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GRUNSKY-NEHARI INEQUALITIES FOR A SUBCLASS OF BOUNDED UNIVALENT FUNCTIONS

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Abstract. Let D_1 be the class of regular analytic functions $F(z)$ in the disc $U=\{z : |z|<1\}$ for which $F(0)>0$, $|F(z)|<1$, and $F(z)+F(\zeta)\neq 0$ for all z, $\zeta \in U$. Inequalities of the Grunsky-Nehari type are obtained for the univalent functions in D_1, the proof being based on the area method. By subordination it is shown univalency is unnecessary for certain special cases of the inequalities. Employing a correspondence between D_1 and the class S_1 of bounded univalent functions, the results can be reinterpreted to apply to this latter class.

1. **Introduction of the function classes.** In this section we provide motivation for the introduction of the function classes which we shall denote in the sequel by D, D_1, and $D(\beta)$. The univalent functions within these classes (which we indicate by a * superscript) have an interesting connection with the following classes of univalent functions defined on the unit disc $U=\{z : |z|<1\}$.

Class S. The functions $g(z)$ which are regular analytic and univalent in U, and whose Taylor series expansion about the origin is of the form

$$g(z) = z+a_2z^2+\cdots+a_nz^n+\cdots. \tag{1}$$

Class S_1. The functions $f(z)$ which are regular analytic and univalent in U, have a Taylor series expansion about the origin of the form

$$f(z) = b_1z+b_2z^2+\cdots+b_nz^n+\cdots, \qquad b_1 > 0, \tag{2}$$

and which are bounded (by unity):

$$|f(z)| < 1. \tag{3}$$

Class $S(b_1)$. The functions $f(z)$ in S_1 which have a prescribed first derivative $f'(0)=b_1$ at the origin, $0<b_1\leqq 1$.

It should be noted that if $f(z) \in S(b_1)$ then $b_1^{-1}f(z) \in S$. Under this renormalization S_1 can be considered to be a dense subset of S, and the coefficients $a_n\equiv b_1^{-1}b_n$ belong to the S function $b_1^{-1}f(z)=b_1^{-1}(b_1z+\cdots+b_nz^n+\cdots)$. Here dense refers to the topology of uniform convergence on the compact subsets of the disc U.

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Let $g(z) \in S$ and suppose u is a complex number for which $g(z) \neq 1/u$. We can then form the function

$$G(z) = \sqrt{(1-ug(z))}, \qquad G(0) = 1, \tag{4}$$

which is again regular analytic and univalent in U. Moreover, the univalency of $g(z)$ assures us that if $G(z)$ takes a value w then it must omit its negative $-w$. This property is characteristic of a function class studied by S. A. Guelfer [3].

Class D (*Guelfer*). The functions $G(z)$ which are regular analytic in U, have a Taylor series expansion about the origin of the form

$$G(z) = 1+\alpha_1 z+\cdots+\alpha_n z^n+\cdots, \tag{5}$$

and which satisfy the property

$$G(z)+G(\zeta) \neq 0 \tag{6}$$

for all pairs of numbers $z, \zeta \in U$.

Guelfer proved a number of distortion and rotation theorems for D and $D^* \equiv \{G(z) \in D : G \text{ univalent}\}$, and also derived certain estimates for the coefficients α_n. Inverting the transformation (4) he obtained some of the well-known results for functions $g(z)$ in S which omit a value $1/u$. For the proofs Guelfer relied on the close relationship which exists between D and the class C of Bieberbach-Eilenberg functions (see [4]).

Class C (*Bieberbach-Eilenberg*). The functions $E(z)$ which are regular analytic in U, and with the properties that $E(0)=0$ and

$$E(z)E(\zeta) \neq 1 \tag{7}$$

for all pairs $z, \zeta \in U$.

If $G(z) \in D$, we easily see

$$E(z) = (1-G(z))/(1+G(z)) \tag{8}$$

is a function in C. Thus results known for the class C can be utilized to obtain information about the functions of D.

