of B. Suppose b_i, b'_i $(1 \leqq i \leqq n)$ are points such that

(3.10) $b_i \in B_i - B_{i-1}$,

(3.11) $b'_i \in \bar{B}_i - J(B - b_i)$.

Then

$$b'_i \notin J\Big(\Big(B - \bigcup_{j=1}^{i} b_j\Big) \cup \bigcup_{j=1}^{i-1} b'_j\Big) \quad \textit{for } 1 \leqq i \leqq n.$$

Proof. Let

$$B'_i = \Big(B_i - \bigcup_{j=1}^{i} b_j\Big) \cup \bigcup_{j=1}^{i} b'_j.$$

We first show that $\bar{B}'_i = \bar{B}_i$. By (3.11), $b'_1 \notin J(B_1 - b_1)$, so (3.10) and the exchange property (2.1) imply the result for $i=1$. The proof proceeds by induction. Let

$$C_i = \Big(B_i - \bigcup_{j=1}^{i-1} b_j\Big) \cup \bigcup_{j=1}^{i-1} b'_j = (B_i - B_{i-1}) \cup B'_{i-1}.$$

Then

$$\begin{aligned} \bar{C}_i &= J((B_i - B_{i-1}) \cup B'_{i-1}) = J((B_i - B_{i-1}) \cup \bar{B}'_{i-1}) \\ &= J((B_i - B_{i-1}) \cup \bar{B}_{i-1}) \qquad \text{(by hypothesis)} \\ &= J((B_i - B_{i-1}) \cup B_{i-1}) = \bar{B}_i, \end{aligned}$$

and by a similar argument,

$$J(C_i - b_i) = J((B_i - B_{i-1} - b_i) \cup B'_{i-1}) = J(B_i - b_i).$$

It follows now from (3.11) and the exchange property that

$$\bar{B}_i = \bar{C}_i = J((C_i - b_i) \cup b'_i) = \bar{B}'_i.$$

Thus

$$J\Big(\Big(B - \bigcup_{j=1}^{i} b_j\Big) \cup \bigcup_{j=1}^{i-1} b'_j\Big) = J((B - B_{i-1} - b_i) \cup B'_{i-1}) = J(B - b_i),$$

and the lemma follows by (3.11).

LEMMA 4. *Let A be an independent set and*

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$$

a strictly increasing sequence of subsets of A. Suppose a_i, a'_i $(1 \leqq i \leqq n)$ are points such that

(3.12) $a_i \in A_i - A_{i-1}$,

(3.13) $a'_i \in J(A - A_{i-1}) - J(A - a_i)$.

Then

$$a'_i \notin J\Big(\Big(A - \bigcup_{j=1}^{i} a_j\Big) \cup \bigcup_{j=1}^{i-1} a'_j\Big) \quad \textit{for } 1 \leqq i \leqq n.$$

Proof. By (3.13), $a'_1 \notin J(A - a_1)$, so assume inductively that the lemma holds up to $i-1$, where $2 \leqq i \leqq n$. Then

$$C_i = \Big(A - \bigcup_{j=1}^{i-1} a_j\Big) \cup \bigcup_{j=1}^{i-1} a'_j$$

is independent. We can write $C_i = (C_i - a_i) \cup (A - A_{i-1})$ and apply Lemma 1, obtaining

$$J(C_i - a_i) \cap J(A - A_{i-1}) = J(A - A_{i-1} - a_i).$$

Since $J(A-A_{i-1}-a_i)\subseteq J(A-a_i)$, it follows by (3.13) that $a_i'\notin J(C_i-a_i)$, and the proof is complete.

