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A REPRESENTATION THEOREM FOR LARGE AND SMALL ANALYTIC SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS IN SECTORS(1)

BY
STEVEN BANK

Abstract. In this paper, we treat first-order algebraic differential equations whose coefficients belong to a certain type of function field. In the particular case where the coefficients are rational functions, our main result states that for any given sector S in the plane, there exists a positive real number N, depending only on the equation and the angle opening of S, such that any solution $y(z)$, which is meromorphic in S and satisfies the condition $z^{-N}y \to \infty$ as $z \to \infty$ in S, must be of the form $\exp \int cz^m(1+o(1))$ in subsectors, where c and m are constants. (From this, we easily obtain a similar representation for analytic solutions in S, which are not identically zero, and for which $z^K y \to 0$ as $z \to \infty$ in S, where the positive real number K again depends only on the equation and the angle opening of S.)

1. **Introduction.** In [5], G. H. Hardy proved the following representation theorem: If $y(x)$ is a real-valued function on an interval $(x_0, +\infty)$, which is a solution on $(x_0, +\infty)$ of a first order equation $Q(x, y, dy/dx)=0$ (where Q is a polynomial), and which for no value of α is $o(x^\alpha)$ as $x \to +\infty$, then the function $y(x)$ or its negative must be of the form $\exp(cx^\gamma(1+\varepsilon(x)))$ where c and γ are fixed constants and $\varepsilon(x) \to 0$ as $x \to +\infty$. The techniques used by Hardy in the proof are valid only for real-valued solutions and, in fact, it is known [12], [13] that Hardy's result does not hold for arbitrary complex-valued solutions. (In [12], it was shown that for any real-valued increasing function $\Phi(x)$ on $(0, +\infty)$, it is possible to find a complex function $h(z)$, which is analytic in a region containing $(0, +\infty)$ and satisfies a first order polynomial equation $Q(z, y, dy/dz)=0$, and which has the property that $|h(x)| > \Phi(x)$ at a sequence of real x tending to $+\infty$. The solutions $h(z)$ constructed in [12] are of the form $P(\lambda z)$, where P is the Weierstrass P-function and λ is a constant depending on Φ.) In this paper, we treat (for a broader class of first order equations), solutions which are meromorphic in some sectorial region, and we obtain a representation analogous to that obtained by Hardy in the real-valued case, for those solutions $y_0(z)$ which have a sufficiently large rate of growth as $z \to \infty$ in the region. (We also obtain a similar result for those solutions which have a sufficiently small rate of growth.)

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More specifically, we treat equations $\Omega(z, y, dy/dz)=0$, where Ω is a polynomial in y and dy/dz, whose coefficients belong to a certain type of field of meromorphic functions which was investigated by W. Strodt in [9] and which we call a logarithmic field of rank p. This is a field of functions, each defined and analytic in a sectorial region approximately of the form

$$a < \arg(z-\beta e^{i(a+b)/2}) < b \tag{1}$$

(for fixed a and b in $(-\pi, \pi)$, and some $\beta \geqq 0$), which contains all logarithmic monomials of rank $\leqq p$ (i.e. all functions of the form

$$M(z) = Kz^{\gamma_0}(\log z)^{\gamma_1}(\log\log z)^{\gamma_2}\cdots(\log_p z)^{\gamma_p}, \tag{2}$$

for real γ_j and complex $K \neq 0$), and which has the property that for every element f in the field except zero, there is a logarithmic monomial M of rank $\leqq p$ such that $f/M \to 1$ as $z \to \infty$ over a filter base (denoted $F(a, b)$), which consists essentially of the sectors (1) as $\beta \to +\infty$. (The filter base $F(a, b)$ was introduced in [8, §94] and is reviewed in §2 below for the reader's convenience.) The set of all rational combinations of logarithmic monomials of rank $\leqq p$ is the simplest example of a logarithmic field of rank p. (Since any such field (e.g. for $p=0$) contains the field of rational functions, our results include, as a special case, the case where Ω has polynomial coefficients.) In part (a) of our main result (§3), we prove the existence of a positive real number N_0, which depends only on Ω and the sector angles a and b, with the property that any solution $y_0(z)$ of $\Omega(z, y, dy/dz)=0$, which is meromorphic in an element of $F(a, b)$ and for which $z^{-N_0}y_0 \to \infty$ as $z \to \infty$ over $F(a, b)$, must be of the form $y_0(z)=\exp\int M(z)(1+\varepsilon(z))$, where $M(z)$ is a logarithmic monomial of rank $\leqq p$ and $\varepsilon(z)$ is an analytic function in an element of $F(a, b)$ such that $\varepsilon(z) \to 0$ as $z \to \infty$ over $F(a, b)$. As an easy consequence of part (a), we obtain in part (b), a similar representation for "sufficiently small" solutions. The first step in the proof of the main result is to show (§4) the existence of a positive number N_1 (again depending only on Ω, a and b) with the property that any solution $y_1(z)$ of $\Omega(z, y, dy/dz)=0$, which is analytic and not identically zero in an element of $F(a, b)$, and for which $z^{N_1}y_1 \to 0$ as $z \to \infty$ over $F(a, b)$, must be free of zeros in some element of $F(a, b)$. The number N_0 (and the corresponding number for "sufficiently small" solutions) are exhibited in the proof of the main result, and their explicit calculation in specific examples is discussed in §6.

§7 is devoted to the proof of a technical result in complex variables which is needed in the proof of the lemma in §4. It is put at the end of the paper to avoid unduly interrupting the main line of thought.

