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ANALYTIC CONTINUATION OF THE SERIES $\sum (m+nz)^{-s}$

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Abstract. The series $\sum (m+nz)^{-s}$, m, n ranging over all integers except both zero, for s an integer greater than two is well known from the theory of elliptic functions and modular forms. In this paper, we show that this series defines an analytic function $G(z, s)$ for $\operatorname{Im} z>0$ and $\operatorname{Re} s>2$ which has an analytic continuation to all values of s. It is then shown that G satisfies a functional equation under the transformation $z \to -1/z$, and finally as an application some numerical results are obtained.

Throughout, except where otherwise noted, if $w \in \boldsymbol{C}$, the complex numbers, and $w \neq 0$, we define arg w to be that value of the argument such that $-\pi \leqq \arg w < \pi$. Then $\log w = \log |w| + i \arg w$ and for complex u, $w^u = e^{u \log w}$. Also we use the standard notations $z=x+iy$, $s=\sigma+it$ for complex variables. With these conventions define

$$G(z, s) = \sum_{m,n}{}' \frac{1}{(m+nz)^s} \tag{1}$$

The $'$ on the summation sign indicates that m, n range over all integers except $m=n=0$. Each individual term of the sum is an analytic function of $(z, s) \in \{y>0\} \times \boldsymbol{C}$ and it is known that the series in (1) converges absolutely and uniformly on compact subsets of $\{y>0, \sigma>2\}$ so that G is an analytic function of two complex variables in this product of half-planes. Because of the absolute convergence the terms of the series may be arranged in any order of summation.

It is evident that $G(z+1, s) = G(z, s)$ so that G has a Fourier expansion of the form $G(z, s) = \sum_{k=-\infty}^{\infty} a_k(s) e^{2\pi i k z}$. To find the coefficients write the sum in (1) as

$$\sum_{m,n}{}' = \sum_{m=1,n=0}^{\infty} + \sum_{m=-\infty,n=0}^{-1} + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} + \sum_{n=-\infty}^{-1} \sum_{m=-\infty}^{\infty}$$

and recalling our convention concerning the argument, e.g., for $m>0$, $(-m)^s = e^{-\pi i s} m^s$, we obtain

$$G(z, s) = (1+e^{\pi i s})\zeta(s) + \sum_{n=1}^{\infty} F(nz, s) + \sum_{n=1}^{\infty} F(-nz, s),$$

where $\zeta(s)$ is the Riemann zeta function and $F(z, s) = \sum_{m=-\infty}^{\infty} (z+m)^{-s}$. The series for F converges absolutely and uniformly on compact subsets of $\{y \neq 0, \sigma>1\}$ and

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since for $y>0$ we have $(-z+m)^s=e^{-\pi is}(z-m)^s$ it follows that $F(-z,s)=e^{\pi is}F(z,s)$. Thus we have

$$G(z,s) = (1+e^{\pi is})\left(\zeta(s)+\sum_{n=1}^{\infty} F(nz,s)\right). \tag{2}$$

From now on we consider F as defined only for $y>0$.

Since $F(z+1,s)=F(z,s)$ and $\operatorname{Lim}_{y\to+\infty} F(z,s)=0$, F has a Fourier expansion of the form $\sum_{k=1}^{\infty} b_k(s)e^{2\pi ikz}$. Application of the Poisson summation formula gives $b_k(s)=e^{2\pi ky}\int_{-\infty}^{\infty} e^{-2\pi iku}/(u+iy)^s\,du$, where y is any fixed positive number. Setting $y=1$, the integral may be evaluated by the calculus of residues in the complex u-plane and yields $b_k(s)=((-2\pi i)^s/\Gamma(s))k^{s-1}$ so that

$$F(z,s) = \frac{(-2\pi i)^s}{\Gamma(s)}\sum_{k=1}^{\infty} k^{s-1}e^{2\pi ikz}.$$