If transformations (4) and (8) are combined, we obtain a correspondence between S and C^* which has been considered recently by Hummel and Schiffer [4]. By the area technique they derive for C^* functions some general inequalities which are equivalent (via the aforementioned correspondence) to the Garabedian-Schiffer inequalities [2] within the class S. The approach through Bieberbach-Eilenberg functions thus provides a proof of elementary character for the Garabedian-Schiffer inequalities, in distinction to the original derivation by variational means.

We shall now show how the above considerations can be modified for the case of bounded functions. Let $f(z) \in S_1$ and suppose

$$f(z) \neq \beta^2 e^{-i\varphi}, \qquad 0 < \beta < 1. \tag{9}$$

The function

$$F(z) = \sqrt{\left(\frac{\beta^2 - e^{i\varphi}f(z)}{1-\beta^2 e^{i\varphi}f(z)}\right)}, \qquad F(0) = \beta, \tag{10}$$

is again regular analytic and univalent in U. Moreover, $F(z)$ has the Guelfer property (6) and is bounded: $|F(z)|<1$. In this way we are prompted to define the following two additional function classes.

Class D_1. The functions $F(z)$ which are regular analytic in U and satisfy the following properties:

$$F(0) > 0, \tag{11}$$

$$|F(z)| < 1, \tag{12}$$

$$F(z)+F(\zeta) \neq 0 \tag{13}$$

for all pairs $z, \zeta \in U$.

Class $D(\beta)$. The functions $F(z)$ in D_1 which map the origin to a prescribed point $\beta \in (0, 1)$, so that

$$F(0) = \beta. \tag{14}$$

Of course $\beta^{-1}F(z) \in D$ whenever $F(z) \in D(\beta)$, and D_1 is dense in D under this renormalization. More importantly it is to be observed that the transformation (10) may be inverted in a unique way when $F(z)$ is univalent. Indeed, let $F(z) \in D^*(\beta)$ and suppose

$$F(z) = \beta+\beta_1 z+\cdots+\beta_n z^n+\cdots. \tag{15}$$

Then if $\alpha_1=\beta^{-1}\beta_1=-|\alpha_1|e^{i\varphi}$ we can form

$$f(z) = e^{i\varphi}(\beta^2-F^2(z))/(1-\beta^2F^2(z)). \tag{16}$$

The defining properties (12) and (14) assure us $f(z)$ is a bounded regular function which vanishes at the origin. To check univalency, an easy calculation shows that

$$f(z)-f(\zeta) = -e^{-i\varphi}\frac{(F(z)-F(\zeta))(F(z)+F(\zeta))(1-\beta^4)}{(1-\beta^2F^2(z))(1-\beta^2F^2(\zeta))}. \tag{17}$$

The univalency of $F(z)$, along with the Guelfer property (13), implies (17) vanishes only if $z=\zeta$. Finally we notice that

$$f'(0) = (2\beta^2/(1-\beta^4))|\alpha_1| > 0 \tag{18}$$

and so we conclude $f(z) \in S(b_1)$, with $f(z)\neq\beta^2 e^{-i\varphi}$ and

$$b_1 = (2\beta^2/(1-\beta^4))|\alpha_1|. \tag{19}$$

In the next section we obtain a general area inequality for D_1^*. By means of the correspondence (16) we show how this provides an elementary derivation of inequalities recently obtained by Nehari [5] and DeTemple [1] for the class S_1. It is to be noted that although Nehari's proof is of the area type to be considered here, his formulation of the inequalities omits the important complementing role of the Guelfer functions D_1^*; in DeTemple on the other hand, the characteristic properties of the D_1^* class are emphasized, but the variational proof reverts attention again to the class S_1.