Finally, we conclude with three remarks. First, an existence theorem for solutions y_0 of $\Omega=0$ which satisfy the condition $z^{-\alpha}y_0 \to \infty$ for all $\alpha \geqq 0$, as $z \to \infty$ over $F(a, b)$, was proved by the author in [1, §3]. Secondly, as a partial converse to this existence theorem, a much weaker result along the line of part (a) of our main result here was proved in [1, §6B]. This result in [1], which dealt with solutions satisfying the condition $z^{-\alpha}y_0 \to \infty$ for all $\alpha \geqq 0$, imposed severe restrictions on the form of the differential polynomial Ω. We emphasize that no restrictions on the form of Ω are imposed in our results here. Thirdly, the author would like to acknowledge valuable conversations with Robert Kaufman and Gilbert Stengle.

2. **Preliminaries.** (a) [8, §94]. Let $-\pi \leqq a < b \leqq \pi$. For each nonnegative real-valued function ψ on $(0, (b-a)/2)$, let $T(\psi)$ be the union (over $\delta \in (0, (b-a)/2)$) of all sectors

$$a+\delta < \arg(z-\psi(\delta)\exp(i(a+b)/2)) < b-\delta. \tag{3}$$

The set of all $T(\psi)$ (for all choices of ψ) is denoted $F(a, b)$ and is a filter base which converges to ∞ by [8, §95]. Each $T(\psi)$ is simply-connected by [8, §93]. If $W(z)$ is analytic in $T(\psi)$, then the symbol $\int W$ will stand for a primitive of W in $T(\psi)$.

(b) [8, §13]. If f is meromorphic in some $T(\psi)$ and λ is a complex number, then $f \to \lambda$ over $F(a, b)$ means that for any $\varepsilon > 0$, there is a ψ_1 such that $|f(z)-\lambda| < \varepsilon$ for all z in $T(\psi_1)$. Similarly, $f \to \infty$ over $F(a, b)$ means that for any $N > 0$, there is a ψ_1 such that $|f(z)| > N$ for all z in $T(\psi_1)$. We will occasionally use the notation $f \ll g$ to mean $f/g \to 0$ over $F(a, b)$. From the Cauchy formula for derivatives, it follows [4, p. 309] that if $f \to 0$ over $F(a, b)$ then $zf'(z) \to 0$ over $F(a, b)$.

(c) The set of all logarithmic monomials of rank $\leqq p$ (i.e. all functions of the form (2)) will be denoted by Δ_p. A logarithmic field of rank p over $F(a, b)$ is a set L of functions, each defined and meromorphic in some $T(\psi)$, with the following properties: (i) L is a field (where, as usual, we identify two elements of L if they agree on an element of $F(a, b)$), (ii) $\Delta_p \subset L$, and (iii) for every element f in L except zero, there exists M in Δ_p such that $f/M \to 1$ over $F(a, b)$.

3. We now state our main result. The proof will be given in §5.

THEOREM. *Let $\Omega(z, y, y') = \sum f_{kj}(z)y^k(y')^j$ be a polynomial in y and y' whose coefficients f_{kj} belong to a logarithmic field of rank p over $F(a, b)$. Then there exist positive real numbers N_0 and N_1 (each depending only on Ω, a and b) such that the following conclusions hold:*

(a) *If $y_0(z)$ is a solution of $\Omega(z, y, y')=0$ which is defined and meromorphic in an element of $F(a, b)$ and for which $z^{-N_0}y_0 \to \infty$ over $F(a, b)$, then there exist a logarithmic monomial M of rank $\leqq p$ and a function W, analytic in an element of $F(a, b)$, such that $W/M \to 1$ over $F(a, b)$ and $y_0 = \exp \int W$.*

(b) *If $y_1(z)$ is a solution of $\Omega(z, y, y')=0$ which is defined, analytic and not identically zero in an element of $F(a, b)$, and for which $z^{N_1}y_1 \to 0$ over $F(a, b)$, then there exist a logarithmic monomial M_1 of rank $\leqq p$ and a function V, analytic in an element of $F(a, b)$, such that $V/M_1 \to 1$ over $F(a, b)$ and $y_1 = \exp \int V$.*

4. LEMMA. *Let Ω be as in the statement of the above theorem. Then there exists a positive real number N (depending only on Ω, a and b), such that if $y(z)$ is a solution of $\Omega = 0$ which is defined, analytic and not identically zero in an element of $F(a, b)$, and*

for which $z^N y \to 0$ *over* $F(a, b)$, *then there exists an element* S *of* $F(a, b)$ *in which* $y(z)$ *is analytic and has no zeros.*

Proof. We will impose the necessary conditions on N as the proof proceeds. Initially, let $N>0$ and let $y(z)$ be a solution as described in the hypothesis (i.e. $z^N y \to 0$ over $F(a, b)$).

Set $q=\min\{k+j : f_{kj}\not\equiv 0\}$ and $t=\max\{j : f_{q-j,j}\not\equiv 0\}$. By dividing the relation $\Omega(z, y(z), y'(z))\equiv 0$ through by $(y(z))^q$, we obtain

$$\sum_{j=0}^{t} f_{q-j,j}(z)(y'(z)/y(z))^j \equiv \Phi(z), \tag{4}$$

where

$$\Phi(z) = -\sum_{k+j>q} f_{kj}(z)(y'(z)/y(z))^j(y(z))^{k+j-q}. \tag{5}$$

From condition (iii) for a logarithmic field, it easily follows that there exist real numbers $C>0$ and μ, and an element S_1 of $F(a, b)$ such that, for $z\in S_1$,

(6) $|f_{kj}(z)|\leqq |z|^C$ for each (k, j), and

(7) $|f_{kj}(z)|\geqq |z|^\mu$ if $f_{kj}\not\equiv 0$.