Now replace z by nz and substitute in (2), collecting like powers of $e^{2\pi inz}$—which is permissible because the resulting double series is absolutely convergent—to obtain

$$\frac{G(z,s)}{1+e^{\pi is}} = \zeta(s)+\frac{(-2\pi i)^s}{\Gamma(s)}\sum_{n=1}^{\infty} \sigma_{s-1}(n)e^{2\pi inz} \tag{3}$$

where $\sigma_{s-1}(n)=\sum_{k\mid n} k^{s-1}$, k running over the positive divisors of n. It is convenient to define $H(z,s)=G(z,s)/1+e^{\pi is}$ and $A(z,s)=\sum_{n=1}^{\infty}\sigma_{s-1}(n)e^{2\pi inz}$, so that (3) becomes

$$H(z,s) = \zeta(s)+((-2\pi i)^s/\Gamma(s))A(z,s). \tag{4}$$

Now the series for $A(z,s)$ is absolutely and uniformly convergent for (z,s) in any compact subset of $\{y>0\}\times \boldsymbol{C}$ so that we have found the analytic continuation of $G(z,s)$ for all values of s.

The formula (4) is known from the theory of modular forms for the special case where s is an even integer $2r\geqq 4$. In this case $G(z,2r)$ is a modular form of weight r. To determine the behavior of $G(z,s)$ under the modular group we study the transformation $z\to -1/z$ which along with $z\to z+1$ generates the group. To do this we rearrange the terms of the sum (1) as follows:

$$\sum_{m,n}{}' = \left(\sum_{m>0,\,n=0} + \sum_{m<0,\,n=0}\right)+\left(\sum_{m=0,\,n>0} + \sum_{m=0,\,n<0}\right)$$
$$+\left(\sum_{m>0,\,n>0} + \sum_{m<0,\,n<0}\right)+\left(\sum_{m<n,\,n>0} + \sum_{m>0,\,n<0}\right).$$

In each pair of summations the second sum is $e^{\pi is}$ times the first. Thus, recalling the definition of H, we have

$$H(z,s) = \zeta(s)+z^{-s}\zeta(s)+K(z,s)+L(z,s) \tag{5}$$

where $K(z,s)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m+nz)^{-s}$, $L(z,s)=\sum_{m=-\infty}^{-1}\sum_{n=1}^{\infty}(m+nz)^{-s}$. Now, for $y>0$, $\arg(-1/z)=\pi-\arg z$, $\arg(m+n(-1/z))=\arg(mz-n)-\arg z$, if $m>0$, $n>0$, and $\arg(m+n(-1/z))=\arg(-mz+n)-\arg z+\pi$, if $m<0$, $n>0$. Thus $(-1/z)^s$

$=e^{\pi is}/z^s$, $K(-1/z, s)=z^sL(z, s)$ and $L(-1/z, s)=z^se^{-\pi is}K(z, s)$. Using this in (5) gives, after a little rearrangement,

$$H(-1/z, s) = z^sH(z, s)+z^s(e^{-\pi is}-1)(\zeta(s)+K(z, s)). \tag{6}$$

Solving for $A(-1/z, s)$ gives

$$A(-1/z, s) = z^sA(z, s)+\frac{(z^se^{-\pi is}-1)}{(-2\pi i)^s}\Gamma(s)\zeta(s)+\frac{z^s(e^{-\pi is}-1)}{(-2\pi i)^s}\Gamma(s)K(z, s). \tag{7}$$

(6), (7) hold for $\sigma>2$, as derived; however, as all the functions occurring, other than K, have continuations to all values of s so does K and the formulas hold for all s.

To apply these formulas we obtain another expression for K. Recall the relation, for $x>0$, $\sigma>0$, $\Gamma(s)/z^s=\int_0^\infty u^{s-1}e^{-zu}\,du$ so that for $x>0$, $y>0$, $\sigma>2$ we have

$$\Gamma(s)K(z, s) = \sum_{m=1}^\infty\sum_{n=1}^\infty\int_0^\infty u^{s-1}e^{-(m+nz)u}\,du.$$

Because of absolute convergence we can interchange the order of summation and integration and, summing the resulting geometric series, we have

$$\Gamma(s)K(z, s) = \int_0^\infty \frac{u^{s-1}}{(e^u-1)(e^{zu}-1)}\,du. \tag{8}$$