PROPOSITION 3. *If a matching $\phi_M\colon A\to B$ does not admit an augmenting chain, there exists a support $(\bar{C}, \bar{D})$, where $C\subseteq A$, $D=\phi_M(A-C)$.*

Proof. Let $E_0=S-\bar{A}$. Then $R(E_0)\subseteq\bar{B}$, for otherwise there would exist a trivial augmenting chain consisting of one element of R. Let B_1 be the minimal subset of B, defined according to Lemma 2, such that $R(E_0)\subseteq\bar{B}_1$. Let $A_1=\phi_M^{-1}(B_1)$, $E_1=S-J(A-A_1)$. In general, having defined E_{i-1}, we define B_i as the minimal subset of B such that $R(E_{i-1})\cap\bar{B}\subseteq\bar{B}_i$, and set $A_i=\phi_M^{-1}(B_i)$, $E_i=S-J(A-A_i)$. Since $A_{i-1}\subseteq E_{i-1}$, $\phi_M(A_{i-1})\subseteq R(E_{i-1})\cap\bar{B}$, but $\phi_M(A_{i-1})\nsubseteq J(B-b)$ for any $b\in B_{i-1}$. Thus by Lemma 2, $B_{i-1}\subseteq B_i$, and so $A_{i-1}\subseteq A_i$, $E_{i-1}\subseteq E_i$. It is clear, moreover, that each of the sequences A_i, B_i, E_i is *strictly* increasing up to and including some index m, after which the process terminates. Thus

$$R(E_m)\cap\bar{B}\subseteq\bar{B}_m,$$

but

$$R(E_i)\cap\bar{B}\nsubseteq\bar{B}_i$$

for $0\leqq i\leqq m-1$, where $B_0=\varnothing$.

We shall prove that $R(E_i)\subseteq\bar{B}$ for all i, $0\leqq i\leqq m$. Assuming otherwise, let n be the smallest integer, $1\leqq n\leqq m$, for which $R(E_n)\nsubseteq\bar{B}$. Then there exists $(a_n', b_{n+1}')\in R$ such that

$$b_{n+1}'\in T-\bar{B},\qquad a_n'\in E_n-E_{n-1}=J(A-A_{n-1})-J(A-A_n).$$

If $a_n'\in J(A-a)$ for all $a\in A_n$, then $a_n'\in J(A-A_n)$ by Lemma 1. Hence there exists $a_n\in A_n$ such that $a_n'\notin J(A-a_n)$. Since $a_n'\in J(A-A_{n-1})$, $a_n'\in J(A-a)$ for all $a\in A_{n-1}$, and therefore $a_n\in A_n-A_{n-1}$. Let $b_n=\phi_M(a_n)$, then $b_n\in B_n-B_{n-1}$. By Lemma 2 and the definition of B_n, there exists $(a_{n-1}', b_n')\in R$ such that $a_{n-1}'\in E_{n-1}$ and $b_n'\in\bar{B}-J(B-b_n)$. Thus $b_n'\notin\bar{B}_{n-1}$, and so $b_n'\in\bar{B}_n-\bar{B}_{n-1}$. Since $R(E_{n-2})\subseteq\bar{B}_{n-1}$ by assumption, it follows that

$$a_{n-1}'\in E_{n-1}-E_{n-2}\quad\text{and}\quad(a_{n-1}', b_n')\notin M.$$

We can now repeat the above argument beginning with a_{n-1}'. The process terminates when we arrive finally at $a_0'\in E_0$, having constructed a sequence

$$(b_{n+1}', a_n'), (a_n, b_n), (b_n', a_{n-1}'), \ldots, (a_1, b_1), (b_1', a_0'),\tag{3.14}$$

where

$$\begin{aligned}(a_i, b_i)&\in M, & 1\leqq i\leqq n,\\ (a_i', b_{i+1}')&\in R-M, & 0\leqq i\leqq n.\end{aligned}\tag{3.15}$$

$$a_0'\in S-\bar{A},\qquad b_{n+1}'\in T-\bar{B},\tag{3.16}$$

$$\begin{aligned}a_i&\in A_i-A_{i-1}, & a_i'&\in J(A-A_{i-1})-J(A-a_i),\\ b_i&\in B_i-B_{i-1}, & b_i'&\in\bar{B}_i-J(B-b_i),\end{aligned}\qquad 1\leqq i\leqq n.\tag{3.17}$$

Now (3.15) and (3.16) are restatements of (3.7) and (3.8) and by Lemmas 3 and 4, therefore, (3.17) implies (3.9). It follows that the sequence (3.14) is an augmenting chain (in reverse order), contradicting the hypothesis. Thus $R(E_{i-1}) \subseteq \bar{B}_i$ for $1 \leqq i \leqq m$, and $R(E_m) \subseteq \bar{B}_m$. Since $E_m = S - J(A - A_m)$, the pair $(\bar{C}, \bar{D})$ with $C = A - A_m$, $D = B_m$ constitutes a support as required, and the proposition follows.

