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AREA MEASURE AND RADÓ'S LOWER AREA[1]

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Abstract. The theory of Geöcze area for two-dimensional surfaces in three-dimensional space had been essentially completed by the mid 1950's. The only hypothesis needed for all theorems in this case is the finiteness of the area. See [2] for an account of this theory. In the early 1960's, H. Federer established, in his paper [6], fundamental facts concerning his integral geometric area for higher dimensional area theory by employing the theory of normal and integral currents. These facts employ not only the finiteness of area as a basic hypothesis but certain other hypotheses as well. The extensions of Geöcze type area to higher dimensions also employ not only the finiteness of area but certain added hypotheses. These hypotheses are of such a nature as to allow the use of the theory of quasi-additivity [3], [11].

The present paper concerns these added hypotheses which play such an important part of higher-dimensional area theory of today. It is shown that Radó's lower area is the best Geöcze type area to describe these added hypotheses. That is, it is shown that the quasi-additivity hypotheses of Geöcze area in [11] imply the quasi-additivity hypotheses of lower area. Second, it is shown that the quasi-additivity hypotheses for lower area imply that the surface has the essential cylindrical property defined by J. Breckenridge in [5]. This essential cylindrical property is proved to be equivalent to the existence of area measures on the middle space of the mapping representing the surface. Finally, it is shown that the essential cylindrical property of a surface is equivalent to the quasi-additivity condition for lower area. Thus, an intrinsic property of the surface characterizes the quasi-additivity condition for the lower area of a surface.

0. **Introduction.** Around 1945, T. Radó introduced a Geöcze type area which he named lower area. As he remarked in [12] lower area is the largest of the Geöcze type areas of a surface. Also, around 1951, L. Cesari used surface area to induce a measure over a suitable Borel algebra on the domain of the mapping which represents the surface. The definition of area used by L. Cesari was a Geöcze type area but not lower area. In dimension two, the equality of the two areas to Lebesgue area is known and so Cesari's area measure is applicable to both the lower area and Lebesgue area. For dimensions larger than two the equality of areas is not known. Nonetheless, Cesari's area measure can be constructed from his Geöcze area under suitable additional hypothesis and Radó's lower area is well defined.

It is the purpose of this paper to investigate measures induced by areas and the lower area of Radó. This investigation is prompted by certain "cylindrical"

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properties of mappings which were investigated in [5], [9] and [2]. These cylindrical properties relate the mapping to its corresponding flat mappings. It is not surprising that such cylindrical properties are important; for, the area measure of a surface should be intimately tied to the area measures of the flat mappings corresponding to the surface. Indeed, it is shown in the present paper that the essential cylindrical property as defined by J. Breckenridge is precisely the condition needed to define an area measure of a mapping which is consistent with the area measures of its corresponding flat mappings.

§1 establishes some notation and sets forth some concepts and theorems concerning flat mappings. In §2, we define Radó's lower area and relate it to the concept of quasi-additivity introduced by L. Cesari in 1962. §3 concerns the area measure induced by the quasi-additivity property of lower area for a certain class of mappings. Finally, we establish in §4 a criterion under which lower area will induce an area measure on the surface; and also an intrinsic characterization of the quasi-additivity condition for lower area is established.

1. **Preliminaries.** Let k and n be integers with $2 \leqq k \leqq n$.

$\mathscr{T}(n, k)$ will denote the class of all continuous transformations f whose domain $(f) \subset R^k$ and range $(f) \subset R^n$. In general we will only consider those f with domain (f) possessing nice properties such as being locally connected, locally compact and k-dimensional at each of its points.

A mapping will be called *flat* if $f \in \mathscr{T}(k, k)$.

If $f \in \mathscr{T}(n, k)$ and $A \subset$ domain (f), then $L(f, A)$ will denote the Lebesgue area of f on A. (We, of course, mean k-dimensional area.)

$\Lambda(n, k)$ will denote the set of all increasing k-termed sequences in $\{1, 2, \dots, n\}$. For each $\lambda = (\lambda_1, \dots, \lambda_k) \in \Lambda(n, k)$, $P^\lambda \colon R^n \to R^k$ will be the usual orthogonal projection given by $P^\lambda(y) = (y_{\lambda_1}, \dots, y_{\lambda_k})$, where $y = (y_1, \dots, y_n) \in R^n$.

For convenience of notation, s will denote a function defined as follows: For each pair of sets A and B,

$$s(A, B) = 0 \quad \text{if } A \not\subset B, \qquad s(A, B) = 1 \quad \text{if } A \subset B.$$

For the remainder of this section we will assume f is a flat mapping and $X = \text{domain}\,(f)$. For each simple polyhedral region $\pi \subset X$, we have the usual topological index $O(f, \pi, y)$, $y \in R^k$. Let $A \subset X$. For each finite system $\mathfrak{S}$ of non-overlapping simple polyhedral regions $\pi \subset X$, we have the numbers

$$(*) \qquad \begin{gathered} \sum_{\pi \in \mathfrak{S}} s(\pi, A) |O(f, \pi, y)|, \\ \sum_{\pi \in \mathfrak{S}} s(\pi, A) O^+(f, \pi, y), \quad \text{and} \\ \sum_{\pi \in \mathfrak{S}} s(\pi, A) O^-(f, \pi, y), \end{gathered}$$

where O^+ and O^- denote the positive and negative parts of O. The suprema over all such systems $\mathfrak{S}$ are denoted by $N(f, A, y)$, $N^+(f, A, y)$ and $N^-(f, A, y)$, respectively. It is known that each of the functions in (*) is lower semicontinuous on R^k and $N(f, A, y) = N^+(f, A, y) + N^-(f, A, y)$ for almost every $y \in R^k$. The suprema over $\mathfrak{S}$ of the integrals of the three functions of (*) will be denoted by $V(f, A)$, $V^+(f, A)$ and $V^-(f, A)$, respectively. The flat mapping f is said to be of *bounded variation* (BV) if $\int N(f, A, y)\, dy < \infty$.