Since $y\to 0$ over $F(a, b)$, and since $F(a, b)$ is a filter base converging to ∞, we may assume, in addition, that y is analytic on S_1 and, for $z\in S_1$,

(8) $|y(z)|<1$ and $|z|>\max\{1, 2t\}$.

We now assert that there exist a complex number θ with $\arg\theta=(a+b)/2$, and a sequence $\{z_n\}\to\infty$ lying on $\arg(z-\theta)=(a+b)/2$, such that, for each n,

$$|y'(z_n)/y(z_n)| < |z_n|^{14\delta+1}, \quad \text{where } \delta = (5/3)\pi(b-a)^{-1}. \tag{9}$$

To prove (9), we note first (see §2(a)) that S_1 contains a sector $R\colon a+\sigma<\arg(z-\theta)<b-\sigma$, where $\sigma=(b-a)/5$ and $\arg\theta=(a+b)/2$. Set $a_1=a+\sigma$ and $b_1=b-\sigma$. For $z\in R$, let $f(z)=G(e^{-i\delta\lambda}(z-\theta)^\delta)$, where $G(\zeta)=(\zeta-1)/(\zeta+1)$, $\delta=\pi/(b_1-a_1)$ and $\lambda=(b_1+a_1)/2$. Then it is easily verified that f is a univalent analytic mapping of R onto the unit disk. Now writing $\zeta=u+iv$, it is easily verified that $|G(\zeta)|^2\leqq 1-|\zeta|^{-2}$ when $u\geqq 1$. It easily follows that

$$|G(\zeta)| \leqq 1-(1/2)|\zeta|^{-2} \quad \text{when } \operatorname{Re}\zeta \geqq 1. \tag{10}$$

Let R_1 be the closed sector $a_1+\sigma\leqq\arg(z-\theta)\leqq b_1-\sigma$. For $z\in R_1$, say $z=\theta+re^{i\varphi}$ where $a_1+\sigma\leqq\varphi\leqq b_1-\sigma$, we have $\operatorname{Re}(e^{-i\delta\lambda}(z-\theta)^\delta)=r^\delta(\operatorname{Cos}(\delta(\varphi-\lambda)))$. Clearly $\delta(\varphi-\lambda)$ lies in the closed interval $[-(\pi/2)+\delta\sigma, (\pi/2)-\delta\sigma]$, and on this closed interval $\operatorname{Cos} x$ has a strictly positive minimum. Thus clearly, in view of (10), it follows that, for all sufficiently large r, $|f(z)|\leqq 1-(1/2)r^{-2\delta}$. Hence there exists K_0 such that if $z\in R_1$ and $|z|\geqq K_0$, we have

$$(1-|f(z)|)^{-1} \leqq 2^{2\delta+1}|z|^{2\delta}. \tag{11}$$

Now let g be the inverse of f. Then by (8), the function $\varphi(w)=y(g(w))$ is an analytic function on $|w|<1$ such that $|\varphi(w)|<1$ on $|w|<1$. Let $a_1, a_2, \ldots$ be the sequence of zeros (if any) of φ in $0<|w|<1$, and for each n, let D_n be the disk

$|w-a_n|<(1-|a_n|^2)^4$. Let D be the union of the D_n. Then by the lemma of §7, there exist real numbers $r_0 \in [0, 1)$ and $K_1>0$ such that, for $r \in [r_0, 1)$,

$$|\varphi'(w)/\varphi(w)| \leqq K_1(1-r)^{-6} \quad \text{on } |w| = r \quad \text{if } w \notin D. \tag{12}$$

Since φ is a bounded analytic function, we have $\sum (1-|a_n|)<\infty$ (by [6, p. 240]), so in the terminology of Tsuji [10, p. 7], the exponent of convergence of $\{a_n\}$ is zero. Thus by [10, p. 14], there exists a sequence $\{r_n\}$ in [0, 1) which converges to 1 and such that the circle $|w|=r_n$ is disjoint from D for each n. Let $z_n=g(r_n)$. Hence by (12), for all sufficiently large n,

$$|y'(z_n)g'(r_n)/y(z_n)| \leqq K_1(1-|f(z_n)|)^{-6}. \tag{13}$$

Now $(g(w)-g(0))/g'(0)$ is a normalized univalent analytic function on the unit disk and so by the Koebe distortion theorem [6, p. 351] it follows that, for some constant $K_2>0$, $|g'(w)| \geqq K_2(1-|w|)$ for all w. Hence from (13), for all sufficiently large n,

$$|y'(z_n)/y(z_n)| \leqq (K_1/K_2)(1-|f(z_n)|)^{-7}. \tag{14}$$

It is easily verified that $(z_n-\theta)^\delta=e^{i\delta\lambda}((1+r_n)/(1-r_n))$, and so since $\{r_n\}$ is a sequence in [0, 1) tending to 1, it follows that $\arg(z_n-\theta)=\lambda=(a+b)/2$ for each n, and $\{z_n\}\to\infty$. Since $\sigma<(b-a)/4$, it follows that z_n belongs to R_1 for each n. It is easy to see that (9) now follows from (11) and (14).