Our preceding results are summarized in

THEOREM 1. *A matching is maximum if and only if it does not admit an augmenting chain.*

Proof. The necessity of the condition is stated in Proposition 1. If there does not exist an augmenting chain, then the support guaranteed by Proposition 3 has rank equal to the cardinality of the matching, which together with Proposition 2 establishes the maximality of the matching.

COROLLARY. *If a matching $\phi_M: A \to B$ is not maximum, there exists a matching $\phi_{M'}: A' \cup a \to B' \cup b$, where $\bar{A}' = \bar{A}$, $\bar{B}' = \bar{B}$, and $a \notin \bar{A}$, $b \notin \bar{B}$.*

Proof. By Theorem 1, there exists an augmenting chain with respect to M, and the matching M' may be constructed as in the proof of Proposition 1.

The following theorem, an immediate consequence of our preceding results, provides a generalization of the König-Egerváry theorem to geometric relations.

THEOREM 2. *The maximum cardinality of a matching in $(G(S), R, G'(T))$ is equal to the minimum rank of a support*(2).

Proof. By Theorem 1 a maximum matching M satisfies the hypothesis of Proposition 3, so there exists a support of rank $\nu(M)$. By Proposition 2, this support is minimum.

4. **Deficiency and critical sets.** The results of §3 may be applied directly to obtain an expression for the cardinality of a maximum matching in a geometric relation $(G(S), R, G'(T))$, from which a generalization of the Marriage Theorem of Hall and Rado follows as a corollary. Before establishing these results, however, we consider in this section the notion of a *deficiency function* on subsets of S. The concept was introduced by Ore [8] for bipartite graphs, and may be extended to geometric relations as follows.

For any subset A of S, define the *deficiency* $\delta(A)$ of A by

$$\delta(A) = r(S) - r(S-A) - r'(R(A)). \tag{4.1}$$

Since $S-(A_1 \cup A_2)$, $S-(A_1 \cap A_2)$ are, respectively, identical to $(S-A_1) \cap (S-A_2)$, $(S-A_1) \cup (S-A_2)$, it follows from (2.6) and (3.5) that δ is lower semimodular:

$$\delta(A_1 \cup A_2) + \delta(A_1 \cap A_2) \geqq \delta(A_1) + \delta(A_2). \tag{4.2}$$

(2) It can be shown that Theorem 2 is equivalent to a result obtained independently by Edmonds [5].

The rank functions r, r' are finite by (2.2), so the deficiency $\delta(A)$ is finite for all $A \subseteq S$. Since δ is integer-valued and bounded above by $r(S)$, there exists a *maximum deficiency*

$$\eta = \max_{A \subseteq S} \delta(A). \tag{4.3}$$

Subsets A of S satisfying $\delta(A) = \eta$ will be called *critical sets*. Since $\delta(\varnothing) = 0$, $\eta \geqq 0$, and $\eta > 0$ if and only if all critical sets are nonempty. An immediate consequence of (4.2) is

PROPOSITION 4. *If A_1, A_2 are critical sets in $G(S)$, then $A_1 \cup A_2$, $A_1 \cap A_2$ are critical sets.*

It follows from Proposition 4 that the family of critical sets is closed under finite unions and intersections, and thus forms a ring of sets. The open sets in this ring will be of particular importance below. In investigating their structure, it is convenient to consider the *coclosure* operator K induced by the closure operator J of $G(S)$. For any subset $A = S - B$, we define

$$K(A) = S - J(B).$$

Clearly K is a coclosure operator: $K(A) \subseteq A$, $K^2(A) = K(A)$, and $A_1 \supseteq A_2$ implies $K(A_1) \supseteq K(A_2)$. The coclosed sets are the open sets of $G(S)$. Suprema and infima in the lattice $M(S)$ of open sets are given by

$$A_1 \vee A_2 = A_1 \cup A_2, \qquad A_1 \wedge A_2 = K(A_1 \cap A_2). \tag{4.4}$$