Suppose f is BV and π is a simple polyhedral region contained in X. Then $O(f, \pi, y)$ is summable and we will denote its integral by

$$u(f, \pi) = \int O(f, \pi, y)\, dy.$$

For each $A \subset X$, we define

$$U(f, A) = \operatorname{Sup} \sum_{\pi \in \mathfrak{S}} s(\pi, A)|u(f, \pi)|,$$

$$U^+(f, A) = \operatorname{Sup} \sum_{\pi \in \mathfrak{S}} s(\pi, A)u^+(f, \pi),$$

and

$$U^-(f, A) = \operatorname{Sup} \sum_{\pi \in \mathfrak{S}} s(\pi, A)u^-(f, \pi),$$

where the suprema are taken over all finite systems $\mathfrak{S}$ of nonoverlapping simple polyhedral regions $\pi \subset X$.

The following theorem is true [10], [2].

1.1. THEOREM. *For each flat mapping f and $A \subset$* domain (f),

$$L(f, A) = \int N(f, A, y)\, dy = V(f, A),$$

$$\int N^\pm(f, A, y)\, dy = V^\pm(f, A).$$

Furthermore, if f is BV then $U(f, A)$ and $U^\pm(f, A)$ can be added to the above equalities.

For each BV flat mapping f and $A \subset$ domain (f), we define the *relative area of f on A* as

$$\nu(f, A) = U^+(f, A) - U^-(f, A).$$

For a BV flat mapping f with $X =$ domain (f) we define two mesh functions δ and $\bar{\delta}$. Let $\mathcal{D}$ be a finite system of nonoverlapping bounded domains D with closure $\bar{D} \subset X$, and $\mathfrak{S}$ be a finite system of nonoverlapping simple polyhedral regions $\pi \subset X$. Then,

$$\delta(\mathcal{D}) = \operatorname{Max}\{\operatorname{diam}[f(D)] : D \in \mathcal{D}\} + \Big[U(f, X) - \sum_{D \in \mathcal{D}} |\nu(f, D)|\Big],$$

and

$$\bar{\delta}(\mathfrak{S}) = \operatorname{Max}\{\operatorname{diam}[f(\pi)] : \pi \in \mathfrak{S}\} + \Big[U(f, X) - \sum_{\pi \in \mathfrak{S}} |u(f, \pi)|\Big].$$

Each $\mathfrak{S}$ can be considered as a $\mathcal{D}$ by just taking the interior of each $\pi \in \mathfrak{S}$. The following lemmas are valid.

1.2. LEMMA. $\delta(\mathfrak{S}) \leqq 2\delta(\mathfrak{S})$.

Proof. Observe that $U(f, \pi) = U^+(f, \pi) + U^-(f, \pi)$, $u(f, \pi) = u^+(f, \pi) - u^-(f, \pi)$ and $U^\pm(f, \pi) \geqq u^\pm(f, \pi)$ for $\pi \in \mathfrak{S}$. Hence it follows that $\delta(\mathfrak{S}) \leqq 2\delta(\mathfrak{S})$.

1.3. LEMMA. *If* $A \subset$ domain (f), *then*

$$U(f, A) - \sum_{D \in \mathscr{D}} s(D, A)|\nu(f, D)| \to 0$$

as $\delta(\mathscr{D}) \to 0$.

Proof. From [11, §6.5] we infer that for each $\varepsilon > 0$ there is an $\eta > 0$ and a set B, open in X, such that

(i) $B \supset X - A$,

(ii) $U(f, A) + U(f, B) < U(f, X) + \varepsilon$,

(iii) $S \subset X$, $S \cap (X-A) \neq \varnothing \neq S \cap (X-B)$ imply diam $[f(S)] \geqq \eta$.

Suppose $\delta(\mathscr{D}) < \eta$. Then, for each $D \in \mathscr{D}$, we have diam $[f(D)] < \eta$. Hence, $D \in \mathscr{D}$ and $D \cap (X-A) \neq \varnothing$ imply $D \subset B$. So, we have

$$\begin{aligned} U(f, A) - \sum_{D \in \mathscr{D}} s(D, A)|\nu(f, D)| \\ &\leqq U(f, A) - \sum_{D \in \mathscr{D}} s(D, A)|\nu(f, D)| + U(f, B) - \sum_{D \in \mathscr{D}} s(D, B)|\nu(f, D)| \\ &\leqq U(f, X) - \sum_{D \in \mathscr{D}} |\nu(f, D)| + \varepsilon \leqq \delta(\mathscr{D}) + \varepsilon, \end{aligned}$$

since $s(D, A) + s(D, B) \geqq 1$ for all $D \in \mathscr{D}$.

2. **Radó's lower area and quasi-additivity.** Let $f \in \mathscr{T}(n, k)$ and $X =$ domain (f).

For each $A \subset X$ we follow Radó and define the *lower area* of f on A as the supremum of the set of numbers

$$\sum_{D \in \mathscr{D}} s(D, A) \Big(\sum_{\lambda \in \Lambda(n,k)} [L(P^\lambda \circ f, D)]^2 \Big)^{1/2},$$

where $\mathscr{D}$ is a finite system of nonoverlapping domains $D \subset X$. The lower area of f on A is denoted by $R(f, A)$.

The following theorem is easily proved.

2.1. THEOREM. *Let* $f \in \mathscr{T}(n, k)$ *and* $A \subset X =$ domain (f). *Then*

(i) $R(f, A) \leqq L(f, A)$;

(ii) $R(f, A) = L(f, A)$ *iff f is a flat mapping*;

(iii) $(\sum_{\lambda \in \Lambda(n,k)} [R(P^\lambda \circ f, A)]^2)^{1/2} \leqq R(f, A) \leqq \sum_{\lambda \in \Lambda(n,k)} R(P^\lambda \circ f, A)$; *and*

(iv) $R(f, X) < \infty$ *if and only if* $P^\lambda \circ f$ *is BV for each* $\lambda \in \Lambda(n, k)$.