2. **An area theorem for univalent Guelfer functions.** For any $F(z) \in D^*(\beta)$, we may consider the following power series expansions:

$$
(20) \qquad
\begin{aligned}
\log \frac{F(z)-F(\zeta)}{z-\zeta} &= \sum_{m,n=0}^{\infty} k_{mn} z^m \zeta^n, \\
-\log [F(z)+F(\zeta)] &= \sum_{m,n=0}^{\infty} l_{mn} z^m \zeta^n, \\
\log [1+F(z)\overline{F(\zeta)}] &= \sum_{m,n=0}^{\infty} p_{mn} z^m \bar{\zeta}^n, \\
-\log [1-F(z)\overline{F(\zeta)}] &= \sum_{m,n=0}^{\infty} q_{mn} z^m \bar{\zeta}^n.
\end{aligned}
$$

The properties of the class $D^*(\beta)$ guarantee the convergence of these series in the bicylinder $U \times U$. It should be observed that the coefficients k_{mn} and l_{mn} are symmetric in their indices, while p_{mn} and q_{mn} are Hermitian symmetric.

Next we define a sequence of rational functions $\Phi_n(t)$ by means of the generating function

$$
(21) \qquad \log \frac{1-t\beta}{1-tF(\zeta)} = \sum_{n=1}^{\infty} \frac{1}{n} \Phi_n(t) \zeta^n.
$$

The $\Phi_n(t)$ are closely related to the Faber polynomials (see Schiffer [7]). Indeed, if $F_n(s)$ is the nth Faber polynomial for $[F(z)-\beta]$, then

$$
(22) \qquad \Phi_n(t) = F_n(t/(1-\beta t)).
$$

It follows that each $\Phi_n(t)$ has a pole only at $t=\beta^{-1}$.

We utilize the functions $\Phi_n(t)$ to expand the left sides of (20) in powers of ζ, $|\zeta|$ small, holding z fixed. Comparing equal powers of ζ, we deduce the identities

$$
(23) \qquad
\begin{aligned}
\sum_{m=0}^{\infty} k_{m0} z^m &= \log \frac{F(z)-\beta}{z}, \\
n \sum_{m=0}^{\infty} k_{mn} z^m &= z^{-n} - \Phi_n\left[\frac{1}{F(z)}\right] && (n \geqq 1), \\
\sum_{m=0}^{\infty} l_{m0} z^m &= -\log [\beta + F(z)], \\
n \sum_{m=0}^{\infty} l_{mn} z^m &= \Phi_n\left[-\frac{1}{F(z)}\right] && (n \geqq 1), \\
\sum_{m=0}^{\infty} p_{m0} z^m &= \log [1+\beta F(z)], \\
n \sum_{m=0}^{\infty} p_{mn} z^m &= -\overline{\Phi_n[-\overline{F(z)}]} && (n \geqq 1), \\
\sum_{m=0}^{\infty} q_{m0} z^m &= -\log [1-\beta F(z)], \\
n \sum_{m=0}^{\infty} q_{mn} z^m &= \overline{\Phi_n[\overline{F(z)}]} && (n \geqq 1).
\end{aligned}
$$

The identities (23) enable us to define a function $q(w)$ for which the inherently positive integral $\iint |q'(w)|^2\, d\tau$ may be evaluated when taken over an appropriate domain. The basic idea is to consider simultaneously both $F(z) \in D^*(\beta)$ and its negative

$$\hat{F}(z) = -F(z). \tag{24}$$

The ranges of $F(z)$ and $\hat{F}(z)$ are nonoverlapping subdomains of the unit disc $|w|<1$. In particular, by choosing a ρ $(0<\rho<1)$ we may define two nonintersecting simple curves γ and $\hat{\gamma}$ inside $|w|<1$ with the parametric representations

$$w = F(\rho e^{i\theta}), \qquad w = \hat{F}(\rho e^{i\theta}). \tag{25}$$

Along with the unit circumference $|w|=1$ they bound a triply-connected domain Δ_ρ which contains neither $\pm\beta$.