If $A = S - B$ is any subset of S, then $r(\bar{B}) = r(B)$, that is

$$r(S - K(A)) = r(S - A), \tag{4.5}$$

while $K(A) \subseteq A$ implies by (3.1)

$$r'(R(K(A))) \leqq r'(R(A)). \tag{4.6}$$

From (4.5) and (4.6) we have

$$\delta(K(A)) \geqq \delta(A). \tag{4.7}$$

Thus

PROPOSITION 5. *If A is a critical set, then $K(A)$ is a critical open set.*

Propositions 4 and 5, together with (4.4), imply

PROPOSITION 6. *The critical open sets form a sublattice $M_0(S)$ of the lattice $M(S)$ of open sets in $G(S)$.*

In the case where $G(S)$ is a free geometry, all subsets of S are open, and the sublattice $M_0(S)$ of the Boolean algebra $M(S)$ is a ring. By a well-known result (see e.g. [3]), rings are characterized latticially by the distributive property: every distributive lattice is isomorphic to a ring of sets. For arbitrary $G(S)$, we have as analogue

THEOREM 3. *The sublattice $M_0(S)$ of critical open sets is distributive.*

Proof. By a theorem of Birkhoff [3], a lattice is distributive if and only if it contains neither M_1 nor M_2 (Figure 1) as a sublattice.

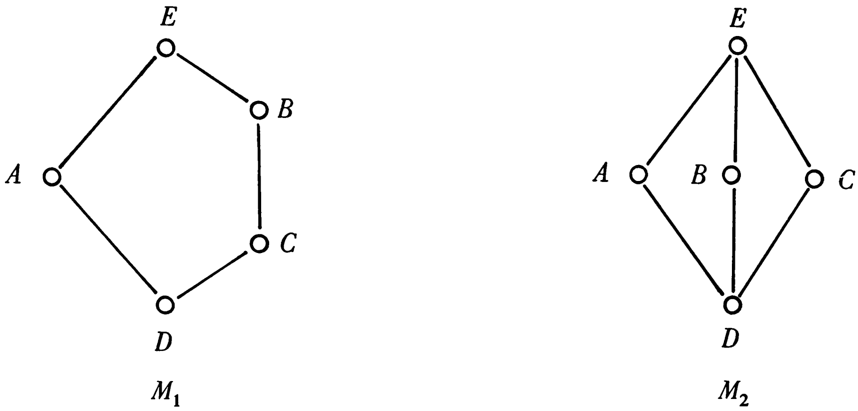

FIGURE 1

We observe that if either M_1 or M_2 is a sublattice of $M_0(S)$, then

$$J(R(A \cap B)) = J(R(D)), \tag{4.8}$$

since $A \cap B$ is a critical set, $D = K(A \cap B)$, and equality must hold in (4.6) whenever both sets are critical.

Suppose first that M_1 is a sublattice of $M_0(S)$. Then by (4.4),

$$A \cup B = A \cup C = E, \tag{4.9}$$

$$K(A \cap B) = K(A \cap C) = D. \tag{4.10}$$

It follows from (4.9) that $B - A = C - A$, so $B - C \subseteq A \cap B$. Then by (4.8),

$$R(B-C) \subseteq R(A \cap B) \subseteq J(R(A \cap B)) = J(R(D)) \subseteq J(R(C)).$$

Thus

$$r'(R(B)) = r'(R(C) \cup R(B-C)) = r'(R(C)),$$

which implies, since $\delta(B) = \delta(C) = \eta$, that $r(S-B) = r(S-C)$. But $B > C$ in M_1, hence in $M(S)$, so in the lattice $L(S)$ of closed sets $S-B < S-C$, and hence $r(S-B) < r(S-C)$, a contradiction.