Let $f \in \mathscr{T}(n, k)$ with $P^\lambda \circ f$ of bounded variation for each $\lambda \in \Lambda(n, k)$. Then for each finite system $\mathscr{D}$ of nonoverlapping bounded domains D with closures $\bar{D} \subset X$ $=$ domain (f), and each $\lambda \in \Lambda(n, k)$, the mesh $\delta_\lambda(\mathscr{D})$ is defined as in §1 above for the flat mapping $P^\lambda \circ f$. We define, for the nonflat case, the mesh δ as

$$\delta(\mathscr{D}) = \text{Max } \{\delta_\lambda(\mathscr{D}) : \lambda \in \Lambda(n, k)\}.$$

Similarly, we define the mesh $\delta(\mathfrak{S})$ of a finite system of nonoverlapping simple polyhedral regions $\pi \subset X$.

$\mathscr{R}^*(n, k)$ will denote the class of those mappings $f \in \mathscr{T}(n, k)$ for which $\delta(\mathscr{D})$ can be made arbitrarily small. Similarly, $\mathscr{T}^*(n, k)$ will denote those mapping $f \in \mathscr{T}(n, k)$ for which $\delta(\mathfrak{S})$ can be made arbitrarily small. The class $\mathscr{T}^*(n, k)$ has been studied in [11], [5] and [8]. Lemma 1.2 implies the following theorem.

2.2. THEOREM. $\mathscr{T}^*(n, k) \subset \mathscr{R}^*(n, k)$.

Let $e_1, \ldots, e_n$ be the standard basis for R^n and for each $\lambda \in \Lambda(n, k)$ let $e_\lambda = e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_k}$. Then for each $f \in \mathscr{T}(n, k)$ with $R(f, X) < \infty$ and $A \subset X =$ domain (f), we have $\nu(P^\lambda \circ f, A)$ as defined in §1. We define the k-vector

$$\nu(f, A) = \sum_{\lambda \in \Lambda(n,k)} \nu(P^\lambda \circ f, A) e_\lambda$$

and denote the Euclidean norm of $\nu(f, A)$ by $|\nu(f, A)|$.

The following theorem generalizes Lemma 1.3.

2.3. THEOREM. *Let $f \in \mathscr{R}^*(n, k)$ and $A \subset X =$ domain (f). Then*

$$R(f, A) = \lim_{\delta(\mathscr{D}) \to 0} \sum_{D \in \mathscr{D}} s(D, A) |\nu(f, D)|.$$

Proof. Let D_0 be any finite system of nonoverlapping domains contained in X. Then, by Lemma 1.3 and Minkowski's inequality, we have

$$\begin{aligned}
\sum_{D_0 \in \mathscr{D}_0} & s(D_0, A) \Big(\sum_{\lambda \in \Lambda(n,k)} [L(P^\lambda \circ f, D_0)]^2 \Big)^{1/2} \\
&= \sum_{D_0 \in \mathscr{D}_0} s(D_0, A) \Big(\sum_{\lambda \in \Lambda(n,k)} [U(P^\lambda \circ f, D_0)]^2 \Big)^{1/2} \\
&= \sum_{D_0 \in \mathscr{D}_0} s(D_0, A) \Big(\sum_{\lambda \in \Lambda(n,k)} \Big[\lim_{\delta(\mathscr{D}) \to 0} \sum_{D \in \mathscr{D}} s(D, D_0) |\nu(P^\lambda \circ f, D)| \Big]^2 \Big)^{1/2} \\
&= \lim_{\delta(\mathscr{D}) \to 0} \sum_{D_0 \in \mathscr{D}_0} s(D_0, A) \Big(\sum_{\lambda \in \Lambda(n,k)} \Big[\sum_{D \in \mathscr{D}} s(D, D_0) |\nu(P^\lambda \circ f, D)| \Big]^2 \Big)^{1/2} \\
&\leqq \liminf_{\delta(\mathscr{D}) \to 0} \sum_{D_0 \in \mathscr{D}_0} s(D_0, A) \sum_{D \in \mathscr{D}} s(D, D_0) |\nu(f, D)| \\
&\leqq \liminf_{\delta(\mathscr{D}) \to 0} \sum_{D \in \mathscr{D}} s(D, A) |\nu(f, D)| \\
&\leqq \limsup_{\delta(\mathscr{D}) \to 0} \sum_{D \in \mathscr{D}} s(D, A) |\nu(f, D)| \\
&\leqq \limsup_{\delta(\mathscr{D}) \to 0} \sum_{D \in \mathscr{D}} s(D, A) \Big(\sum_{\lambda \in \Lambda(n,k)} [L(P^\lambda \circ f, D)]^2 \Big)^{1/2} \\
&\leqq R(f, A).
\end{aligned}$$

The theorem now follows.

We next prove the theorem which establishes the fact that $\varphi(D)=\nu(f, D)$ is a quasi-additive set function in the sense of L. Cesari [3], [4].

2.4. THEOREM. *Let $f\in\mathscr{R}^*(n,k)$. If $\varepsilon>0$ then there is a $\delta_0>0$ such that for each $\mathscr{D}_0$ with $\delta(\mathscr{D}_0)<\delta_0$ there is a $\delta_1>0$ such that $\delta(\mathscr{D})<\delta_1$ implies*

(i)
$$\sum_{D_0\in\mathscr{D}_0}\left|\nu(f, D_0)-\sum_{D\in\mathscr{D}} s(D, D_0)\nu(f, D)\right| < \binom{n}{k}\varepsilon$$

and

(ii)
$$\sum_{D\in\mathscr{D}}\left[1-\sum_{D_0\in\mathscr{D}_0} s(D, D_0)\right]|\nu(f, D)| < 2\binom{n}{k}\varepsilon.$$

That is, $\nu(f, D)$ is a quasi-additive set function.

Proof. There is $\delta_0>0$ such that for all $\lambda\in\Lambda(n,k)$ and $\mathscr{D}$ with $\delta(\mathscr{D})<\delta_0$ we have $0\leqq U(P^\lambda\circ f, X)-\sum_{D\in\mathscr{D}}|\nu(P^\lambda\circ f, D)|<\varepsilon$.