By analogy with the integral representation of the Γ function, we introduce the complex variable $w=u+iv$ and define, for $x>0$, $y>0$,

$$J(z, s) = \int_{\gamma_z}\frac{w^{s-1}}{(e^w-1)(e^{zw}-1)}\,dw.$$

Here γ_z is the path in the w-plane along the real axis from $+\infty$ to δ_z, $0<\delta_z<\min(2\pi, 2\pi/|z|)$, with $\arg w=0$, then along the counterclockwise circle of radius δ_z, with $0\leqq\arg w\leqq 2\pi$, then back along the real axis from δ_z to $+\infty$ with $\arg w=2\pi$. By the residue theorem the value of the integral is independent of the choice of δ_z within the range stated. For z in a compact subset of the first quadrant, the integral converges uniformly for s in any compact subset of C, hence is analytic. Splitting the integral into two, one along the axis, the other along the circle, and noting that $w^{s-1}=u^{s-1}$ on the "upper-edge" of the axis and $w^{s-1}=e^{2\pi is}u^{s-1}$ on the "lower edge", we have

$$J(z, s) = (e^{2\pi is}-1)\int_{\delta_z}^\infty\frac{u^{s-1}}{(e^u-1)(e^{zu}-1)}\,du+\int_{|w|=\delta_z}\frac{w^{s-1}}{(e^w-1)(e^{zw}-1)}\,dw. \tag{9}$$

If $\sigma>2$, the second integral tends to 0 as δ_z tends to 0 so that

$$J(z, s) = (e^{2\pi is}-1)\int_0^\infty\frac{u^{s-1}}{(e^u-1)(e^{zu}-1)}\,du.$$

Comparing this with (8), we see that

$$\Gamma(s)K(z, s) = J(z, s)/(e^{2\pi is}-1). \tag{10}$$

Again, as derived, (10) holds for z in the first quadrant and $\sigma>2$ but we already know that the left side has a continuation to z in the upper half-plane and all s, hence so does $J(z, s)$ and (10) then still persists.

Substituting (10) into (7) and noting $(-2\pi i)^s = e^{-\pi i s}(2\pi i)^s$ yields

$$A(-1/z, s) = z^s A(z, s) + \frac{(z^s - e^{\pi i s})}{(2\pi i)^s}\Gamma(s)\zeta(s) - \frac{z^s J(z, s)}{(2\pi i)^s (e^{\pi i s}+1)} \tag{11}$$

valid for $y>0$, all s.

As an application of these formulas, note that if s an integer k, then by (9), for z in the first quadrant,

$$J(z, k) = \int_{|w|=\delta_z} \frac{w^{k-1}}{(e^w-1)(e^{zw}-1)}\,dw.$$

The integrand here is meromorphic in a neighborhood of the origin and analytic for $0<|w|<\delta_z$ so, by the residue theorem, $J(z,k) = 2\pi i \operatorname{Res}_{w=0} w^{k-1}/(e^w-1)(e^{zw}-1)$. Now

$$\frac{w}{e^w-1} = \sum_{p=0}^{\infty} \frac{b_p}{p!} w^p, \qquad \frac{w}{e^{zw}-1} = \frac{1}{z}\sum_{q=0}^{\infty} \frac{b_q}{q!} z^q w^q,$$

with $b_0=1$, $b_1=-\frac{1}{2}$, $b_{2p+1}=0$ for $p>0$ and b_{2p} the Bernoulli numbers. Thus for $k \geqq 3$, $J(z, k)=0$, while for $k=2-r$, $r\geqq 0$,

$$J(z, 2-r) = 2\pi i \operatorname{Res}_{w=0} w^{-r-1}\cdot\frac{w}{e^w-1}\cdot\frac{w}{e^{zw}-1},$$

hence,

$$J(z, 2-r) = (2\pi i/z)C_r(z) \tag{12}$$

where $C_r(z)$ is the polynomial $\sum_{p+q=r}(b_p/p!)(b_q/q!)z^q$. In particular, for $r=0$, $J(z, 2) = 2\pi i/z$ so (11) becomes, using $\Gamma(2)\zeta(2)=\pi^2/6$,

$$A(-1/z, 2) = z^2 A(z, 2) - z^2/24 + iz/4\pi + 1/24, \tag{13}$$

an interesting functional equation.