If M_2 is a sublattice of $M_0(S)$, then

$$A \cup B = A \cup C = B \cup C = E, \tag{4.11}$$

$$K(A \cap B) = K(A \cap C) = K(B \cap C) = D. \tag{4.12}$$

By (4.11) we have $A-C=B-C=E-C$, so $E-C\subseteq A\cap B$. Then from (4.8)

$$R(E-C)\subseteq R(A\cap B)\subseteq J(R(A\cap B))=J(R(D))\subseteq J(R(C)).$$

Thus

$$r'(R(E))=r'(R(C)\cup R(E-C))=r'(R(C)),$$

which implies as before that $r(S-E)=r(S-C)$, a contradiction. The proof is complete.

The foregoing results may, of course, be applied to subsets of $G'(T)$. In particular, for the converse relation R' to R, a deficiency function δ' is defined on subsets of T by

$$\delta'(B)=r'(T)-r'(T-B)-r(R'(B)),$$

with corresponding maximum deficiency

$$\eta'=\max_{B\subseteq T}\delta'(B).$$

The critical sets and critical open sets in $G'(T)$ are defined in the obvious manner. By our preceding results, the critical open sets of $G'(T)$ form a distributive sublattice $M_0'(T)$ of the lattice $M'(T)$ of open sets in $G'(T)$.

5. **Maximum matchings.** As a consequence of Theorem 2, we may express the cardinality of a maximum matching in $(G(S), R, G'(T))$ in terms of the maximum deficiency $\eta=\max_{A\subseteq S}\delta(A)$ defined in §4. The result, which generalizes a theorem of Ore [8] for bipartite graphs, is

THEOREM 4. *The cardinality of a maximum matching M in $(G(S), R, G'(T))$ is*

$$\nu(M)=r(S)-\eta. \tag{5.1}$$

Proof. By Theorem 2, it is sufficient to prove that $r(S)-\eta$ is the rank of a minimum support. If (C, D) is a minimum support, then clearly $D=J(R(S-C))$, and

$$\begin{aligned}\rho(C, D)&=r(C)+r'(R(S-C))\\&=\min_{B\subseteq S}(r(\bar{B})+r'(R(S-\bar{B})))=\min_{B\subseteq S}(r(S)-\delta(S-\bar{B})).\end{aligned}$$

Since by (4.7), $r(S)-\delta(S-\bar{B})\leqq r(S)-\delta(S-B)$ for any $B\subseteq S$, we may rewrite the equality above as

$$\begin{aligned}\rho(C, D)&=\min_{B\subseteq S}(r(S)-\delta(S-B))\\&=r(S)-\max_{B\subseteq S}\delta(S-B)=r(S)-\eta.\end{aligned}$$

The case $\eta=0$ in Theorem 4 provides a generalization of the Marriage Theorem of P. Hall and Rado to geometric relations:

COROLLARY. *There exists a matching of cardinality* $r(S)$ *in* $(G(S), R, G'(T))$ *if and only if*

$$r(S)-r(S-A) \leqq r'(R(A)) \tag{5.2}$$

for all subsets A *of* S.

Recalling the definition (2.7) of the rank function for a minor, we observe that condition (5.2) may be alternatively stated as follows:

For every subset $A \subseteq S$, the rank of the contraction of $G(S)$ to A does not exceed the rank of the reduction of $G'(T)$ to $R(A)$, i.e.

$$r_{[S-A,S]}(A) \leqq r'_{[\varnothing,R(A)]}(R(A)). \tag{5.3}$$

We close this section by characterizing the minimal critical set, which by (4.7) is also the minimal element in the lattice $M_0(S)$.

PROPOSITION 7. *Let* A_0 *be the minimal critical set in* S. *Then a point* $a \in S$ *belongs to* A_0 *if and only if the maximum deficiency* η *is reduced by one when* $G(S)$ *is contracted to* $S-\bar{a}$. *If* $a \notin A_0$, *the maximum deficiency* η *remains unchanged when contracting* $G(S)$ *to* $S-\bar{a}$.