Let $\mathscr{D}_0$ be such that $\delta(\mathscr{D}_0)<\delta_0$ and N be the number of elements in $\mathscr{D}_0$. There is $\delta_1>0$ such that for all $\lambda\in\Lambda(n,k)$, $D_0\in\mathscr{D}_0$ and $\mathscr{D}$ with $\delta(\mathscr{D})<\delta_1$ we have

$$0 \leqq U(P^\lambda\circ f, D_0)-\sum_{D\in\mathscr{D}} s(D, D_0)|\nu(P^\lambda\circ f, D)| < \varepsilon/N.$$

Now,

$$\begin{aligned}
&\sum_{D_0\in\mathscr{D}_0}\left|\nu(P^\lambda\circ f, D_0)-\sum_{D\in\mathscr{D}} s(D, D_0)\nu(P^\lambda\circ f, D)\right| \\
&\quad\leqq \sum_{D_0\in\mathscr{D}_0}\left|\left[U^+(P^\lambda\circ f, D_0)-\sum_{D\in\mathscr{D}} s(D, D_0)U^+(P^\lambda\circ f, D)\right]\right. \\
&\qquad\qquad\left.-\left[U^-(P^\lambda\circ f, D_0)-\sum_{D\in\mathscr{D}} s(D, D_0)U^-(P^\lambda\circ f, D)\right]\right| \\
&\quad\leqq \sum_{D_0\in\mathscr{D}_0}\left[U(P^\lambda\circ f, D_0)-\sum_{D\in\mathscr{D}} s(D, D_0)U(P^\lambda\circ f, D)\right] \\
&\quad\leqq \sum_{D_0\in\mathscr{D}_0}\left[U(P^\lambda\circ f, D_0)-\sum_{D\in\mathscr{D}} s(D, D_0)|\nu(P^\lambda\circ f, D)|\right] < \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{D\in\mathscr{D}}\left[1-\sum_{D_0\in\mathscr{D}_0} s(D, D_0)\right]|\nu(P^\lambda\circ f, D)| \\
&\qquad\leqq \sum_{D\in\mathscr{D}}|\nu(P^\lambda\circ f, D)|-\sum_{D_0\in\mathscr{D}_0}\sum_{D\in\mathscr{D}} s(D, D_0)|\nu(P^\lambda\circ f, D)| \\
&\qquad\leqq U(P^\lambda\circ f, X)-\sum_{D_0\in\mathscr{D}_0}[U(P^\lambda\circ f, D_0)-\varepsilon/N] \\
&\qquad\leqq U(P^\lambda\circ f, X)-\sum_{D_0\in\mathscr{D}_0}|\nu(P^\lambda\circ f, D_0)|+\varepsilon \\
&\qquad< 2\varepsilon.
\end{aligned}$$

The theorem follows easily.

2.5. Remark. The notion of quasi-additivity has been extensively studied in [3], [4], [5] and [11]. We refer the reader to these papers for some further facts concerning mappings f in the class $\mathscr{R}^*(n, k)$. Notably, we can infer from these papers that for mappings in the class $\mathscr{T}^*(n, k)$ Cesari's Geöcze area equals the lower area. We also refer the reader to [8] for the quasi-additivity approach to the existence of a current valued measure associated with mappings in the class $\mathscr{R}^*(n, k)$.

3. **Area measures.** Let $f \in \mathscr{T}(n, k)$ and, for convenience, assume $X = \text{domain}\,(f)$ is compact and locally connected in addition to those conditions assumed in §1 above. Then the following diagram commutes:

$$(**) \qquad \begin{array}{ccccc} X & \xrightarrow{m} & \mathscr{M} & \xrightarrow{l} & R^n \\ & \searrow^{m_\lambda} & \downarrow \Pi_\lambda & & \downarrow P^\lambda \\ & & \mathscr{M}_\lambda & \xrightarrow[l_\lambda]{} & R^k \end{array}$$

where m, l and m_λ, l_λ are the monotone-light factorizations of f and $P^\lambda \circ f$ and $\mathscr{M}$ and $\mathscr{M}_\lambda$ are the middle spaces of the respective mappings. The mappings Π_λ are also monotone, $\lambda \in \Lambda(n, k)$.

Suppose further that $R(f, X) < \infty$. For each $Z \subset \mathscr{M}$ and $\lambda \in \Lambda(n, k)$, we define the following outer measures on $\mathscr{M}$:

$$\begin{aligned} \mu(Z) &= \inf\{R(f, m^{-1}[B]) : Z \subset B, B \text{ open}\}, \\ \mu_\lambda(Z) &= \inf\{R(P^\lambda \circ f, m^{-1}[B]) : Z \subset B, B \text{ open}\}, \\ \mu_\lambda^\pm(Z) &= \inf\{U^\pm(P^\lambda \circ f, m^{-1}[B]) : Z \subset B, B \text{ open}\}. \end{aligned}$$

We also define the signed set function

$$\sigma_\lambda(Z) = \mu_\lambda^+(Z) - \mu_\lambda^-(Z).$$

3.1. Theorem. *Suppose $f \in \mathscr{R}^*(n, k)$ and $\lambda \in \Lambda(n, k)$. Then μ, μ_λ, $\mu_\lambda^\pm$ are Borel regular measures on $\mathscr{M}$. Furthermore, σ_λ is a signed measure on the algebra of Borel subsets of $\mathscr{M}$ and $(\sigma_\lambda)^\pm = \mu_\lambda^\pm$.*

Proof. This theorem is a consequence of quasi-additivity. See [2], [11], [4] and [5].

Suppose $f \in \mathscr{R}^*(n, k)$. Then we define a k-vector valued measure σ on the algebra of Borel subsets of $\mathscr{M}$ by

$$\sigma(Z) = \sum_{\lambda \in \Lambda(n,k)} \sigma_\lambda(Z) e_\lambda.$$

Then the total variation of σ with respect to the Euclidean norm is μ. This again follows from the quasi-additivity of $\varphi(D) = \nu(f, D)$.

3.2. REMARK. There are natural outer measures $\Pi_{\lambda\#}(\mu_\lambda)$ defined on $\mathscr{M}_\lambda$ by letting $\Pi_{\lambda\#}(\mu_\lambda)=\mu_\lambda \circ \Pi_\lambda^{-1}$. It is a simple matter to show for each $Z\subset\mathscr{M}_\lambda$ that

$$\Pi_{\lambda\#}(\mu_\lambda)(Z) = \inf\{R(P^\lambda \circ f, m_\lambda^{-1}[B]) : Z \subset B, B \text{ open}\}.$$

Similar remarks hold for $\Pi_{\lambda\#}(\mu_\lambda^\pm)$. These measures on $\mathscr{M}_\lambda$ are also Borel regular.