We make a number of simple deductions from (13). First, recalling the definition of A, and taking $z=iy$, $y>0$, gives

$$\sum_{k=1}^{\infty} \sigma_1(k)e^{-2\pi k/y} = -y^2 \sum_{k=1}^{\infty}\sigma_1(k)e^{-2\pi k y} + \frac{y^2}{24} - \frac{y}{4\pi} + \frac{1}{24}. \tag{14}$$

Since $A(iy, 2)\to 0$ as $y\to+\infty$ we note that $\sum_{k=1}^{\infty}\sigma_1(k)e^{-2\pi k/y} \sim y^2/24$ as $y\to+\infty$. Also, by (13) and $A(z+1, 2)=A(z, 2)$, one can evaluate $A(z, 2)$ at any point z fixed under the modular group. For example, putting $z=i$ in (13) gives

$$\sum_{k=1}^{\infty}\sigma_1(k)e^{-2\pi k} = \frac{1}{24} - \frac{1}{8\pi},$$

while putting $z=\omega=e^{2\pi i/3} = -\frac{1}{2}+i\sqrt{3}/2$, in (13), and noting $-1/\omega=\omega+1$ so that $A(-1/\omega, 2) = A(\omega+1, 2) = A(\omega, 2)$ gives, after a little calculation,

$$\sum_{k=1}^{\infty}(-1)^k\sigma_1(k)e^{-\sqrt{3}\pi k} = \frac{1}{24} - \frac{1}{4\sqrt{3}\pi}.$$

To apply (7) or (11) for s an odd integer, one needs more information about K or J. For example, when $s=2k+1$, and $J(z, s)=J(z, 2k+1)+\alpha(s-2k-1)+\cdots$ is the Taylor's series of $J(z, s)$ about $s=2k+1$, $\alpha=\partial J(z, 2k+1)/\partial s$, then if α were known, we could insert this in (11), expand all functions of s about $s=2k+1$, and comparing the constant terms, we would obtain the functional equation for $A(z, 2k+1)$ in terms of known numbers.

In case s is a nonpositive even integer, say $s=-2k$, $k\geqq 0$, then we may simplify (11). For by the functional equation for the ζ-function one has, if $k>0$, $\Gamma(s)\zeta(s)|_{s=-2k}=(-1)^k\zeta(2k+1)/2(2\pi)^{2k}$, while $(e^{-\pi i s}z^s-1)\Gamma(s)\zeta(s)|_{s=0}=(\pi i-\log z)/2$. Also, $J(z, -2k)=J(z, 2-(2k+2))=(2\pi i/z)C_{2k+2}(z)$, by (12). Since $b_2=1/6$, we obtain

$$A(-1/z, 0) = A(z, 0)-\frac{\pi iz}{12}+\frac{\pi i}{4}-\frac{\pi i}{12z}-\frac{\log z}{2}. \tag{15}$$

Note that we cannot use this to evaluate $A(i, 0)$. For $k>0$, we obtain

$$\begin{aligned} &A(-1/z, -2k) \\ &\quad = z^{-2k}A(z, -2k)+\frac{(z^{-2k}-1)}{2}\zeta(2k+1)-\frac{(-1)^k i(2\pi)^{2k+1}}{2z^{2k+1}}C_{2k+2}(z). \end{aligned} \tag{16}$$

If k is odd, then (16) can be used to evaluate $A(i, -2k)$ in terms of the Bernoulli numbers and $\zeta(2k+1)$.

Additional Note. We have just observed that the function $A(z, 0)$ and its functional equation are familiar objects. Recall the definition of Dedekind's η function,

$$\eta(z) = e^{\pi iz/12}\prod_{m=1}^{\infty}(1-e^{2\pi imz})$$

convergent for z in the upper half-plane. Then

$$\log \eta(z) = \frac{\pi iz}{12}+\sum_{m=1}^{\infty}\log(1-e^{2\pi imz}),$$

all logarithms chosen to be real for $z=iy$. Using the Taylor's series for $\log(1-t)$ about $t=0$ in the above infinite series, we obtain

$$\log \eta(z)-\frac{\pi iz}{12} = -\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{2\pi ikmz} = -\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{2\pi inz} = -A(z, 0).$$

Thus, using the transformation formula for $A(z, 0)$ under $z\to -1/z$ we obtain the classical formula

$$\log \eta(-1/z) = \log \eta(z)+(\log z)/2-\pi i/4.$$

This derivation seems to be different from the other proofs in the literature.

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