Proof. The deficiency $\delta_{[\bar{a},S]}(A)$ of a set $A \subseteq S-\bar{a}$ for $(G_{[\bar{a},S]}, R, G'(T))$ is

$$\begin{aligned} \delta_{[\bar{a},S]}(A) &= r_{[\bar{a},S]}(S-\bar{a})-r_{[\bar{a},S]}(S-(A\cup\bar{a}))-r'(R(A)) \\ &= (r(S)-r(\bar{a}))-(r(S-A)-r(\bar{a}))-r'(R(A)) = \delta(A). \end{aligned} \tag{5.4}$$

If $a \notin A_0$, then $a \in S-A_0$, so $\bar{a} \subseteq S-A_0$ since $S-A_0$ is closed. Thus $a \notin A_0$ implies $A_0 \subseteq S-\bar{a}$ and hence, by (5.4), $(G_{[\bar{a},S]}, R, G'(T))$ has maximum deficiency η.

If $a \in A_0$, however, it follows from (5.4) and the minimality of A_0 that the maximum deficiency is reduced when $G(S)$ is contracted to $S-\bar{a}$. It is sufficient to show that the set $A_0-\bar{a}$ has deficiency $\eta-1$ in $(G(S), R, G'(T))$. Now since $\bar{a} \subseteq A_0$,

$$\begin{aligned} \delta(A_0-\bar{a}) &= r(S)-r((S-A_0)\cup\bar{a})-r'(R(A_0-\bar{a})) \\ &= r(S)-r(S-A_0)-1-r'(R(A_0-\bar{a})) \\ &\geqq r(S)-r(S-A_0)-r'(R(A_0))-1 \\ &= \delta(A_0)-1 = \eta-1. \end{aligned}$$

But $\delta(A_0-\bar{a}) \leqq \eta-1$ since A_0 is minimal, so equality holds, and the proof is complete.

6. **The structure of minimum supports.** Let $L(S), L'(T)$ denote the geometric lattices of closed sets in $G(S)$, $G'(T)$, respectively. The same symbol will be used to denote an arbitrary closed set of $G(S)$ when it is regarded as an element of the lattice $L(S)$, and similarly for $G'(T)$.

Consider now a pair of functions $\sigma: L(S) \to L'(T)$, $\sigma': L'(T) \to L(S)$ defined by

$$\sigma(C) = J(R(S-C)), \qquad \sigma'(D) = J(R'(T-D)), \tag{6.1}$$

where $R' \subseteq T \times S$ is the converse relation to R. Clearly σ, σ' are order-inverting:

$$\begin{aligned} &C_1 \leqq C_2 \quad \text{implies} \quad \sigma(C_1) \geqq \sigma(C_2), \\ &D_1 \leqq D_2 \quad \text{implies} \quad \sigma'(D_1) \geqq \sigma'(D_2). \end{aligned} \tag{6.2}$$

Furthermore, we have for any $C \in L(S)$, $D \in L'(T)$,

$$\sigma'\sigma(C) \leqq C, \qquad \sigma\sigma'(D) \leqq D, \tag{6.3}$$

as is easily verified.

We deduce from (6.2) and (6.3) that the pair of functions σ, σ' forms a dual Galois connection between the lattices $L(S)$ and $L'(T)$, that is, a Galois connection between their dual lattices. It follows from the theory of Galois connections (cf. Ore [7]) that the composite functions $\sigma'\sigma$, $\sigma\sigma'$ are coclosure operators on $L(S)$, $L'(T)$, respectively, and that the quotient lattices $Q(S)$, $Q'(T)$ of coclosed elements are anti-isomorphic, with the restriction of σ to $Q(S)$ providing a canonical anti-isomorphism. Furthermore, for any subset $\{C_i : i \in I\}$ of Q,

$$\overset{Q}{\bigvee_{i\in I}} C_i = \overset{L}{\bigvee_{i\in I}} C_i, \qquad \overset{Q}{\bigwedge_{i\in I}} C_i = \sigma'\sigma\Big(\overset{L}{\bigwedge_{i\in I}} C_i\Big),$$

and similarly for $Q'(T)$ and $L'(T)$.

The preceding observations may be related to supports of $(G(S), R, G'(T))$ as follows. Let us define a support (C, D) to be *irredundant* if no pair (C_1, D_1) of closed sets, such that $C_1 \cup D_1$ is a proper subset of $C \cup D$, is a support. Clearly any minimum support is irredundant.