3.3. REMARK. Since $f \in \mathscr{T}^*(k, k)$ if and only if f is BV, we have that μ, $\mu^\pm$ are Borel regular measures on $\mathscr{M}$ whenever f is a BV flat mapping. Also, μ^+ and μ^- are mutually singular. Finally, we have, for $A\subset X=\text{domain}(f)$, that $R(f, A) =\mu[m(A)-m(\text{boundary } A)]$, where boundary A is computed relative to R^k. See [11].

4. **Essential cylindrical property.** For the sake of convenience all mappings f in this section will be such that $X=\text{domain}(f)$ is compact and locally connected.

First, let f be a BV flat mapping and let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "m"'] & & R^k \\ & \mathscr{M} \arrow[ur, "l"'] & \end{tikzcd}$$

be the monotone-light factorization of f. As remarked in 3.3 above there are three Borel regular measures μ, μ^+ and μ^- on $\mathscr{M}$ and μ^+ and μ^- are mutually singular. For each set A open in X, the functions $N(f, A, \cdot)$, $N^\pm(f, A, \cdot)$ are nonnegative-integer valued lower-semicontinuous functions on R^k such that $N(f, A, y) =N^+(f, A, y)+N^-(f, A, y)$ for almost every y in R^k and $\int N(f, A, y)\, dy<\infty$.

4.1. PROPOSITION. *Let f be a BV flat mapping. Then there are two Borel subsets E^+ and E^- of $\mathscr{M}$ with the following properties*:

(i) $E^+ \cap E^- = \varnothing$;

(ii) $\mu(E^\pm \cap B)=\mu^\pm(E^\pm \cap B)=\mu^\pm(B)$ *for Borel subsets B of $\mathscr{M}$*; *and*

(iii) *if $z \in E^\pm$, G is an open neighborhood of z and $y=l(z)$ then* $1 \leqq N(f, m^{-1}(G), y)$.

Proof. Let $\mathscr{B}$ be a countable basis for the topology of $\mathscr{M}$. Then $\mu(B)= \int N(f, m^{-1}(B), y)\, dy$ for each $B \in \mathscr{B}$. Let $N_B=l^{-1}\{y : N(f, m^{-1}(B), y)=0\} \cap B$. N_B is a closed subset of B and hence is an F_σ subset of $\mathscr{M}$. It is easily shown that $\mu(N_B)=0$. Let $S=\mathscr{M}-\bigcup\{N_B : B \in \mathscr{B}\}$. S is a G_δ subset of $\mathscr{M}$.

Next, observe that there are disjoint Borel subsets S^+ and S^- of $\mathscr{M}$ which satisfy the condition $\mu^\pm(S^\pm)=\mu^\pm(\mathscr{M})$ and $\mu^\pm(S^\mp)=0$. Since $\mu=\mu^+ +\mu^-$, we have $\mu(S^+ \cup S^-)=\mu(\mathscr{M})$. So, $\mu(S^\pm)=\mu^\pm(S^\pm)=\mu^\pm(\mathscr{M})$.

Let $E^\pm = S \cap S^\pm$. (i) is obviously true. Since $\mu(\mathscr{M}-S)=0$ and $\mu(\mathscr{M}-(S^+ \cup S^-)) =0$, (ii) follows. To prove (iii), let $B \in \mathscr{B}$ with $z \in B\subset G$. Then $z \notin N_B$ and hence $1 \leqq N(f, m^{-1}(B), y) \leqq N(f, m^{-1}(G), y)$.

The proposition is now proved.

In the remainder of this paper we will need to refer to the Lebesgue measure of subsets A of R^k. We will denote it by $\mathscr{L}^k[A]$.

4.2. LEMMA. *If $E^{\pm}$ are the Borel subsets of $\mathscr{M}$ in Proposition* 4.1 *and* $Z\subset E^{\pm}$ *with* $\mu^{\pm}(Z)=0$ *then* $\mathscr{L}^k[l(Z)]=0$.

Proof. $Z\subset E^{\pm}$ and $\mu^{\pm}(Z)=0$ implies $\mu(Z)=0$. Let G be any open subset of $\mathscr{M}$ with $Z\subset G$. Then $\mathscr{L}^k[l(Z)]\leqq\int N(f, m^{-1}(G), y)\,dy=R(f, m^{-1}(G))$ and the lemma follows from the definition of $\mu(Z)=0$.

4.3. REMARK. If $E=E^+\cup E^-$ then E satisfies the following two conditions:

(i) $\mu(E)=\mu(\mathscr{M})$.

(ii) If $Z\subset E$ and $\mu(Z)=0$ then $\mathscr{L}^k[l(Z)]=0$.

Borel subsets of $\mathscr{M}$ with the above two properties are called essential sets. See [5].

Next suppose f is a mapping in the class $\mathscr{T}(n, k)$ with $P^\lambda\circ f$ of bounded variation for each $\lambda\in\Lambda(n, k)$ and consider the commutative diagram (**). For each $\lambda\in\Lambda(n, k)$, $P^\lambda\circ f$ has an essential set E_λ. We say f has the *essential cylindrical property* if

$$\mathscr{L}^k\{y\in R^k : \exists z\in E_\lambda \ni l_\lambda(z) = y \text{ and diam } [l(\Pi_\lambda^{-1}(z))] > 0\} = 0$$

for each $\lambda\in\Lambda(n, k)$. It is clear that the notion of essential cylindrical property is independent of choice of the essential sets E_λ, $\lambda\in\Lambda(n, k)$.

4.4. THEOREM. *If $f\in\mathscr{R}^*(n, k)$ then f possesses the essential cylindrical property.*

Proof. This theorem is an easy consequence of the next theorem due to the definition of μ_λ, Theorem 3.1 and Remark 3.2.

In the remainder of this section we investigate the relationship between measures on $\mathscr{M}$ and the essential cylindrical property of f where $f\in\mathscr{T}(n, k)$ with $P^\lambda\circ f$ of bounded variation, $\lambda\in\Lambda(n, k)$. Consider again the commutative diagram (**). For each flat mapping $P^\lambda\circ f$, $\lambda\in\Lambda(n, k)$, we have the Borel regular measures $\bar\mu_\lambda$, $\bar\mu_\lambda^{\pm}$ on the middle space $\mathscr{M}_\lambda$ of $P^\lambda\circ f$.