Returning to the relation (4), let

$$m = 1+\max\{j : f_{kj} \not\equiv 0 \text{ for some } k\}, \tag{15}$$

and let $d>1$ be a constant which is greater than the number of coefficients f_{kj} which are not identically zero.

We now distinguish two cases:

Case I. $t=0$. In this case, relation (4) is

$$f_{q0}(z) \equiv -\sum_{k+j>q} f_{kj}(y'(z)/y(z))^j(y(z))^{k+j-q}. \tag{16}$$

Now let N be any positive real number such that

$$N > C-\mu+Am+1, \quad \text{where } A = 14\delta+1. \tag{17}$$

We now assert that in this case (i.e. $t=0$), there can be no solutions $y(z)$ of $\Omega=0$, which are defined, analytic and not identically zero in an element of $F(a, b)$, and satisfy $z^N y\to 0$ over $F(a, b)$.

To prove this assertion, we assume the contrary, and let S_1 have the properties stated in (6), (7) and (8). Since $z^N y\to 0$, there exists an element S of $F(a, b)$ such that S is contained in S_1 and

$$|z^N y(z)| < 1 \quad \text{and} \quad |z| > d \quad \text{for all } z \in S. \tag{18}$$

Now since S is in $F(a, b)$, S contains a sector R_2 of the form

$$a+\lambda < \arg(z-\psi_0) < b-\lambda, \tag{19}$$

where arg $\psi_0=(a+b)/2$ and $\lambda=(b-a)/4$. Consider the sequence $\{z_n\}$ whose existence was proved in (9). Since arg $\theta=\arg\psi_0=\arg(z_n-\theta)$ and $|z_n-\theta|\to+\infty$, it follows easily that for sufficiently large n, $\arg(z_n-\psi_0)=(a+b)/2$, and hence $z_n\in R_2$ (where R_2 is given by (19)). Thus there exists an index n_0 such that

(20) $z_{n_0}\in S$, and

(21) $|y'(z_{n_0})/y(z_{n_0})|<|z_{n_0}|^A$ (by (9)).

We now estimate the right side of (16) for $z=z_{n_0}$. Since $k+j\geqq q+1$, and since $|y(z_{n_0})|<1$ (by (8)), we have (by (18)), $|y(z_{n_0})|^{k+j-q}<|z_{n_0}|^{-N}$. In view of (6) and (7) (for $z=z_{n_0}$) and (21), relation (16) gives rise to the inequality

$$|z_{n_0}|^{\mu} \leqq |z_{n_0}|^{C-N}\sum_{k+j>q}|z_{n_0}|^{Aj}. \tag{22}$$

Since $|z_{n_0}|>1$ (by (8)), we obtain $|z_{n_0}|^{\mu}\leqq d|z_{n_0}|^{C-N+Am}$ (where d and m are as in (15)). Hence,

$$|z_{n_0}|^{N-C+\mu-Am} \leqq d. \tag{23}$$

But in view of the condition (17) on N, $N-C+\mu-Am>1$, and so since $|z_{n_0}|>d$ by (18), we see that (23) is impossible. This contradiction proves the assertion that when N is chosen so as to satisfy condition (17), then there are no solutions of the type considered in the statement of the lemma in the case when $t=0$.

Case II. $t>0$. Set $H(z,v)=\sum_{j=0}^{t}f_{q-j,j}(z)v^j$.

We now assert that if E is any positive integer satisfying $E\geqq 1+C-\mu$, and if $v(z)$ is any meromorphic function on S_1, then for any $z\in S_1$,

(24) $|v(z)|\geqq|z|^E$ implies $|H(z,v(z))|\geqq(1/2)|z|^{Et+\mu}$.

To prove (24), let E be any positive integer which satisfies $E\geqq 1+C-\mu$. We may write

$$H(z,v) = v^t(f_{q-t,t}(z)+\Gamma(z,v)), \tag{25}$$

where $\Gamma(z,v)=\sum_{j=0}^{t-1}f_{q-j,j}(z)v^{j-t}$. Now let $z\in S_1$ and let $|v(z)|\geqq|z|^E$. Since $|z|>1$ (by (8)), we have $|v(z)|^{j-t}\leqq|z|^{-E}$ for $j<t$. Hence in view of (6), $|\Gamma(z,v(z))|\leqq t|z|^{C-E}$. Since $\mu-C+E\geqq 1$ and $|z|>2t$ (by (8)), we thus have $|\Gamma(z,v(z))|\leqq(1/2)|z|^{\mu}$. Since $|f_{q-t,t}(z)|\geqq|z|^{\mu}$ by (7), it is clear that (24) now follows from (25). Now let N be any positive number which is greater than $1+C+(E+A)m-(Et+\mu)$, where $A=14\delta+1$. Since $z^Ny\to 0$, there exists an element S of $F(a,b)$ such that $S\subset S_1$ and

(26) $|z^Ny(z)|<1$ and $|z|>2d$ for all $z\in S$.

We assert that

(27) $y(z)$ has no zeros in S.