Proposition 8. *A pair (C, D) of closed sets in $G(S)$, $G'(T)$ is an irredundant support if and only if $C \in Q(S)$, $D = \sigma(C)$, or equivalently $D \in Q'(T)$, $C = \sigma'(D)$.*

Proof. Suppose (C, D) is an irredundant support. By the definition of a support, $D \supseteq J(R(S-C)) = \sigma(C)$, and irredundancy implies that $D = \sigma(C)$. By symmetry, $C = \sigma'(D)$, i.e. $C = \sigma'\sigma(D)$, so $C \in Q(S)$.

Conversely, if $C \in Q(S)$, $D = \sigma(C)$, then (C, D) is a support, but (C, D_1) is not a support for any proper closed subset $D_1 \subset D$. Hence if (C, D) is not irredundant, there exists a closed set $C_1 \subsetneq C$ such that $C_1 \supseteq J(R'(T-D)) = \sigma'(D)$. But then in $L(S)$, $\sigma'\sigma(C) < C$, which contradicts $C \in Q(S)$.

Consider now the minimum supports. If $L_0(S)$ is the subset of $L(S)$ consisting of those flats in $G(S)$, which are elements of minimum supports, then $L_0(S) \subseteq Q(S)$ by Proposition 8, and the corresponding subset $L_0'(T) \subseteq Q'(T)$ is order-anti-isomorphic to $L_0(S)$ under σ. Thus every minimum support is of the form (C, D), where $C \in L_0(S)$, $D = \sigma(C)$, or equivalently, $D \in L_0'(T)$, $C = \sigma'(D)$. The structure of the subsets $L_0(S)$, $L_0'(T)$ will become apparent from our results in §4 through

Proposition 9. *A closed set C in $G(S)$ is an element of a minimum support if and only if its complement $S - C$ is a critical open set, and similarly for closed sets D in $G'(T)$.*

Proof. Suppose $S-C$ is a critical open set. Then

$$\rho(C, \sigma(C)) = r(C)+r'(R(S-C)) = r(S)-\delta(S-C) = r(S)-\eta,$$

so $(C, \sigma(C))$ is a minimum support by Theorems 2 and 4.

Conversely, if $(C, \sigma(C))$ is a minimum support, then

$$\begin{aligned}\delta(S-C) &= r(S)-r(C)-r'(R(S-C))\\ &= r(S)-\rho(C, \sigma(C)) = \eta,\end{aligned}$$

so $S-C$ is a critical open set.

The same result holds for a closed set D in $G'(T)$ by symmetry.

The quotients $Q(S)$, $Q'(T)$ are lattices in which suprema coincide with suprema in $L(S)$, $L'(T)$, respectively. For the subsets $L_0(S)$, $L_0'(T)$, the result is considerably stronger.

PROPOSITION 10. *The subsets $L_0(S)$, $L_0'(T)$ are anti-isomorphic, distributive sublattices of $L(S)$, $L'(T)$, respectively, and the restriction of σ to $L_0(S)$ provides a canonical anti-isomorphism.*

Proof. By Theorem 3 the critical open sets in $G(S)$, $G'(T)$ form distributive sublattices of the lattices $M(S)$, $M'(T)$ of open sets, respectively. Since distributivity is preserved under dualization, the complements of critical open sets in $G(S)$, $G'(T)$ form distributive sublattices of $L(S)$, $L'(T)$. By Proposition 9 the complements of critical open sets in $G(S)$, $G'(T)$ are precisely the elements of $L_0(S)$, $L_0'(T)$. That σ is an anti-isomorphism from $L_0(S)$ to $L_0'(T)$ follows from the Galois connection theory, since $L_0(S)\subseteq Q(S)$ and $L_0'(T)$ is the image of $L_0(S)$ under σ.