The function $q(w)$ is now defined as follows. Let x_0, y_0 be any two real numbers and let x_n, y_n $(n=1, 2, \ldots, N)$ be any $2N$ complex numbers. Then let

$$\begin{aligned} q(w) = {} & x_0 \log \frac{w-\beta}{1-\beta w} - y_0 \log \frac{w+\beta}{1+\beta w} \\ & - \sum_{m=1}^{N} \left[\frac{x_m}{m}\, \Phi_m(w^{-1}) - \frac{\overline{x_m}}{m}\, \overline{\Phi_m(\overline{w})}\right] \\ & + \sum_{m=1}^{N} \left[\frac{y_m}{m}\, \Phi_m(-w^{-1}) - \frac{\overline{y_m}}{m}\, \overline{\Phi_m(-\overline{w})}\right]. \end{aligned} \tag{26}$$

We observe that $q(w)$ is analytic in Δ_ρ and has there a single-valued real part in view of the realness of x_0 and y_0. Using the complex version of Green's identity, we can write

$$0 \leqq \iint_{\Delta_\rho} |q'(w)|^2\, d\tau = \frac{1}{i} \oint_{\partial\Delta_\rho} \operatorname{Re}\{q(w)\}\, dq(w), \tag{27}$$

where $d\tau$ is the area element in Δ_ρ. From (26) we note that $\operatorname{Re}\{q(w)\}=0$ on $|w|=1$. Thus from the parametric representations (25) for the remaining two components γ and $\hat{\gamma}$ of the boundary $\partial\Delta_\rho$, we see (27) goes over to

$$I_1 + I_2 \leqq 0, \tag{28}$$

where

$$\begin{aligned} I_1 &= \frac{1}{i} \int_0^{2\pi} \operatorname{Re}\{q[F(\rho e^{i\theta})]\}\, \frac{dq[F]}{d\theta}\, d\theta, \\ I_2 &= \frac{1}{i} \int_0^{2\pi} \operatorname{Re}\{q[\hat{F}(\rho e^{i\theta})]\}\, \frac{dq[\hat{F}]}{d\theta}\, d\theta. \end{aligned} \tag{29}$$

The identities (23) enable us to express these integrals in terms of the coefficients of the series (20). Indeed, we find after straightforward calculation that

$$q[F(z)] = x_0 \log z - \sum_{n=1}^{N} \frac{x_n}{n z^n} + \sum_{m=0}^{\infty} A_m z^m, \tag{30}$$

where

$$A_m = \sum_{n=0}^{N} [k_{mn}x_n + q_{mn}\bar{x}_n + l_{mn}y_n + p_{mn}\bar{y}_n]. \tag{31}$$

Similarly, we can calculate

$$q[\hat{F}(z)] = -y_0 \log z + \sum_{n=1}^{N} \frac{y_n}{nz^n} - \sum_{m=0}^{\infty} B_m z^m + \text{imaginary constant}, \tag{32}$$

where

$$B_m = \sum_{n=0}^{N} [k_{mn}y_n + q_{mn}\bar{y}_n + l_{mn}x_n + p_{mn}\bar{x}_n]. \tag{33}$$

In view of the developments (30) and (32), the integrals I_1 and I_2 may be evaluated to give

$$\begin{aligned} I_1 &= 2\pi x_0^2 \log \rho - \pi \sum_{n=1}^{N} \frac{1}{n} |x_n|^2 \rho^{-2n} \\ &\quad + \pi \sum_{n=1}^{\infty} n |A_n|^2 \rho^{2n} + 2\pi x_0 \operatorname{Re}\{A_0\}, \\ I_2 &= 2\pi y_0^2 \log \rho - \pi \sum_{n=1}^{N} \frac{1}{n} |y_n|^2 \rho^{-2n} \\ &\quad + \pi \sum_{n=1}^{\infty} n |B_n|^2 \rho^{2n} + 2\pi y_0 \operatorname{Re}\{B_0\}. \end{aligned} \tag{34}$$

From (28) we conclude

$$\begin{aligned} &2x_0 \operatorname{Re}\{A_0\} + 2y_0 \operatorname{Re}\{B_0\} + \sum_{n=1}^{\infty} n[|A_n|^2 + |B_n|^2]\rho^{2n} \\ &\qquad \leqq \sum_{n=1}^{N} \frac{1}{n} [|x_n|^2 + |y_n|^2]\rho^{-2n} + 2(x_0^2 + y_0^2) \log \rho^{-1}. \end{aligned} \tag{35}$$