To prove (27), we assume the contrary, and let w_0 be a zero of y in S. Then the function $z^{-(E+A)}y'(z)/y(z)$ has a pole at w_0 and so there is a deleted neighborhood of w_0, lying in S, containing no zeros of y and on which the function in modulus is greater than 1. Letting w_1 be a point in this deleted neighborhood, we thus have

$$w_1\in S \quad\text{and}\quad 1 < |w_1^{-(E+A)}y'(w_1)/y(w_1)| < \infty. \tag{28}$$

As in Case I (see (20)), the sequence $\{z_n\}$, whose existence was proved in (9), eventually lies in S. Thus there exists an index n_0 such that $z_{n_0} \in S$ and (9) holds for $n=n_0$. Since $E \geqq 1$ and $|z_{n_0}|>1$ (by (8)), we have $|z_{n_0}|^A < |z_{n_0}|^{E+A}$, and hence, by (9),

$$(29) \qquad |y'(z_{n_0})/y(z_{n_0})| < |z_{n_0}|^{E+A}.$$

Since S is connected, clearly there exists a curve, lying in S, joining z_{n_0} to w_1 and not passing through any zeros of y. Along this curve, $|z^{-(E+A)}y'(z)/y(z)|$ is a continuous real-valued function which by (29) is <1 at z_{n_0} and by (28) is >1 at w_1. Hence somewhere along the curve the function assumes the value 1. Thus there exists a point w_2 such that

$$(30) \qquad w_2 \in S \quad \text{and} \quad |y'(w_2)/y(w_2)| = |w_2|^{E+A}.$$

We now consider the function $\Phi(z)$, given by (5), evaluated at $z=w_2$. By (8), $|y(w_2)|<1$, so if $k+j>q$ (i.e. $k+j \geqq q+1$), we have $|y(w_2)|^{k+j-q} \leqq |y(w_2)| < |w_2|^{-N}$ by (26). In view of (6) and (30), it thus follows that

$$(31) \qquad |\Phi(w_2)| \leqq |w_2|^{C-N} \sum_{k+j>q} |w_2|^{j(E+A)},$$

which by definition of m and d (see (15)), leads to $|\Phi(w_2)| \leqq d|w_2|^{C-N+m(E+A)}$. By definition of N, we thus obtain $|\Phi(w_2)| \leqq d|w_2|^{Et+\mu-1}$. But $|w_2|>2d$ (by (26)) and hence we obtain

$$(32) \qquad |\Phi(w_2)| < (1/2)|w_2|^{Et+\mu}.$$

Now the equation (4) is $H(z, y'(z)/y(z)) = \Phi(z)$, and so by (32), $|H(w_2, y'(w_2)/y(w_2))| < (1/2)|w_2|^{Et+\mu}$. Hence by (24), we must have $|y'(w_2)/y(w_2)| < |w_2|^E$. But in view of (30), this implies $|w_2|^{E+A} < |w_2|^E$ which is impossible since $A>0$ and $|w_2|>1$ (by (8)). This contradiction thus establishes the assertion (27) and so the proof of the lemma is now complete.

REMARK. It is clear that the N constructed in the above lemma depends only on Ω, a and b, and can be explicitly calculated in any specific example.

5. **Proof of the main result** (§3). Part (a)—If $y_0(z)$ is a solution of $\Omega=0$, then by dividing the relation $\Omega(z, y_0(z), y_0'(z)) \equiv 0$ through by $(y_0(z))^n$, where $n = \max\{k+j : f_{kj} \not\equiv 0\}$, and setting $v_0 = y_0'/y_0$, we obtain

$$(33) \qquad G(z, v_0(z)) \equiv g(z),$$

where

$$(34) \qquad G(z, v) = \sum_{j=0}^{m} f_{n-j,j}(z)v^j \quad (\text{where } m = \max\{j : f_{n-j,j} \not\equiv 0\}),$$

and where

$$(35) \qquad g(z) = -\sum_{k+j<n} f_{kj}(z)(y_0'(z))^j/(y_0(z))^{n-k}.$$

Now $G(z, v)$ is an algebraic polynomial in v of degree m, whose coefficients belong to a logarithmic field of rank p over $F(a, b)$. It follows from [9, Theorem II,

p. 244] (by applying this result to, in the terminology of [9, p. 246], the logarithmic quadruple $(F, E_0(0, F), R, S_p)$, where $F=F(a, b)$ and R is the set of real numbers) that there exists a logarithmic field of rank p over $F(a, b)$ in which $G(z, v)$ factors completely. Hence there exist distinct functions, $B_1, \dots, B_q$, each defined and meromorphic in some element T of $F(a, b)$, such that the following three conclusions hold:

(36) If $B_j \not\equiv 0$, there exists M_j in Δ_p such that $B_j/M_j \to 1$ over $F(a, b)$.

(37) If $i \neq j$, there exists M_{ij} in Δ_p such that over $F(a, b)$, $(B_i - B_j)/M_{ij} \to 1$ (since $B_i - B_j$ belongs to a logarithmic field of rank p and is not the zero element).

(38) There exist positive integers $m_1, \dots, m_q$ such that

$$G(z, v) = f_{n-m,m}(z)(v-B_1(z))^{m_1} \cdots (v-B_q(z))^{m_q}$$

for all meromorphic functions $v=v(z)$ defined on T.

In view of (36), all the functions $B_j(z)$ are analytic in some element of $F(a, b)$, and we may assume that T has this property. In addition, we may assume $|z|>1$ for all z in T.

Now since the M_{ij} in (37) are logarithmic monomials, there exists a positive real number λ such that $z^{\lambda}M_{ij} \to \infty$ over $F(a, b)$ for all (i, j) with $i \neq j$. Then clearly,

(39) $z^{\lambda}(B_i - B_j) \to \infty$ over $F(a, b)$ if $i \neq j$.

Since the M_j in (36) are logarithmic monomials, there exists a real number $Q < -1$ such that

(40) $z^{-Q}M_j \to \infty$ over $F(a, b)$ for each j.