Area measures should be defined on $\mathscr{M}$ rather than $\mathscr{M}_\lambda$. To each flat mapping $P^\lambda\circ f$, there should be a measure ψ_λ on $\mathscr{M}$ satisfying the following natural conditions:

(i) ψ_λ is a Borel regular measure on $\mathscr{M}$.

(ii) If G is an open subset of $\mathscr{M}$ then $R(P^\lambda\circ f, m^{-1}(G))=\psi_\lambda(G)$.

(iii) $\Pi_{\lambda\#}(\psi_\lambda)=\bar\mu_\lambda$.

We remark that conditions (i) and (ii) imply (iii). Condition (iii) is included to explicitly emphasize the interplay of $\mathscr{M}$ and $\mathscr{M}_\lambda$.

4.5. THEOREM. *Let $f\in\mathscr{T}(n, k)$ with $P^\lambda\circ f$ of bounded variation for $\lambda\in\Lambda(n, k)$. Then, natural measures ψ_λ on $\mathscr{M}$, $\lambda\in\Lambda(n, k)$, exist if and only if f possesses the essential cylindrical property.*

Proof. Suppose ψ_λ, $\lambda\in\Lambda(n, k)$, are natural measures on $\mathscr{M}$. Let E_λ be essential subsets of $\mathscr{M}_\lambda$ for $P^\lambda\circ f$. For each positive integer m and $\lambda\in\Lambda(n, k)$, let $H_m^\lambda=\{z\in E_\lambda : \text{diam } [\Pi_\lambda^{-1}(z)]\geqq 1/m\}$. H_m^λ is a Borel set. Consider any finite family

$\{B_1, B_2, \ldots, B_t\}$ of closed sets of $\mathscr{M}$ with diam $[B_j] < 1/m$ and $\bigcup_{j=1}^t B_j$ equal to the closure of $\Pi_\lambda^{-1}(H_m^\lambda)$. Since ψ_λ is Borel regular, there is for every j and $\varepsilon > 0$, an open set $G_j \supset B_j$ with diam $(G_j) < 1/m$ and $\psi_\lambda(G_j - B_j) < \varepsilon/t$. By Remark 3.3 and the monotonicity of Π_λ, we have

$$\psi_\lambda(G_j) = R(P^\lambda \circ f, m^{-1}[G_j]) = R(P^\lambda \circ f, m^{-1}[G_j - B_j]) = \psi_\lambda(G_j - B_j) < \varepsilon/t.$$

Hence

$$\psi_\lambda(\Pi_\lambda^{-1}[H_m^\lambda]) \leqq \sum_{j=1}^t \psi_\lambda(B_j) \leqq \sum_{j=1}^t \psi_\lambda(G_j) < \varepsilon.$$

That is,

$$0 = \psi_\lambda(\Pi_\lambda^{-1}[H_m^\lambda]) = \Pi_{\lambda\#}(\psi_\lambda)(H_m^\lambda) = \bar{\mu}_\lambda(H_m^\lambda).$$

It now follows easily that $\bar{\mu}_\lambda\{z \in E_\lambda : \text{diam } [\Pi_\lambda^{-1}(z)] > 0\} = 0$ from which we infer that $\mathscr{L}^k[\{y \in R^k : \exists z \in E_\lambda \ni y = l_\lambda(z) \text{ and diam } [l(\Pi_\lambda^{-1}(z))] > 0\}] = 0$.

Conversely, suppose f has the essential cylindrical property. Using the mapping Π_λ and the Carathéodory construction, we can induce on $\mathscr{M}$ a Borel regular measure ψ_λ such that, for any Borel subset B of $\mathscr{M}$,

$$\psi_\lambda(B) = \int n(\Pi_\lambda | B, z)\, d\bar{\mu}_\lambda(z),$$

where $n(\Pi_\lambda | B, z)$ is the number of elements in $\Pi_\lambda^{-1}(z) \cap B$ (possibly $+\infty$). See [7, §2.10.10].

Since Π_λ is a monotone mapping, we have for each Borel subset B of $\mathscr{M}$ that $\bar{\mu}_\lambda[\{z : n(\Pi_\lambda | B, z) \geqq 2\}] = 0$. Hence, for Borel subsets A of $\mathscr{M}_\lambda$ we have

$$\Pi_{\lambda\#}(\psi_\lambda)(A) = \psi_\lambda[\Pi_\lambda^{-1}(A)] = \int n(\Pi_\lambda | \Pi_\lambda^{-1}(A), z)\, d\bar{\mu}_\lambda(z)$$
$$= \int_A d\bar{\mu}_\lambda(z) = \bar{\mu}_\lambda(A).$$

Let G be an open subset of $\mathscr{M}$. Since $m^{-1}(G) \subset m_\lambda^{-1}[\Pi_\lambda(G)]$, we have $R(P^\lambda \circ f, m^{-1}(G)) \leqq \bar{\mu}_\lambda[\Pi_\lambda(G)] = \int n(\Pi_\lambda | G, z)\, d\bar{\mu}_\lambda(z) = \psi_\lambda(G)$.

For each $\varepsilon > 0$, we next establish $R(P^\lambda \circ f, m^{-1}[G]) > \psi_\lambda(G) - \varepsilon$. Since $\bar{\mu}_\lambda[B] = 0$ where $B = \{z : \text{diam } [\Pi_\lambda^{-1}(z)] > 0\}$, we have $\psi_\lambda(\Pi_\lambda^{-1}[B]) = 0$. Therefore, there is a compact subset F of $G - \Pi_\lambda^{-1}[B]$ such that $\psi_\lambda(F) > \psi_\lambda(G) - \varepsilon$. Since F is a subset of the set on which Π_λ is one-to-one, we have $\Pi_\lambda(\mathscr{M} - G) \cap \Pi_\lambda(F) = \varnothing$. Hence there is an open subset K of $\mathscr{M}_\lambda$ such that $K \supset \Pi_\lambda(F)$ and $m_\lambda^{-1}(K) \subset m^{-1}(G)$. Consequently,

$$\begin{aligned} R(P^\lambda \circ f, m^{-1}[G]) &\geqq R(P^\lambda \circ f, m_\lambda^{-1}[K]) = \bar{\mu}_\lambda(K) \\ &\geqq \bar{\mu}_\lambda(\Pi_\lambda[F]) = \int n(\Pi_\lambda | F, z)\, d\bar{\mu}_\lambda(z) \\ &= \psi_\lambda(F) > \psi_\lambda(G) - \varepsilon. \end{aligned}$$

The theorem is now completely proved.