This inequality holds for all $\rho < 1$, so we may pass to the limit, $\rho \uparrow 1$. We obtain an area theorem for bounded univalent Guelfer functions

$$2x_0 \operatorname{Re}\{A_0\} + 2y_0 \operatorname{Re}\{B_0\} + \sum_{n=1}^{\infty} n[|A_n|^2 + |B_n|^2] \leqq \sum_{n=1}^{N} \frac{1}{n} [|x_n|^2 + |y_n|^2]. \tag{36}$$

It now becomes useful to introduce a change of variables. Let

$$\lambda_n = \tfrac{1}{2}(x_n + y_n), \qquad \mu_n = \tfrac{1}{2}(x_n - y_n) \tag{37}$$

and denote

$$\begin{aligned} a_{mn} &= k_{mn} + l_{mn}, & \alpha_{mn} &= k_{mn} - l_{mn}, \\ b_{mn} &= q_{mn} + p_{mn}, & \beta_{mn} &= q_{mn} - p_{mn}. \end{aligned} \tag{38}$$

Finally we define the new vectors

$$C_m \equiv \tfrac{1}{2}(A_m+B_m) = \sum_{n=0}^{N} [a_{mn}\lambda_n + b_{mn}\bar{\lambda}_n], \tag{39}$$
$$D_m \equiv \tfrac{1}{2}(A_m-B_m) = \sum_{n=0}^{N} [\alpha_{mn}\mu_n + \beta_{mn}\bar{\mu}_n].$$

Observing that

$$x_0A_0 + y_0B_0 = 2\lambda_0 C_0 + 2\mu_0 D_0 \tag{40}$$

we find from (36) that

$$2\lambda_0 \operatorname{Re}\{C_0\} + 2\mu_0 \operatorname{Re}\{D_0\} + \sum_{n=1}^{\infty} n[|C_n|^2 + |D_n|^2] \leqq \sum_{n=1}^{N} \frac{1}{n}[|\lambda_n|^2 + |\mu_n|^2]. \tag{41}$$

Here we have utilized the parallelogram law for arbitrary complex numbers a, b, namely,

$$|(a+b)/2|^2 + |(a-b)/2|^2 = \tfrac{1}{2}[|a|^2 + |b|^2].$$

Having now separated the variables in the inequality, (41) may be split into two independent conditions. For example, taking $\mu_0 = \mu_1 = \cdots = \mu_N = 0$ we find

$$2\lambda_0 \operatorname{Re}\{C_0\} + \sum_{n=1}^{\infty} n|C_n|^2 \leqq \sum_{n=1}^{N} \frac{|\lambda_n|^2}{n}. \tag{42}$$

Similarly we find

$$2\mu_0 \operatorname{Re}\{D_0\} + \sum_{n=1}^{\infty} n|D_n|^2 \leqq \sum_{n=1}^{N} \frac{|\mu_n|^2}{n}. \tag{43}$$

We utilize our inequalities as follows. First consider the quantity

$$\Phi[F] = \operatorname{Re}\left\{\sum_{m,n=0}^{N} a_{mn}\lambda_m\lambda_n\right\} + \sum_{m,n=0}^{N} b_{mn}\lambda_m\bar{\lambda}_n. \tag{44}$$

From (39) we find

$$\begin{aligned}
\Phi[F] &= \lambda_0 \operatorname{Re}\{C_0\} + \operatorname{Re}\left\{\sum_{m=1}^{N} \lambda_m C_m\right\} \\
&\leqq \lambda_0 \operatorname{Re}\{C_0\} + \left[\left(\sum_{m=1}^{N} \frac{|\lambda_m|^2}{m}\right)\left(\sum_{m=1}^{N} m|C_m|^2\right)\right]^{1/2} \\
&\leqq \lambda_0 \operatorname{Re}\{C_0\} + \left[\left(\sum_{m=1}^{N} \frac{|\lambda_m|^2}{m}\right)^2 - 2\lambda_0 \sum_{m=1}^{N} \frac{|\lambda_m|^2}{m} \operatorname{Re}\{C_0\}\right]^{1/2} \\
&\leqq \lambda_0 \operatorname{Re}\{C_0\} + \sum_{m=1}^{N} \frac{|\lambda_m|^2}{m} - \lambda_0 \operatorname{Re}\{C_0\}.
\end{aligned} \tag{45}$$