By condition (iii) for a logarithmic field, clearly there exist real numbers $C>0$ and μ, such that

(41) $f_{kj} \ll z^C$ over $F(a, b)$ for each (k, j), and

(42) $z^{\mu} \ll f_{kj}$ over $F(a, b)$ if $f_{kj} \not\equiv 0$.

Let μ_1 be a real number such that

(43) $\mu_1 \leqq \mu - \lambda m$ and $\mu_1 < \min \{m_j(Q+\lambda) - \lambda m + \mu : 1 \leqq j \leqq q\}$.

Let I be the set of all (k, j) with $k+j<n$ and $f_{kj} \not\equiv 0$. Let N_1 be a real number greater than 1 such that

(44) $N_1 > (C - \mu_1 - j)/(n-k-j)$ for all $(k, j) \in I$.

Finally, if y_0 is a solution of $\Omega = 0$, and if we set $w_0 = 1/y_0$, then it is easily verified that w_0 is a solution of the equation

$$\sum f_{kj}(z)(-1)^j w^{\sigma-(k+2j)}(w')^j = 0, \tag{45}$$

where $\sigma = \max \{k+2j : f_{kj} \not\equiv 0\}$. By the previous lemma (applied to equation (45)), there is a positive real number N_2, such that any solution $w_0(z)$ of equation (45), which is defined, analytic and not identically zero in an element of $F(a, b)$, and for which $w_0 \ll z^{-N_2}$ over $F(a, b)$, must be free of zeros in some element of $F(a, b)$.

We now assert that if we set

$$N_0 = \max \{N_1, N_2\}, \tag{46}$$

then N_0 is the number required in part (a) of the theorem.

To prove this, let $y_0(z)$ be a solution of $\Omega=0$, which is defined and meromorphic in an element of $F(a, b)$, and for which

(47) $z^{N_0} \ll y_0$ over $F(a, b)$.

Set $w_0 = 1/y_0$, so clearly (by (46)), $w_0 \ll z^{-N_2}$. Since w_0 is a solution of (45), we have by construction of N_2, that there is an element of $F(a, b)$ in which

(48) w_0 has no zeros (and so y_0 is analytic).

Thus, for any $\varepsilon > 0$, there exists an analytic branch h_ε of $(z^{N_0}w_0)^\varepsilon$ in some element of $F(a, b)$. Since $h_\varepsilon \to 0$ over $F(a, b)$ (by (47)), we have by Cauchy's formula (see §2(b)) that $zh'_\varepsilon \to 0$ over $F(a, b)$. But clearly, $zh'_\varepsilon = N_0\varepsilon h_\varepsilon - \varepsilon z^{N_0\varepsilon+1}(y'_0/y_0^{1+\varepsilon})$, and so it follows that

(49) $z^{N_0\varepsilon+1}(y'_0/y_0^{1+\varepsilon}) \to 0$ over $F(a, b)$ for any $\varepsilon > 0$.

We now assert that

(50) $g \ll z^{\mu_1}$ over $F(a, b)$ (where μ_1 is as in (43)).

To see this, we refer to (35) and prove that each term $f_{kj}(y'_0)^j/y_0^{n-k}$ (where $k+j<n$) in g is $\ll z^{\mu_1}$. When $j=0$, this term is $\ll z^{C-N_1(n-k)}$ and so $\ll z^{\mu_1}$ by (44). If $j>0$, this term can be written $f_{kj}(y'_0/y_0^{1+\varepsilon})^j$, where $\varepsilon = -1+((n-k)/j)$. Hence by (49), this term is $\ll z^{C-j(N_0\varepsilon+1)}$, which by (44) and (46) is $\ll z^{\mu_1}$, thus proving (50).

From (50), we conclude that the degree m of G must be strictly positive if $\Omega=0$ has such a solution y_0, for if $m=0$, (33) would be impossible since the left side is $\gg z^{\mu_1}$ by (42).

Now set $v_0 = y'_0/y_0$ and $u_0 = z^\lambda v_0$. Then in view of (38), the relation (33) can be written

$$(51) \qquad (u_0(z) - z^\lambda B_1(z))^{m_1} \cdots (u_0(z) - z^\lambda B_q(z))^{m_q} \equiv g_1(z) \quad \text{on } T,$$

where $g_1(z) = z^{\lambda m} g(z)/f_{n-m,m}(z)$, noting that by (34) and (38),

$$(52) \qquad m_1 + \cdots + m_q = m.$$

In view of (42) and (50), clearly $g_1 \ll z^{\lambda m + \mu_1 - \mu}$, and so by (43), $g_1 \to 0$ over $F(a, b)$. From this, together with (39), (48) and the fact that $y_0 \to \infty$, it follows that there exists an element T_1 of $F(a, b)$, with $T_1 \subset T$, such that y_0 and $v_0 = y'_0/y_0$ are analytic on T_1, and for all z in T_1,

(53) $|g_1(z)| \leqq (1/2)^m$, and,

(54) $|z^\lambda B_i(z) - z^\lambda B_j(z)| > 2$ if $i \neq j$.