We conclude this paper with our main theorem.

4.6. THEOREM. *Let $f \in \mathscr{T}(n, k)$ with $P^\lambda \circ f$ of bounded variation for $\lambda \in \Lambda(n, k)$. Then $f \in \mathscr{R}^*(n, k)$ if and only if f possesses the essential cylindrical property.*

Proof. One half of the theorem has been proved as Theorem 4.4. We prove the converse. Suppose f possesses the essential cylindrical property. Let $\varepsilon > 0$.

For each $\lambda \in \Lambda(n, k)$ there are disjoint Borel subsets E_λ^+ and E_λ^- of $\mathscr{M}_\lambda$ such that $E_\lambda = E_\lambda^+ \cup E_\lambda^-$ is an essential set for $P^\lambda \circ f$, $\bar{\mu}_\lambda \mathbin{L} E_\lambda^\pm = \bar{\mu}_\lambda^\pm$, and $z \in E_\lambda$ implies $\operatorname{diam} \Pi_\lambda^{-1}(z) = 0$. ($\bar{\mu}_\lambda \mathbin{L} E_\lambda^\pm$ means the measure given by $(\bar{\mu}_\lambda \mathbin{L} E_\lambda^\pm)(B) = \bar{\mu}_\lambda(E_\lambda^\pm \cap B)$.) Let $\mathscr{P}$ be a finite Borel partition of $\mathscr{M}$ such that each of the Borel sets $\Pi_\lambda^{-1}(E_\lambda^+)$, $\Pi_\lambda^{-1}(E_\lambda^-)$ and $\mathscr{M} - \Pi_\lambda^{-1}(E_\lambda)$, $\lambda \in \Lambda(n, k)$, is the union of sets in $\mathscr{P}$. Let M be the number of sets S in the partition $\mathscr{P}$.

Suppose $\lambda \in \Lambda(n, k)$ and $S \in \mathscr{P}$ are such that $\Pi_\lambda(S) \subset E_\lambda^+$. We will employ the measure $\psi_\lambda(B) = \int n(\Pi_\lambda | B, z)\, d\bar{\mu}_\lambda(z)$, given by Theorem 4.5 above. Since ψ_λ is Borel regular, there is an open set G and a closed set F such that $F \subset S \subset G$ such that $\psi_\lambda(G - F) < \varepsilon/M$. Since F is contained in $\Pi_\lambda^{-1}(E_\lambda)$, F is a compact subset of the set points at which Π_λ is one-to-one. We have from Theorem 4.5 that $\psi_\lambda(G) = \int N(P^\lambda \circ f, m^{-1}(G), y)\, dy$. Let C be a compact totally disconnected subset of R^k such that

$$\begin{aligned} \varepsilon/M &> \int_{R^k - C} N(P^\lambda \circ f, m^{-1}(G), y)\, dy \\ &= \int N(P^\lambda \circ f, [m^{-1}(G) - (P^\lambda \circ f)^{-1}(C)], y)\, dy. \end{aligned}$$

$l_\lambda^{-1}(C)$ is a zero-dimensional subset of $\mathscr{M}_\lambda$ and hence $\Pi_\lambda(F) \cap l_\lambda^{-1}(C)$ is also a compact zero-dimensional set. Since Π_λ is a homeomorphism on F, $T = F \cap (l_\lambda \circ \Pi_\lambda)^{-1}(C)$ is a compact zero-dimensional subset of S. Also, $\Pi_\lambda(\mathscr{M} - G) \cap \Pi_\lambda(F) = \varnothing$. Consequently, we have

$$\begin{aligned} 0 &\leqq \psi_\lambda(S) - \psi_\lambda(T) \leqq \psi_\lambda(G) - \psi_\lambda(T) = \psi_\lambda(G - F) + \psi_\lambda(F - T) \\ &\leqq \varepsilon/M + \bar{\mu}_\lambda[\Pi_\lambda(F - T)] = \varepsilon/M + \bar{\mu}_\lambda[\Pi_\lambda(F) - l_\lambda^{-1}(C)] \\ &= \varepsilon/M + \int_{R^k - C} N(P^\lambda \circ f, m^{-1}(G), y)\, dy \leqq 2\varepsilon/M. \end{aligned}$$

Let $T(\lambda, S)$ denote the compact zero-dimensional subset T of S and $G(\lambda, S)$ denote the open set G containing S constructed above. Similar constructions can be made for λ and S with $\Pi_\lambda(S) \subset E_\lambda^-$.

Let $\tilde{\mathscr{D}}$ be any finite system of nonoverlapping domains $\tilde{D}$ in $\mathscr{M}$ such that $\operatorname{diam}(\tilde{D}) < \delta(\lambda, S)$ where $\delta(\lambda, S)$ is the distance from $\mathscr{M} - G(\lambda, S)$ and $T(\lambda, S)$.