Here we have used first Schwarz's inequality, then our inequality (42), and finally we have completed the square under the radical. This computation gives us

THEOREM 1. *Let $F(z) \in D_1^*$ and let a_{mn} and b_{mn} be defined by*

$$\log \frac{F(z)-F(\zeta)}{[F(z)+F(\zeta)](z-\zeta)} = \sum_{m,n=0}^{\infty} a_{mn} z^m \zeta^n, \tag{46}$$

$$\log \frac{1+F(z)\overline{F(\zeta)}}{1-F(z)\overline{F(\zeta)}} = \sum_{m,n=0}^{\infty} b_{mn} z^m \bar{\zeta}^n. \tag{47}$$

Then for any real number λ_0 and any N complex numbers $\lambda_1, \ldots, \lambda_N$,

$$\operatorname{Re}\left\{\sum_{m,n=0}^{N} a_{mn}\lambda_m\lambda_n\right\} + \sum_{m,n=0}^{N} b_{mn}\lambda_m\bar{\lambda}_n \leqq \sum_{m=1}^{N} \frac{|\lambda_m|^2}{m}. \tag{48}$$

Equality is possible only when

$$\operatorname{Re}\left\{\sum_{n=0}^{N} \left[a_{0n}\lambda_n + b_{0n}\bar{\lambda}_n\right]\right\} = 0, \tag{49}$$

$$\begin{aligned} \sum_{n=0}^{N} [a_{mn}\lambda_n + b_{mn}\bar{\lambda}_n] &= \frac{\bar{\lambda}_m}{m}, \qquad m = 1, 2, \ldots, N, \\ &= 0, \qquad m = N+1, N+2, \ldots, \end{aligned} \tag{50}$$

and in this case the complement of the union of the domains $F(U)$ and $-F(U)$ with respect to the unit disc has zero area.

Similarly, we find beginning from (43) the

THEOREM 2. *Let $F(z) \in D_1^*$ and let α_{mn} and β_{mn} be defined by*

$$\log \frac{F^2(z)-F^2(\zeta)}{z-\zeta} = \sum_{m,n=0}^{\infty} \alpha_{mn} z^m \zeta^n, \tag{51}$$

$$-\log [1-F^2(z)\overline{F^2(\zeta)}] = \sum_{m,n=0}^{\infty} \beta_{mn} z^m \bar{\zeta}^n. \tag{52}$$

Then for any real number μ_0 and any N complex numbers $\mu_1, \ldots, \mu_N$,

$$\operatorname{Re}\left\{\sum_{m,n=0}^{N} \alpha_{mn}\mu_m\mu_n\right\} + \sum_{m,n=0}^{N} \beta_{mn}\mu_m\bar{\mu}_n \leqq \sum_{n=1}^{N} \frac{|\mu_n|^2}{n}. \tag{53}$$

Equality is possible only when

$$\operatorname{Re}\left\{\sum_{n=0}^{N} [\alpha_{0n}\mu_n + \beta_{0n}\bar{\mu}_n]\right\} = 0, \tag{54}$$

$$\begin{aligned} \sum_{n=0}^{N} [\alpha_{mn}\mu_n + \beta_{mn}\bar{\mu}_n] &= \frac{\bar{\mu}_m}{m}, \qquad m = 1, \ldots, N, \\ &= 0, \qquad m = N+1, N+2, \ldots, \end{aligned} \tag{55}$$

and in this case the complement of the union of the domains $F(U)$ and $-F(U)$, taken with respect to the unit disc, has zero area.