The anti-isomorphism σ from $L_0(S)$ to $L_0'(T)$ implies that for any set $\{C_i : i\in I\}$ of lattice elements in $L_0(S)$,

$$\sigma\Big(\bigvee_{i\in I} C_i\Big) = \bigwedge_{i\in I} \sigma(C_i), \qquad \sigma\Big(\bigwedge_{i\in I} C_i\Big) = \bigvee_{i\in I} \sigma(C_i),$$

where suprema and infima are as in $L(S)$, $L'(T)$, since $L_0(S)$, $L_0'(T)$ are sublattices. Our results on the structure of minimum supports are summarized in

THEOREM 5. *Let $M_0(S)$, $M_0'(T)$ be the sublattices of $M(S)$, $M'(T)$ whose elements are the critical open sets in $G(S)$, $G'(T)$, respectively. Let $L_0(S)$, $L_0'(T)$ be the sublattices of $L(S)$, $L'(T)$ whose elements are the set-theoretic complements of the elements of $M_0(S)$, $M_0'(T)$. Then the sublattices $L_0(S)$, $L_0'(T)$ are distributive and anti-isomorphic. The minimum supports of $(G(S), R, G'(T))$ consist of all pairs $(C, \sigma(C))$, $C\in L_0(S)$, and, for any set $\{(C_i, \sigma(C_i)) : i\in I\}$ of minimum supports, the pairs*

$$\Big(\bigvee_{i\in I} C_i, \bigwedge_{i\in I} \sigma(C_i)\Big), \qquad \Big(\bigwedge_{i\in I} C_i, \bigvee_{i\in I} \sigma(C_i)\Big)$$

are minimum supports.

We conclude with an example illustrating the foregoing theory. Take the geometric relation consisting of the triple $(G(S), I, G(S))$, where $G(S)$ is an arbitrary open pregeometry and I the identity relation. Since in this case the maximum cardinality of a matching obviously equals $r(S)$, we have $\eta=0$ by Theorem 4. Hence the critical sets $A\subseteq S$ are those which satisfy

$$r(A)+r(S-A) = r(S). \tag{6.4}$$

Now it is easy to see that any such set A is closed, and hence by symmetry open as well. Thus the lattices of critical and critical open sets coincide and, since according to (6.4) this lattice is uniquely complemented, it is by Theorem 3 a Boolean algebra. The dual lattice $L_0(S)$ is hence also a Boolean algebra and every minimum support of $(G(S), I, G(S))$ is of the form $(C, S-C)$, C satisfying (6.4). Since (6.4) characterizes the separators of an open pregeometry, we thus obtain the well-known result (see e.g. [4]) that the separators of a geometric lattice L form a Boolean algebra which is a sublattice of L.

Acknowledgment. It is a pleasure to record our gratitude to Professor G.-C. Rota who suggested much of this work and kept an encouraging interest throughout.

References

1. M. Aigner and T. A. Dowling, *Matching theorems for combinatorial geometries*, Bull. Amer. Math. Soc. **76** (1970), 57–60.

2. C. Berge, *Théorie des graphes et ses applications*, Dunod, Paris, 1958; English transl., Methuen, London; Wiley, New York, 1962. MR **21** #1608; MR **24** #A2381.

3. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1940; rev. ed., 1948, 1967. MR **1**, 325; MR **10**, 673.

4. H. H. Crapo and G.-C. Rota, *On the foundations of combinatorial theory: Combinatorial geometries*, M.I.T. Press, Cambridge, Mass., 1970.

5. J. Edmonds, *Submodular functions, matroids, and certain polyhedra*, Proc. of Calgary Inter. Conf. on Combinatorial Structures, Gordon and Breach, New York, 1970.

6. L. Mirsky and H. Perfect, *Applications of the notion of independence to problems of combinatorial analysis*, J. Combinatorial Theory **2** (1967), 327–357. MR **37** #1268.

7. O. Ore, *Galois connexions*, Trans. Amer. Math. Soc. **55** (1944), 493–513. MR **6**, 36.

8. ———, *Graphs and matching theorems*, Duke Math. J. **22** (1955), 625–639. MR **17**, 394.

9. R. Rado, *A theorem on independence relations*, Quart. J. Math. Oxford Ser. **13** (1942), 83–89. MR **4**, 269.

University of North Carolina,
Chapel Hill, North Carolina 27514