Suppose further that each $\tilde{D}$ meets $T(\lambda, S)$ and $\tilde{\mathscr{D}}$ is a cover of $T(\lambda, S)$. Then, if $\Pi_\lambda(S) \subset E_\lambda^+$, we have

$$\begin{aligned}
\sum |\nu(P^\lambda \circ f, m^{-1}(\tilde{D}))| &\geqq \left|\sum \nu(P^\lambda \circ f, m^{-1}(\tilde{D}))\right| \\
&\geqq |\nu(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}])| \\
&= U(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) - 2U^-(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) \\
&= \psi_\lambda(\bigcup \tilde{\mathscr{D}}) - 2U^-(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) \\
&\geqq \psi_\lambda(T) - 2U^-(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) \\
&\geqq \psi_\lambda(S) - 2\varepsilon/M - 2U^-(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]),
\end{aligned}$$

where the summation is extended over $\tilde{\mathscr{D}}$. We next show $U^-(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) < \varepsilon/M$. To see this, consider

$$\begin{aligned}
U^-(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) &= U(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) - U^+(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) \\
&\leqq \psi_\lambda(\bigcup \tilde{\mathscr{D}}) - U^+(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) \\
&\leqq \psi_\lambda(G) - U^+(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) \\
&= \psi_\lambda(G-T) + \psi_\lambda(T) - U^+(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) \\
&< \varepsilon/M - (U^+(P^\lambda \circ f, m^{-1}[\bigcup \tilde{\mathscr{D}}]) - \bar{\mu}_\lambda^+(\Pi_\lambda[T])) \\
&\leqq \varepsilon/M
\end{aligned}$$

where the last inequality is valid because $\Pi_\lambda[\mathscr{M} - \bigcup \tilde{\mathscr{D}}] \cap \Pi_\lambda(T) = \varnothing$ and $\bar{\mu}_\lambda(\Pi_\lambda[T]) = \bar{\mu}_\lambda^+(\Pi_\lambda[T])$. Consequently, we have

$$\psi_\lambda(S) - \sum |\nu(P^\lambda \circ f, m^{-1}[\tilde{D}])| < 4\varepsilon/M.$$

The same inequality holds when λ and S are such that $\Pi_\lambda(S) \subset E_\lambda^-$.

Consider the compact zero-dimensional subset

$$Z = \bigcup \{T(\lambda, S) : \Pi_\lambda(S) \subset E_\lambda^+ \text{ or } \Pi_\lambda(S) \subset E_\lambda^-, S \in \mathscr{P}, \lambda \in \Lambda(n, k)\},$$

and the positive number δ with $\delta < 4\varepsilon$ and

$$\delta < \min \{\delta(\lambda, S) : \Pi_\lambda(S) \subset E_\lambda^+ \text{ or } \Pi_\lambda(S) \subset E_\lambda^-, S \in \mathscr{P}, \lambda \in \Lambda(n, k)\}.$$

Let $\tilde{\mathscr{D}}$ be any finite cover of Z by nonoverlapping open connected sets with diam $(\tilde{D}) < \delta$, $\tilde{D} \in \tilde{\mathscr{D}}$. Such covers $\tilde{\mathscr{D}}$ exist because Z is a compact zero-dimensional set. Let $\lambda \in \Lambda(n, k)$ and let $\mathscr{P}_\lambda = \{S \in \mathscr{P} : \Pi_\lambda(S) \subset E_\lambda\}$. Then $\bigcup \mathscr{P}_\lambda = E_\lambda$. Let $\tilde{\mathscr{D}}_{\lambda,S} = \{\tilde{D} \in \tilde{\mathscr{D}} : \tilde{D} \cap T(\lambda, S) \neq \varnothing\}$. Our earlier calculations now yield

$$\begin{aligned}
U(P^\lambda \circ f, X) - \sum_{\tilde{D} \in \tilde{\mathscr{D}}} &|\nu(P^\lambda \circ f, m^{-1}[\tilde{D}])| \\
&\leqq U(P^\lambda \circ f, X) - \sum_{S \in \mathscr{P}_\lambda} \sum_{\tilde{D} \in \tilde{\mathscr{D}}_{\lambda,S}} |\nu(P^\lambda \circ f, m^{-1}[\tilde{D}])| \\
&= \sum_{S \in \mathscr{P}_\lambda} \left[\psi_\lambda(S) - \sum_{\tilde{D} \in \tilde{\mathscr{D}}_{\lambda,S}} |\nu(P^\lambda \circ f, m^{-1}[\tilde{D}])|\right] \\
&< 4\varepsilon.
\end{aligned}$$

Therefore, we can assert the existence of a finite system $\mathscr{D}$ of nonoverlapping domains $D \subset X$ such that $\delta(\mathscr{D}) < 4\varepsilon$ because m is monotone and l is Lipschitzian with Lipschitz constant not exceeding one. Thus we have shown $f \in \mathscr{R}^*(n, k)$.

References

1. L. Cesari, *L'area di Lebesgue come una misura*, Rend. Mat. e Appl. (5) **14** (1955), 655–673. MR **17**, 470.

2. ———, *Surface area*, Ann. of Math. Studies, no. 35, Princeton Univ. Press, Princeton, N. J., 1956. MR **17**, 596.

3. ———, *Quasiadditive set functions and the concept of integral over a variety*, Trans. Amer. Math. Soc. **102** (1962), 94–113. MR **26** #292.

4. ———, *Extension problem for quasiadditive set functions and Radon-Nikodym derivatives*, Trans. Amer. Math. Soc. **102** (1962), 114–146. MR **26** #293.

5. J. Breckenridge, *Geöcze k-area and measure theoretical methods in surface area theory*, Thesis, University of Michigan, Ann Arbor, Mich., 1969.

6. H. Federer, *Currents and area*, Trans. Amer. Math. Soc. **98** (1961), 204–233. MR **23** #A1006.

7. ———, *Geometric measure theory*, Springer-Verlag, New York, 1969.

8. R. Gariepy, *Current valued measures and Geöcze area*, Thesis, Wayne State University, Detroit, Mich., 1969.

9. T. Nishiura, *The Geöcze k-area and a cylindrical property*, Proc. Amer. Math. Soc. **12** (1961), 795–800. MR **23** #A3238.

10. ———, *The Geöcze k-area and flat mappings*, Rend. Circ. Mat. Palermo (2) **11** (1962), 105–125. MR **26** #6359.

11. ———, *Integrals over a product variety and Fubini theorems*, Rend. Circ. Mat. Palermo (2) **14** (1965), 207–236. MR **33** #5848.

12. T. Radó, *On surface area*, Proc. Nat. Acad. Sci. U.S.A. **31** (1945), 102–106. MR **6**, 204.

13. ———, *Length and area*, Amer. Math. Soc. Colloq. Publ., vol. 30, Amer. Math. Soc., Providence, R. I., 1948. MR **9**, 505.

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