Now let z_0 be a fixed element of T_1, so $|g_1(z_0)| \leqq (1/2)^m$ by (53). In view of (51) (for $z=z_0$) and (52), it is clearly impossible that $|u_0(z_0) - z_0^\lambda B_j(z_0)| > (1/2)$ for all j. Hence there exists an index $t \in \{1, \dots, q\}$ (t depending on z_0), such that

$$(55) \qquad |u_0(z_0) - z_0^\lambda B_t(z_0)| \leqq (1/2).$$

We now show that the index t will work for every z in T_1, by showing that

$$(56) \qquad |u_0(z) - z^\lambda B_t(z)| \leqq 1 \quad \text{for all } z \text{ in } T_1.$$

To prove (56), we assume the contrary, i.e. there exists $z_1 \in T_1$ such that $|u_0(z_1) - z_1^\lambda B_t(z_1)| > 1$. Since T_1 is pathwise connected, there is a curve, lying in T_1, joining z_0 to z_1. Along this curve, the function $|u_0(z)-z^{\lambda}B_t(z)|$ is a real-valued continuous function which by (55) is $\leqq 1/2$ at z_0 and by assumption is >1 at z_1. Hence, somewhere along the curve the function assumes the value 1. Thus there exists a point $z_2 \in T_1$ such that

$$(57) \qquad |u_0(z_2)-z_2^{\lambda}B_t(z_2)| = 1.$$

But by (53), $|g_1(z_2)| \leqq (1/2)^m$, and so as before, it follows from (51) (for $z=z_2$) and (52) that it is impossible that $|u_0(z_2)-z_2^{\lambda}B_j(z_2)|>1/2$ for all j. Hence there exists an index $k \in \{1,\ldots,q\}$ (k depending on z_2), such that $|u_0(z_2)-z_2^{\lambda}B_k(z_2)| \leqq 1/2$. From (57), it follows that $k \neq t$ and $|z_2^{\lambda}B_t(z_2)-z_2^{\lambda}B_k(z_2)| \leqq 3/2$, which clearly contradicts (54) for $z=z_2$. This establishes (56).

In view of (54) and (56), we clearly have for $j \neq t$, $|u_0(z)-z^{\lambda}B_j(z)|>1$ for all $z \in T_1$, and so it follows from (51) that

$$(58) \qquad |u_0(z)-z^{\lambda}B_t(z)| \leqq |g_1(z)|^{1/m_t} \quad \text{for all } z \text{ in } T_1.$$

Now set $U(z)=u_0(z)-z^{\lambda}B_t(z)$ for $z \in T_1$, and set $V=z^{-\lambda}U$. Then V is analytic on T_1, and since $u_0=z^{\lambda}(y_0'/y_0)$, we have

$$(59) \qquad y_0'/y_0 = B_t+V \quad \text{on } T_1.$$

We now assert that

$$(60) \qquad V \ll z^{Q} \quad \text{over } F(a, b).$$

To prove (60), we observe that by (58), we have $|V(z)| \leqq |z|^{-\lambda}|g_1(z)|^{1/m_t}$ on T_1. But by (42) and (50), $g_1 \ll z^{\lambda m+\mu_1-\mu}$ in $F(a, b)$. Since $\mu_1<m_t(Q+\lambda)-\lambda m+\mu$ (by (43)), it easily follows that, in some element of $F(a, b)$, we have $|V(z)|<|z|^{Q-\varepsilon}$ for some $\varepsilon>0$, from which (60) follows.

We now assert that $B_t \not\equiv 0$ on T_1. If this were not the case, then by (59), $y_0'=Vy_0$ on T_1. Setting $\varphi=z^{-1}y_0$, we would obtain $\varphi(z)=\varphi(z_0)\exp\int_{z_0}^{z}(V(\zeta)-\zeta^{-1})\,d\zeta$ on T_1, for some convenient fixed point z_0. But $V \ll z^{-1}$ by (60) since $Q<-1$, and it would thus follow from [8, Lemma 103] that $\varphi \to 0$ over $F(a, b)$. Hence we would have $y_0 \ll z$, which contradicts $y_0 \gg z^{N_0}$, since $N_0>1$.

Thus $B_t \not\equiv 0$, so by (36), there is a logarithmic monomial M_t of rank $\leqq p$ such that $B_t/M_t \to 1$ over $F(a, b)$. By (60) and (40), $V/M_t \to 0$, so if we set $W=B_t+V$, then $W/M_t \to 1$ over $F(a, b)$, and by (59), $y_0=\exp\int W$. This concludes the proof of part (a).

Part (b). If $y_1(z)$ is a solution of $\Omega=0$, then $w_1=1/y_1$ is a solution of (45). Let N_1 be the positive number for the equation (45), whose existence was proved in part (a) (applied to (45)). Then clearly N_1 is the number required in part (b).

6. **Remark.** Concerning the explicit calculation of N_0 for a given Ω relative to a given $F(a, b)$, it is easily seen that all quantities involved in the calculation of N_0, with the exception of λ, m_j and Q, can be immediately deduced from Ω, a and b. But a suitable value for Q can also be easily determined, since by [9, §36, p. 237],

the logarithmic monomials M_j (see (36)) can be found by applying to $G(z, v)$ the algorithm in [9, §28, p. 236] (or, if the coefficients of G are of the type treated in [2], then in view of [2, §5], one could also use the algorithm in [2, §26]). However, values for λ (see (39)) and the multiplicities m_j are not easily accessible since to find these numbers requires much deeper information on the exact roots B_j of $G(z, v)$ than just knowing the asymptotic behavior of these roots. In certain cases, however, these numbers can be determined. For example, we first calculate the resultant R of G and $\partial G/\partial v$ (see [14, p. 84]). R depends only on the coefficients of G and belongs to the same logarithmic field. It is well known [14, p. 87] that R is related to the discriminant D of G (see [14, p. 82]) by the relation

$$R = f_{n-m,m}D. \tag{61}$$

Thus $R \not\equiv 0$ is a necessary and sufficient condition for all the roots of G to be simple. Hence if $R \not\equiv 0$, then each $m_j = 1$, and by definition of D,