Each of Theorems 1 and 2 gives analogous results for the unbounded class D^* by an obvious limiting procedure. If these inequalities are transformed by (8) into statements on Bieberbach-Eilenberg functions, these results coincide with those obtained by Hummel and Schiffer [4]. For definiteness we state

THEOREM 3. *Let $G(z) \in D^*$ and let the coefficients a_{mn} be defined by*

$$\log \frac{G(z)-G(\zeta)}{[G(z)+G(\zeta)](z-\zeta)} = \sum_{m,n=0}^{\infty} a_{mn} z^m \zeta^n. \tag{56}$$

Then for λ_0 real and $\lambda_1, \ldots, \lambda_n$ any N complex numbers,

$$\operatorname{Re}\left\{\sum_{m,n=0}^{N} a_{mn}\lambda_m\lambda_n\right\} \leqq \sum_{m=1}^{N} \frac{|\lambda_m|^2}{m}. \tag{57}$$

To compare with Hummel-Schiffer we need only notice from (8) that

$$(G(z)-G(\zeta))/(G(z)+G(\zeta)) = (E(z)-E(\zeta))/(1-E(z)E(\zeta)). \tag{58}$$

If in turn $G(z)$ is related to a function $g(z) \in S$ by means of (4), then inequality (57) becomes equivalent to that of Garabedian-Schiffer.

3. **Applications.** Given a function $f(z) \in S_1$ which omits a value $\beta^2 e^{-i\varphi}$, we have seen that the $F(z)$ given by (10) is in $D^*(\beta)$. Hence the inequalities of Theorems 1 and 2 can be restated for these subclasses of S_1 functions. We find that Theorem 1 becomes equivalent to inequalities of Nehari [5, pp. 325–327] and DeTemple [1], while Theorem 2 is equivalent to the Nehari inequalities as generalized by Schiffer and Tammi [8].

Our inequalities of the preceding section may also be utilized to obtain coefficient and growth estimates for functions $F(z)$ in the class $D(\beta)$. Since the inequalities apply only to univalent functions, the following lemma for the subordination of $D(\beta)$ functions is essential.

LEMMA. *Let $F(z) \in D(\beta)$. Then there exists for $F(z)$ a univalent majorant $F^*(z) \in D^*(\beta)$. That is*

$$F(z) = F^*[\omega(z)] \tag{59}$$

where $F^* \in D(\beta)$ and $\omega(z)$ is analytic on U with $\omega(0)=0$, $|\omega(z)|<1$.

The proof, which we omit, can be achieved by the obvious modifications of procedures found in either Rogosinski [6] or Guelfer [3].

A special case of inequality (48) now gives us

THEOREM 4. *Let $F(z)=\beta(1+\alpha_1 z+\cdots+\alpha_n z^n+\cdots) \in D(\beta)$. Then*

$$|\alpha_1| \leqq 2(1-\beta^2)/(1+\beta^2), \tag{60}$$

and equality holds only for the functions $w=F(z)$ mapping U onto the half-disc $\{|w|<1, \operatorname{Re} w>0\}$; that is, the functions given by

$$\frac{F(z)-\beta}{1-\beta F(z)}\frac{1+\beta F(z)}{F(z)+\beta} = \varepsilon z, \qquad |\varepsilon| = 1. \tag{61}$$

Proof. Choose $\lambda_0=1$ and $\lambda_1=\lambda_2=\cdots=\lambda_n=0$ in (48). Then for any function $F^*(\zeta)=\beta(1+\alpha_1^*\zeta+\cdots)\in D^*(\beta)$ we have

$$\operatorname{Re}\{a_{00}\}+b_{00} = \log\frac{|\alpha_1^*|}{2}+\log\frac{1+\beta^2}{1-\beta^2} \leqq 0, \tag{62}$$

which is equivalent to (60) in the case of univalent functions. For a general $F(z)\in D(\beta)$ we can write $F(z)=F^*(\omega(z))$ by the lemma. But then

$$|\alpha_1| = |\beta^{-1}F'(0)| = |\beta^{-1}F^{*\prime}(0)|\,|\omega'(0)| \leqq |\alpha_1^*| \tag{63}$$

with equality only if $\omega(z)=\varepsilon z$, $|\varepsilon|=1$.