$$D = (f_{n-m,m})^{2m-2} \prod_{i<k} (B_i - B_k)^2. \tag{62}$$

Since $R \not\equiv 0$, there is a logarithmic monomial M^* such that $R/M^* \to 1$ over $F(a, b)$. Since $B_j/M_j \to 1$, if we let γ be a real number such that $M_j \ll z^\gamma$ for all j, then clearly $B_i - B_k \ll z^\gamma$ for all i and k. Hence from (61) and (62), clearly for any pair (i_0, k_0), with $i_0 < k_0$, we have, $(B_{i_0} - B_{k_0})^2 \gg M^* z^{(1-2m)C}/z^{\gamma d}$ where $d = m(m-1)-2$, and thus a suitable value of λ satisfying (39) can be determined explicitly. Hence N_0 can be explicitly calculated when $R \not\equiv 0$.

Of course, even when N_0 cannot be explicitly calculated for a given Ω, our result still provides a representation theorem for all meromorphic solutions y_0 of $\Omega = 0$ for which $z^{-\alpha} y_0 \to \infty$ over $F(a, b)$ for all $\alpha \geqq 0$, and for all solutions y_1 for which $z^\alpha y_1 \to 0$ over $F(a, b)$ for all $\alpha \geqq 0$.

7. Lemma. *Let $\varphi(z)$ be an analytic function on $|z| < 1$ such that $|\varphi(z)| < 1$ on $|z| < 1$. Let $a_1, a_2, \ldots$ be the sequence of zeros of φ in $0 < |z| < 1$, and let D_n be the disk, $|z - a_n| < (1 - |a_n|^2)^4$. Let D be the union of the D_n. Then there exist real numbers $r_0 \in [0, 1)$ and $K_1 > 0$ such that, for $r \in [r_0, 1)$,*

$$|\varphi'(z)/\varphi(z)| \leqq K_1(1-r)^{-6} \quad \textit{on } |z| = r \quad \textit{if } z \notin D. \tag{63}$$

Proof. Set

$$S(z) = \prod_{n \geqq 1} (1 - ((1 - |a_n|^2)/(1 - \bar{a}_n z))) \exp((1 - |a_n|^2)/(1 - \bar{a}_n z)). \tag{64}$$

It is proved in [3, §4] (using the fact that $\sum_{n \geqq 1} (1 - |a_n|)$ converges) that $S(z)$ represents an analytic function in $|z| < 1$, whose sequence of zeros is $\{a_n\}$, and that there exist real numbers $r_1 \in [0, 1)$ and $K_2 > 0$ such that, for $r \in [r_1, 1)$,

$$|S'(z)/S(z)| \leqq K_2(1-r)^{-6} \quad \text{on } |z| = r \quad \text{if } z \notin D. \tag{65}$$

(An analogous estimate for canonical products in the whole plane is developed in [11, p. 75].)

Now by using the same estimates developed by Tsuji [10, p. 12] (where, for our purposes, we take his number p to be 1), it follows that for some $K_3>0$, we have, for $r \in [\frac{1}{2}, 1)$,

$$\log |1/S(z)| \leqq K_3(1-r)^{-2} \quad \text{on } |z| = r \quad \text{if } z \notin D. \tag{66}$$

Let $\varphi(z)$ have a k-fold root at $z=0$. Then clearly there exists an analytic function $\psi(z)$ on $|z|<1$, such that

$$\varphi(z) = z^k e^{\psi(z)} S(z) \quad \text{on } |z| < 1. \tag{67}$$

Since $|\varphi(z)|<1$ on $|z|<1$, it follows easily from (66) that there exists $r_2 \in [\frac{1}{2}, 1)$, such that for $r \in [r_2, 1)$,

$$\operatorname{Re}(\psi(z)) \leqq (K_3+1)(1-r)^{-2} \quad \text{on } |z| = r \quad \text{if } z \notin D. \tag{68}$$

Let $A(r)=\max_{|z|=r} \operatorname{Re}(\psi(z))$. Then it is well known [7, p. 338] that $A(r)$ is increasing. Now by [10, p. 14], there exists $r_3 \in [r_2, 1)$ such that, for any r in $[r_3, 1)$, there is an r', with $r \leqq r' \leqq (r+1)/2$, such that the circle $|z|=r'$ is disjoint from D. Hence from (68), $A(r) \leqq (K_3+1)(1-r')^{-2}$ and so

$$A(r) \leqq 4(K_3+1)(1-r)^{-2} \quad \text{for all } r \text{ in } [r_3, 1). \tag{69}$$

Let $M(r, \psi)=\max_{|z|=r} |\psi(z)|$. By an inequality of Carathéodory [7, p. 338], $M(r, \psi) \leqq 4(1-r)^{-1}(A((1+r)/2)+|\psi(0)|)$, and so $M(r, \psi) \leqq K_4(1-r)^{-3}$ for $r \in [r_3, 1)$, where K_4 is some positive constant. Now applying the Cauchy formula for derivatives (using the contour $|\zeta-z|=(1-|z|)/2$), we easily obtain, for $r \in [r_3, 1)$,

$$|\psi'(z)| \leqq K_5(1-r)^{-4} \quad \text{on } |z| = r, \tag{70}$$

where K_5 is a positive constant. Since by (67), $\varphi'/\varphi=(k/z)+\psi'+(S'/S)$, the result now follows immediately from (65) and (70).

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University of Illinois,
Urbana, Illinois 61801