For equality to hold in (60), we note from (49) and (50) that

$$C_0 = a_{00}+b_{00} = \text{imaginary}, \qquad C_m = a_{m0}+b_{m0} = 0, \qquad m = 1,2,\ldots. \tag{64}$$

Subtracting (42) from (40) we then find

$$q[F(z)]-q[-F(z)] = 2\log z+2C_0. \tag{65}$$

On the other hand we have from (26) that

$$q[F(z)]-q[-F(z)] = 2\log\frac{F(z)-\beta}{1-\beta F(z)}\frac{1+\beta F(z)}{F(z)+\beta}+\text{imaginary}. \tag{66}$$

Comparing (65) and (66) we conclude upon taking real parts that

$$\log\left|\frac{F(z)-\beta}{1-\beta F(z)}\cdot\frac{1+\beta F(z)}{F(z)+\beta}\cdot\frac{1}{z}\right| \equiv 0. \tag{67}$$

From (67) the statements for the extremal functions follow easily. □

For a given $f(z)\in S(b_1)$ we can apply Theorem 4 to the function $F(z)$ given by transformation (10). Thus if $f(z)\neq d$ we obtain the well-known inequality

$$|d| \geqq b_1(1+\sqrt{(1-b_1)})^{-2}, \tag{68}$$

with equality holding only for the function given by

$$f(z)(1+\eta f(z))^{-2} = 4|d|(1+|d|)^{-2}z(1+\eta z)^{-2}, \qquad \eta = \bar{d}/|d|. \tag{69}$$

THEOREM 5. *Let $F(z)\in D_1$. Then*

$$\left|\frac{F'(z)}{F(z)}\right|\frac{1+|F(z)|^2}{1-|F(z)|^2} \leqq \frac{2}{1-|z|^2} \tag{70}$$

holds for all $z \in U$. *Equality, at a point* $\zeta \in U$, *holds only for the functions*

$$F(z) = \frac{\overline{F_1(-\zeta)}}{|F_1(-\zeta)|} F_1\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right) \tag{71}$$

where $F_1(z)$ *has the form given by* (61).

Proof. Let $F(z) \in D_1$, $\zeta \in U$, and form

$$F_1(z) = \frac{\overline{F(\zeta)}}{|F(\zeta)|} F\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right). \tag{72}$$

Then $F_1(z) \in D(|F(\zeta)|)$ and

$$F_1'(0) = \frac{\overline{F(\zeta)}}{|F(\zeta)|} F'(\zeta)(1-|\zeta|^2). \tag{73}$$

The result now follows by application of Theorem 4 to $F_1(z)$. $\square$

THEOREM 6. *Let* $F(z) \in D(\beta)$. *Then*

$$\left|\log \frac{F(z)}{\beta} \frac{1-\beta^2}{1-F^2(z)}\right| \leqq \log \frac{1+|z|}{1-|z|} \tag{74}$$

and in particular

$$\frac{1-|z|}{1+|z|} \leqq \frac{\beta}{1-\beta^2}\left|\frac{1}{F(z)}-F(z)\right| \leqq \frac{1+|z|}{1-|z|}, \tag{75}$$

$$\left|\arg\left(\frac{1}{F(z)}-F(z)\right)\right| \leqq \log \frac{1+|z|}{1-|z|}. \tag{76}$$

Proof. Integrate $(dF/F)((1+F^2)/(1-F^2))$ over the line segment $\langle 0, z\rangle$, and use inequality (70). $\square$

For $F(z) \in D(\beta)$ we have seen $G(z) = \beta^{-1}F(z) \in D$. If we replace $F(z)$ by $\beta G(z)$ in our inequalities of this section, then results of Guelfer for the class D are obtained upon sending $\beta \to 0$.

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